CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

CANDIDATE NAME

CLASS

INDEX NUMBER

MATHEMATICS

Paper 1

9758/01
02 Sep 2025
3 hours

Additional Materials: Printed Answer Booklet
List of Formulae (MF27)

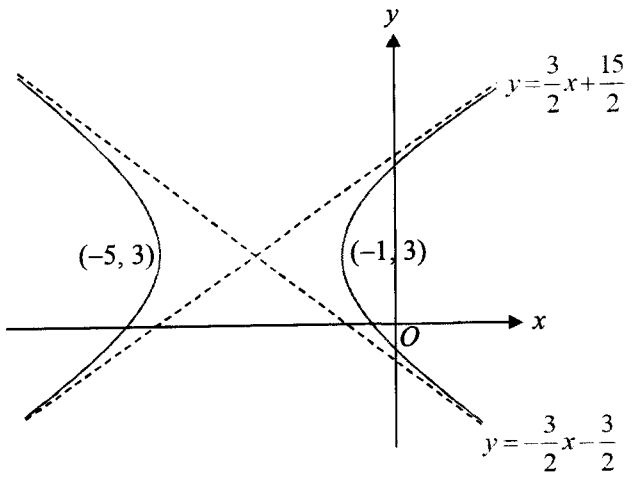
READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

1



The diagram above shows the curve C with equation $\frac{(x+p)^2}{r}-\frac{(y+q)^2}{9}=1$, where p, q and r are constants. The vertices are $(-5,3)$ and $(-1,3)$ and the asymptotes are $y=\frac{3}{2}x+\frac{15}{2}$ and $y=-\frac{3}{2}x-\frac{3}{2}$.

(a) Find the values of p, q and r. [3]

Method ①:

Centre of hyperbola is the mid-point of $(-5,3)$ and $(-1,3)$, i.e.

$$\left(\frac{-5-1}{2},\frac{3+3}{2}\right)=(-3,3)$$

$$\frac{(x+p)^2}{r}-\frac{(y+q)^2}{9}=1$$

$$\frac{[x-(-p)]^2}{(\sqrt{r})^2}-\frac{[y-(-q)]^2}{9}=1$$

Centre is $(-p,-q)$

Comparing $p=3$ and $q=-3$

Method ②:

Centre of hyperbola is the point of intersection of the asymptotes.

$$y=\frac{3}{2}x+\frac{15}{2}---(1)$$

$$y=-\frac{3}{2}x-\frac{3}{2}---(2)$$

Solving (1) & (2),

$x=-3$

$y=3$

$$\frac{(x+p)^2}{h}-\frac{(y+q)^2}{9}=1$$

$$\frac{[x-(-p)]^2}{(\sqrt{r})^2}-\frac{[y-(-q)]^2}{9}=1$$

Centre is $(-p,-q)$

Comparing $p=3$ and $q=-3$

x-distance from vertex to centre of hyperbola is 2, therefore $\sqrt{r}=2 \Rightarrow r=4$

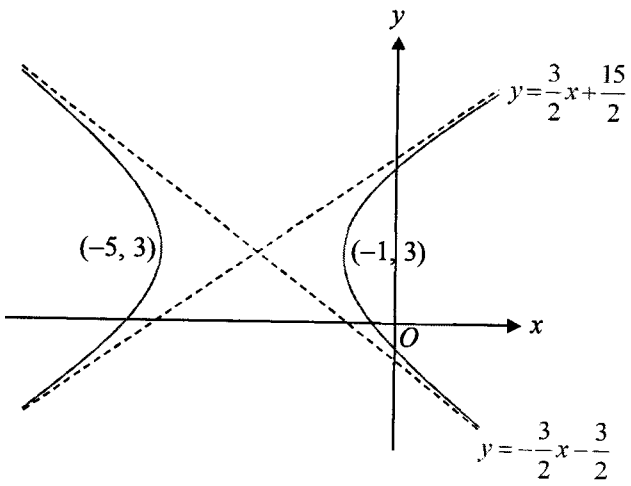
Examiners Feedback:
Candidates generally had no difficulty identifying the centre of the hyperbola $(-3,3)$. However, a number made sign errors when determining p and q. In addition, when finding r, many candidates incorrectly wrote $r=2$ instead of $\sqrt{r}=2$.

(b) The curve D has equation $(x+3)^2+m(y-3)^2=m$, where m is a positive constant. Find the range of m for which curves C and D do not intersect. [2]

$(x+3)^2+m(y-3)^2=m$

$\dfrac{(x+3)^2}{\left(\sqrt{m}\right)^2}+(y-3)^2=1$

Ellipse, centre (–3, 3)
[Notice that the centre of the ellipse is also the centre of the hyperbola]



$\sqrt{m}<2$

$m<4$

Since m is a positive constant, $0<m<4$.

Examiners Feedback:
A number of candidates attempted to use $b^2-4ac<0$ to find the range of values of m. This often led to lengthy solutions, and many were unsuccessful in identifying the correct range. Candidates who adopted a graphical approach were generally more successful, with many obtaining $m<4$. However, they failed to take into account the requirement that m is a positive constant. In addition, a number of candidates wrote $m<2$ when it should be $\sqrt{m}<2$.

2 The nth term of a sequence is given by $u_n=an^2+bn+c$. The first three terms of this sequence of numbers are 2, 6 and 12.

(a) Find the values of a, b and c. [3]

When $n=1$, $u_1=a(1)^2+b(1)+c=2$

$a+b+c=2$ ---- (1)

When $n=2$, $u_2=a(2)^2+b(2)+c=6$

$4a+2b+c=6$ ---- (2)

When $n=3$, $u_3=a(3)^2+b(3)+c=12$

$9a+3b+c=12$ ---- (3)

Using G.C.,
$a=1$, $b=1$ and $c=0$

Examiners Feedback:
This question was generally well attempted. However, some candidates did not make use of the G.C. in solving this part, which led to unnecessarily lengthy working. Candidates should note that not using the G.C. effectively may result in a loss of valuable time.

(b) Given that $\sum_{n=1}^{N}\frac{1}{u_n}=1-\frac{1}{N+1}$, find $\sum_{n=8}^{\infty}\frac{1}{u_n}$. [3]

Method ①:

As $N\to\infty$, $\frac{1}{N+1}\to 0$.

$\therefore \sum_{n=1}^{\infty}\frac{1}{u_n}=1$

$$\sum_{n=8}^{\infty}\frac{1}{u_n}=\sum_{n=1}^{\infty}\frac{1}{u_n}-\sum_{n=1}^{7}\frac{1}{u_n}$$
$$=1-\left(1-\frac{1}{7+1}\right)$$
$$=\frac{1}{8}$$

Method ②:

$$\sum_{n=8}^{N}\frac{1}{u_n}=\sum_{n=1}^{N}\frac{1}{u_n}-\sum_{n=1}^{7}\frac{1}{u_n}$$
$$=\left(1-\frac{1}{N+1}\right)-\left(1-\frac{1}{7+1}\right)$$
$$=\frac{1}{8}-\frac{1}{N+1}$$

As $N\to\infty$, $\frac{1}{N+1}\to 0$.

$\therefore \sum_{n=8}^{\infty}\frac{1}{u_n}=\frac{1}{8}$

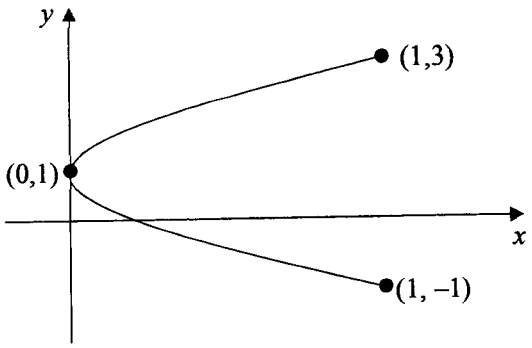
Examiners Feedback:

The presentation for this part needs improvement. Many candidates wrote $\frac{1}{\infty+1}$, which makes no mathematical sense. In addition, a number of candidates appeared to have forgotten the change of base formula in sigma notation.

3 A curve C has parametric equations

$$x=\sin^2 t,\quad y=1+2\sin t,\quad -\frac{\pi}{2}\le t\le\frac{\pi}{2}.$$

(a) Sketch C, stating clearly the coordinates of the endpoints and the coordinates of the y-intercept(s). [2]



Examiners Feedback:

The parametric curve was generally well drawn by many candidates. However, the coordinates of the end points and the y-intercept must be clearly labelled as required by the question, and a number of candidates failed to do so. In addition, candidates who drew open circles at the end points of the curve were penalised.

(b) Find the exact cartesian equation of l, the normal to C at the point $\left(\frac{1}{4},2\right)$. [4]

At $\left(\frac{1}{4},2\right)$,

Since $y=1+2\sin t$,

$1+2\sin t=2$

$\sin t=\frac{1}{2}$

$t=\frac{\pi}{6}$

OR

Since $x=\sin^2 t$

$\sin^2 t=\frac{1}{4}$

$\sin t=\pm\frac{1}{2}$

$t=\frac{\pi}{6}$ or $t=-\frac{\pi}{6}$ (rejected since $y>0$)

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2\cos t}{2\sin t\cos t}=\frac{1}{\sin t}$$

When $t=\frac{\pi}{6}$, $\frac{dy}{dx}=\frac{1}{\frac{1}{2}}=2$

Gradient of normal $=-\frac{1}{2}$

Equation of normal:

$y-2=-\frac{1}{2}\left(x-\frac{1}{4}\right)$

$y=-\frac{1}{2}x+\frac{17}{8}$

or $8y=-4x+17$ (or any other equivalent form)

OR

Substitute $\left(\frac{1}{4},2\right)$ and gradient of normal $=-\frac{1}{2}$ into $y=mx+c$,

$2=-\frac{1}{2}\left(\frac{1}{4}\right)+c$

$c=\frac{17}{8}$

Equation of normal:

$y=-\frac{1}{2}x+\frac{17}{8}$

Examiners Feedback:

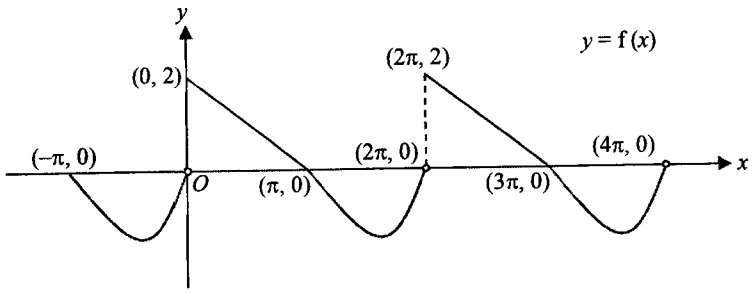
A number of candidates differentiated incorrectly for $\frac{dx}{dt}$ and $\frac{dy}{dt}$, which resulted in an incorrect expression for $\frac{dy}{dx}$. The value of t at the point $\left(\frac{1}{4},2\right)$ was either not found or was found incorrectly due to careless mistakes. Common errors included finding the equation of the tangent instead of the normal, and failing to determine the gradient at $\left(\frac{1}{4},2\right)$ before substituting into the general equation of the normal.

4 It is given that

$$f(x)=\begin{cases} 2-\frac{2}{\pi}x & \text{for} \quad 0\le x<\pi \\ x\sin x & \text{for} \quad \pi\le x<2\pi \end{cases}$$

and that $f(x)=f(x+2\pi)$ for all real values of x.

(a) Sketch the graph of $y=f(x)$ for $-\pi\le x<4\pi$. You do not need to label the coordinates of any stationary point(s). [3]



Examiners Feedback:

Most of the candidates were able to produce part of the graph required. However, many candidates missed out certain important features and made mistakes such as:

- Missing out the domain of $-\pi\le x<4\pi$ and sketched graph for $0\le x<2\pi$ instead.
- Using $2\pi\le x<4\pi$ and sketch, instead of making use of the periodic function.
- Not labelling the end points and intersection with the axes.
- Not indicating the solid circle and hollow circle.

Candidates are reminded to be clear in the sketch, especially when indicating the hollow circle that coincides with the axes as it may look like a solid circle if the hollow circle is drawn too small.

Candidates should review:

- How to use the G.C. to obtain the graph of a piecewise function.
- How to recognise and make use of the periodic function properties to sketch graphs.
- The main features required in any graphs (e.g. asymptotes, intercepts, turning points, end points etc).

(b) Find the exact area bounded by the curve $y = \mathrm{f}(x)$, the x-axis and the lines $x = 0$ and $x = \frac{3\pi}{2}$.

[4]

$$\int_{\pi}^{3\pi/2} x \sin x \, \mathrm{d}x$$

$$= \left[-x\cos x\right]_{\pi}^{3\pi/2} - \int_{\pi}^{3\pi/2} (-\cos x)\, \mathrm{d}x$$

$$= \left[\left(-\frac{3\pi}{2}\cos\frac{3\pi}{2}\right) - (-\pi\cos\pi)\right] + \int_{\pi}^{3\pi/2} \cos x \, \mathrm{d}x$$

$$= -\pi + \left[\sin x\right]_{\pi}^{3\pi/2}$$

$$= -\pi + \sin\frac{3\pi}{2} - \sin\pi$$

$$= -\pi - 1$$

$u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, $v = -\cos x$

$$\int_{0}^{\frac{3\pi}{2}} \mathrm{f}(x)\, \mathrm{d}x = \frac{1}{2}(\pi)(2) - \int_{\pi}^{3\pi/2} x \sin x \, \mathrm{d}x$$

$$= \pi - (-\pi - 1)$$

$$= (2\pi + 1) \text{ units}^2$$

Examiners Feedback:

Majority of the candidates were able to recognise that integration by parts is required to calculate the required area. However, many were unable to formulate the area correctly and/or perform the integration and calculation incorrectly.

Common mistakes observed include:

- Area $= \pi + \int_{\pi}^{3\pi/2} x \sin x \, \mathrm{d}x$, when part of the graph is below the x-axis.
- Area $= \pi + \int_{\pi}^{2\pi} x \sin x \, \mathrm{d}x$, when the question stated that the region is bounded by $x = \frac{3\pi}{2}$.
- Algebraic slips when performing integration by parts and calculation.
- Using G.C. when exact area if required.

Candidates should review:

- The technique of integration by parts.
- How to find the area of a region bounded by curves that are below the x-axis.

To read the question carefully to identify key words such as "bounded by $x = \frac{3\pi}{2}$" and "exact".

5 A sequence of negative real numbers $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = -1 \quad \text{and} \quad u_{n+1} = \frac{-2u_n + 6}{u_n - 1}, \text{ for } n \geq 1.$$

(a) Given that $u_n \to l$ as $n \to \infty$, find the value of l. [3]

As $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$.

$$l = \frac{-2l + 6}{l - 1}$$

Method ①: Algebraic

$l(l-1) = -2l + 6$

$l^2 - l = -2l + 6$

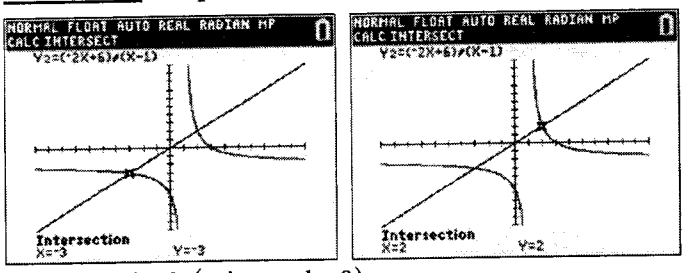
$l^2 + l - 6 = 0$

$(l+3)(l-2) = 0$

$l = -3$ or $l = 2$ (reject $\because l < 0$)

$\therefore l = -3$

Method ②: Graphical



$l = -3$ or $l = 2$ (reject $\because l < 0$)

$\therefore l = -3$

Examiners Feedback:

Majority of the candidates found it challenging to answer this part of the question. Many candidates tried to write down a few values in the attempt to observe some pattern. However, most failed to realise that they can use the G.C. for this purpose, although the intent of the question is to use the algebraic approach.

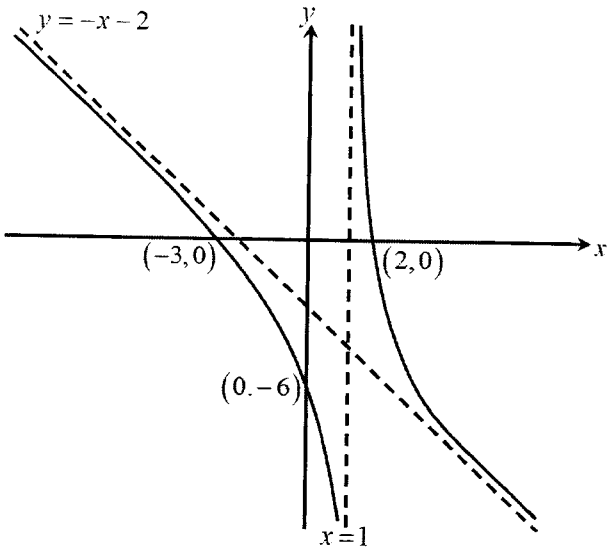
Common mistakes observed include:

- "As $n \to \infty$, $u_{n+1} \to 0$" and proceed to form "$0 = \frac{-2l+6}{l-1} \Rightarrow l = 3$".
- Did not reject $l = 2$ or wrongly reject $l = -3$, when the sequence is negative real numbers.
- Used "$u_n = u_{n+1} - u_1$".

Candidates should review:

- The concept of finding the limit of a sequence algebraically by using the idea that as $n \to \infty$, $u_n \to l$, $u_{n+1} \to l$.

(b) Sketch the graph of $y=\dfrac{-2x+6}{x-1}-x$, stating the equations of the asymptotes and the coordinates of the points where it crosses the axes. [3]



Examiners Feedback:
Most of the candidates were able to sketch the graph well.

Common mistakes observed include:
- Not labelling the intercepts in the coordinates form when question requires it in coordinates form.
- Using $y=-x$ as the oblique asymptote, or not labelling the oblique asymptote entirely.
- Sketching the incorrect graph.

Candidates should review:
- The techniques of curve sketching which includes the features such as asymptotes, intercepts and turning points.
- Read the question carefully to identify any requirements from the question (e.g. coordinates form).

(c) Hence show that $u_{n+1}>u_n$ if $u_n<l$. [2]

$$u_{n+1}-u_n=\frac{-2u_n+6}{u_n-1}-u_n$$

From the graph of $y=\dfrac{-2x+6}{x-1}-x$, when $x<-3$, $y>0$.

If $u_n<-3$, $u_{n+1}-u_n>0$

$u_{n+1}>u_n$ (shown)

Examiners Feedback:
This part of the question was poorly attempted or left blank.

Common mistakes observed include:
- Not making reference to the graph.
- Showing $u_n < l$ instead of showing $u_{n+1} > u_n$.
- Concluding that the sequence is decreasing because it is a sequence of negative numbers.
- Comparing explicit values of u_n and u_{n+1}, i.e. found u_2 and u_3 and showed $u_3 > u_2$ and hence $u_{n+1} > u_n$.
- Comparing $y > x$ when the graph of $y = x$ is not present.

Candidates should review:
- The concept of showing that a sequence is increasing/decreasing by making use of the graph of $y = x_{n+1} - x_n$, followed by making comparison such as "when $x < l$, $y > 0$ means $x_{n+1} - x_n > 0$".

6 Relative to the origin O, the position vectors of points A, B and C are **a**, **b** and **c** respectively. It is given that **a** is a unit vector and **b** is perpendicular to $\mathbf{b} - \mathbf{a}$.

(a) Show that $0 < |\mathbf{b}| < 1$. [3]

Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Since $\underset{\sim}{b}$ and $\underset{\sim}{b} - \underset{\sim}{a}$ are perpendicular,

$\underset{\sim}{b} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$

$\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} = 0$

$|\underset{\sim}{b}|^2 - |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta = 0$

Since $\underset{\sim}{a}$ is a unit vector,

$|\underset{\sim}{b}|^2 - |\underset{\sim}{b}| \cos\theta = 0$

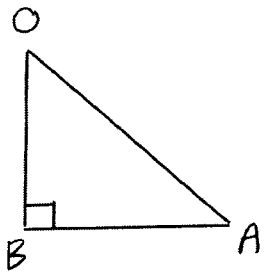
$(|\underset{\sim}{b}|)(|\underset{\sim}{b}| - \cos\theta) = 0$

$|\underset{\sim}{b}| - \cos\theta = 0$, since $|\underset{\sim}{b}| > 0$

$|\underset{\sim}{b}| = \cos\theta$

If $\cos\theta = 1$, $\theta = 0°$. This implies that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel which contradicts the fact that $\underset{\sim}{b}$ and $\underset{\sim}{b} - \underset{\sim}{a}$ are perpendicular. Furthermore, $\cos\theta = |\underset{\sim}{b}| > 0$.

Since $0 < \cos\theta < 1$, $0 < |\underset{\sim}{b}| < 1$ (shown)



OR

Since OAB is a right-angled triangle,

$|a|^2 = |b|^2 + |b - a|^2$

Since $|a|$ is the hypotenuse and $|a| = 1$,

$|b| < 1$

Since $|b| > 0$

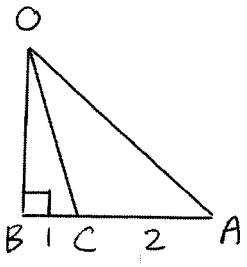
$0 < |\underset{\sim}{b}| < 1$ (shown)

Examiners Feedback:
Generally, students are able to attain $\underset{\sim}{b}\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$, but did not know how to proceed after that. Some did manage to achieve $|\underset{\sim}{b}|=\cos\theta$, but got stuck after that as well. There were some who just stated $0<\cos\theta<1$ with no reasoning. Students need to explain why it is positive, and why $\cos\theta\neq 1$.

It is given further that $|\mathbf{b}|=\frac{1}{2}$ and C lies on AB such that $AC:CB=2:1$.

(b) Find the value of $|\mathbf{a}\cdot\mathbf{c}|$ and state the geometrical interpretation of $|\mathbf{a}\cdot\mathbf{c}|$. [4]

By Ratio Theorem,



$$\underset{\sim}{c}=\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}$$

$$|\underset{\sim}{a}\cdot\underset{\sim}{c}|=\left|\underset{\sim}{a}\cdot\left(\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\right)\right|$$

$$=\left|\frac{1}{3}\underset{\sim}{a}\cdot\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{a}\cdot\underset{\sim}{b}\right|$$

$$=\left|\frac{1}{3}\underset{\sim}{a}\cdot\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\cdot\underset{\sim}{b}\right|, \text{ since } \underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{b} \text{ from } (a)$$

$$=\left|\frac{1}{3}|\underset{\sim}{a}|^2+\frac{2}{3}|\underset{\sim}{b}|^2\right|$$

$$=\left|\frac{1}{3}(1)^2+\frac{2}{3}\left(\frac{1}{2}\right)^2\right|$$

$$=\frac{1}{2}$$

OR

$$BA=\sqrt{1^2-\left(\frac{1}{2}\right)^2}=\frac{\sqrt{3}}{2}$$

$$BC=\frac{BA}{3}=\frac{\sqrt{3}}{6}$$

$$\angle BOC=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{6}}{\frac{1}{2}}\right)=\frac{\pi}{6}$$

$$\angle BOA=\cos^{-1}\left(\frac{\frac{1}{2}}{1}\right)=\frac{\pi}{3}$$

$$\angle AOC=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$|\underset{\sim}{c}| = \sqrt{\left(\frac{\sqrt{3}}{6}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{3}}$$

$$|\underset{\sim}{a}\cdot\underset{\sim}{c}| = \left||\underset{\sim}{a}||\underset{\sim}{b}|\cos\frac{\pi}{6}\right|$$

$$= \left|(1)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2}\right)\right|$$

$$= \frac{1}{2}$$

$|\underset{\sim}{a}\cdot\underset{\sim}{c}| = |\underset{\sim}{c}\cdot\underset{\sim}{a}|$ is the length of projection of $\underset{\sim}{c}$ on $\underset{\sim}{a}$.

Examiners Feedback:
Ratio Theorem was well done by students. Students need to revise their scalar product properties. Common mistakes include $\underset{\sim}{a}\cdot a = a^2$, $\underset{\sim}{a}\cdot b = ab$, $\underset{\sim}{a}\cdot a = 0$. Many students did not observe that they can use the answer from part (a) to proceed.
Students need to write the geometrical interpretation correctly. Loose use of terms such as 'length of projection of line OC on line OA', 'length of project of $\underset{\sim}{c}$ on $\underset{\sim}{a}$', etc were not accepted.

(c) Find the value of $\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}\times\mathbf{c}|}$. [2]

Method ①: Cross Product

$$\frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{|\underset{\sim}{a}\times\underset{\sim}{c}|} = \frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{\left|\underset{\sim}{a}\times\left(\frac{1}{3}\underset{\sim}{a}+\frac{2}{3}\underset{\sim}{b}\right)\right|}$$

$$= \frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{\left|\frac{2}{3}\underset{\sim}{a}\times\underset{\sim}{b}\right|}, \text{ where } \underset{\sim}{a}\times\underset{\sim}{a} = 0$$

$$= \frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{\frac{2}{3}|\underset{\sim}{a}\times\underset{\sim}{b}|}$$

$$= \frac{3}{2}$$

Method ②: Area of triangle

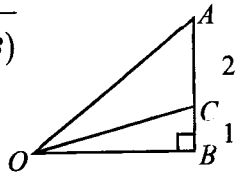
Since $\underset{\sim}{b}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are perpendicular, and C lies on AB,

$$\frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{|\underset{\sim}{a}\times\underset{\sim}{c}|} = \frac{\frac{1}{2}|\underset{\sim}{a}\times\underset{\sim}{b}|}{\frac{1}{2}|\underset{\sim}{a}\times\underset{\sim}{c}|}$$

$$= \frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OAC}$$

$$= \frac{\frac{1}{2}(3)(OB)}{\frac{1}{2}(2)(OB)}$$

$$= \frac{3}{2}$$



Method 3:

$$\frac{|\underset{\sim}{a}\times\underset{\sim}{b}|}{|\underset{\sim}{a}\times\underset{\sim}{c}|} = \frac{|\underset{\sim}{a}||\underset{\sim}{b}|\sin\frac{\pi}{3}}{|\underset{\sim}{a}||\underset{\sim}{c}|\sin\frac{\pi}{6}}$$

$$= \frac{(1)\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{1\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)}$$

$$= \frac{3}{2}$$

Examiners Feedback:
Students need to revise their vector product properties and definitions, and not confuse these properties with scalar product properties. Some students were missing essential working required to obtain the marks.

7 It is given that the curve $y = f(x)$ satisfies the equation $\ln y = \frac{\pi}{4} - \tan^{-1}(e^x)$.

(a) Show that $(1+e^{2x})\frac{dy}{dx} + ye^x = 0$. [2]

$$\ln y = \frac{\pi}{4} - \tan^{-1}(e^x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{e^x}{1+(e^x)^2}$$

$$(1+e^{2x})\frac{dy}{dx} = -ye^x$$

$(1+e^{2x})\frac{dy}{dx} + ye^x = 0$ (shown)

Examiners Feedback:

Candidates who were not successful in part (a) could continue with part (b) using implicit differentiation.

Those who differentiated $\ln y$ implicitly with respect to x were able to work towards the required show result as compared to those who make y the subject i.e. $y = e^{\frac{\pi}{4} - \tan^{-1}(e^x)}$ and differentiate using $e^{f(x)}$ instead. There were also candidates who missed out the e^x when differentiating $\tan^{-1}(e^x)$ i.e. $\frac{d}{dx}(\tan^{-1}(e^x)) = \frac{1}{1+(e^x)^2}$ instead of the correct $\frac{d}{dx}(\tan^{-1}(e^x)) = \frac{e^x}{1+(e^x)^2}$.

There were a handful of candidates who showed the required result by verifying L.H.S. and substituting the expression of $\frac{dy}{dx}$ and y and making it equal to zero = R.H.S, which is not recommended.

(b) Find the Maclaurin series for y, up to and including the term in x^2. [5]

$$\left(1+e^{2x}\right)\frac{dy}{dx}+ye^{x}=0$$

Differentiating implicitly with respect to x,

$$\left(1+e^{2x}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(2e^{2x}\right)+ye^{x}+e^{x}\frac{dy}{dx}=0$$

$$\left(1+e^{2x}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(2e^{2x}+e^{x}\right)+ye^{x}=0$$

When $x=0$,

$$\ln y=\frac{\pi}{4}-\tan^{-1}\left(e^{0}\right)=\frac{\pi}{4}-\frac{\pi}{4}=0$$

$$y=1$$

$$\left(1+e^{2x}\right)\frac{dy}{dx}+ye^{x}=0$$

$$\left(1+e^{0}\right)\frac{dy}{dx}+1=0$$

$$\frac{dy}{dx}=-\frac{1}{2}$$

$$\left(1+e^{0}\right)\frac{d^2y}{dx^2}+\left(-\frac{1}{2}\right)\left(2e^{0}+e^{0}\right)+1=0$$

$$2\frac{d^2y}{dx^2}-\frac{3}{2}+1=0$$

$$\frac{d^2y}{dx^2}=\frac{1}{4}$$

$$y=1-\frac{1}{2}x+\frac{1}{4}\cdot\left(\frac{x^2}{2}\right)+\ldots$$

$$y=1-\frac{1}{2}x+\frac{1}{8}x^2+\ldots$$

Examiners Feedback:

Candidates were generally able to apply implicit differentiation on the result in part (a) with product rule executed. Candidates who were unsuccessful made more algebraic slips in their product rule when differentiating $\left(1+e^{2x}\right)\frac{dy}{dx}$ than ye^{x}. There were also candidates who made $\frac{dy}{dx}$ the subject from part (a) i.e. $\frac{dy}{dx}=-\frac{ye^{x}}{1+e^{2x}}$ and tried to use quotient rule were mostly unsuccessful as they failed to execute product rule correctly on the numerator ye^{x}. Hence this method is not recommended.

Most candidates were able to find the correct values of y and $\frac{dy}{dx}$ when $x=0$.

When finding the value of $\frac{d^2y}{dx^2}$ when $x=0$, there were also a significant number of candidates who solved it this way $2\frac{d^2y}{dx^2}-\frac{1}{2}=0$ ✓

$2\frac{d^2y}{dx^2}=\frac{1}{2}$ ✓

But ending up with $\frac{d^2y}{dx^2}=1$ instead of $\frac{d^2y}{dx^2}=\frac{1}{4}$, thereby affecting the final Maclaurin series.

(c) Deduce the series expansion for $\dfrac{e^{\frac{\pi}{4}-\tan^{-1}(e^x)}}{\sqrt{1-x}}$, up to and including the term in x^2, giving the coefficients of the terms in exact form. [4]

$\ln y = \dfrac{\pi}{4} - \tan^{-1}(e^x)$

$y = e^{\frac{\pi}{4}-\tan^{-1}(e^x)}$

$y \approx 1 - \dfrac{1}{2}x + \dfrac{1}{8}x^2$ from part (b)

$y = \dfrac{e^{\frac{\pi}{4}-\tan^{-1}(e^x)}}{\sqrt{1-x}}$

$= e^{\frac{\pi}{4}-\tan^{-1}(e^x)}(1-x)^{-\frac{1}{2}}$

$= \left(1 - \dfrac{1}{2}x + \dfrac{1}{8}x^2 + \ldots\right)\left[1 + \left(-\dfrac{1}{2}\right)(-x) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \ldots\right]$

$= \left(1 - \dfrac{1}{2}x + \dfrac{1}{8}x^2\right)\left(1 + \dfrac{1}{2}x + \dfrac{3}{8}x^2 + \ldots\right)$

$= 1 - \dfrac{1}{2}x + \dfrac{1}{8}x^2 + \dfrac{1}{2}x - \dfrac{1}{4}x^2 + \dfrac{3}{8}x^2 + \ldots$

$= 1 + \dfrac{1}{4}x^2 + \ldots$

Examiners Feedback:

Most candidates who did not manage to get the correct Maclaurin series in part (b) were able to apply the correct method in this part.

1. Substitute the result in (b) as the series expansion for $y = e^{\frac{\pi}{4}-\tan^{-1}(e^x)}$ in the numerator
2. Make the denominator become $(1-x)^{-\frac{1}{2}}$, multiplied to the $y = e^{\frac{\pi}{4}-\tan^{-1}(e^x)}$ expression in the numerator. Common mistakes include: leaving it as $(1-x)^{\frac{1}{2}}$ in the denominator. Most candidates were stuck with a fraction and cannot proceed.
3. Apply binomial expansion for $(1-x)^{-\frac{1}{2}}$ up to the term in x^2. Common mistake include leaving out the negative sign for the term in x i.e. $1 + \left(-\dfrac{1}{2}\right)x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}x^2 + \ldots$ instead of the correct $1 + \left(-\dfrac{1}{2}\right)(-x) + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \ldots$
4. Lastly, use "rainbow method" to expand all the terms required up to the term in x^2.

8 The plane p passes through the points A, B and C with coordinates $(1,0,2)$, $(2,-1,3)$ and $(-4,-1,0)$ respectively.

(a) Show that a cartesian equation of p is $x-y-2z=-3$. [3]

$$\overrightarrow{OA}=\begin{pmatrix}1\\0\\2\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}2\\-1\\3\end{pmatrix},\ \overrightarrow{OC}=\begin{pmatrix}-4\\-1\\0\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}2\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}-4\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-5\\-1\\-2\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}-5\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\-3\\-6\end{pmatrix}=3\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

A normal of p is $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$.

$$p:\ \mathbf{r}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3$$

A cartesian equation of p is $x-y-2z=-3$. (shown)

Examiners Feedback:

Common mistakes:

1) Errorneously thinking that the normal vector of plane p can be found using $\overrightarrow{OA}\times\overrightarrow{OB}$, $\overrightarrow{OA}\times\overrightarrow{OC}$ or $\overrightarrow{OB}\times\overrightarrow{OC}$. Students, please note that the normal vector, $\mathbf{n}$, of plane p is found by computing the cross product of any 2 (non-parallel) vectors that are **parallel** to the plane p.
e.g. $\mathbf{n}=\overrightarrow{AB}\times\overrightarrow{AC}$.

2) Errorneously using different normals to find the scalar product form (and Cartesian form) of the plane p. e.g. $\mathbf{r}\cdot\underbrace{\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}3\\-3\\-6\end{pmatrix}}_{\text{Can't use different normals!}}=-9$. Students, please note that the same normal must be used e.g.

$$\underbrace{\mathbf{r}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}}_{\text{Use same normal}}\Rightarrow\ \mathbf{r}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3\ \Rightarrow\ \begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3\ \Rightarrow\ x-y-2z=-3.$$

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

(b) Find the acute angle between l and p. [2]

Method ①:

Let the acute angle between l and p be θ.
Let the acute angle between the normal of p and the direction vector of l be α.

$$\alpha = \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$$

$$= \cos^{-1} \frac{5}{\sqrt{6}\sqrt{6}} = 33.55730976°$$

$$\theta = 90° - 33.55730976$$

$$= 56.44269024°$$

$$\approx 56.4° \text{ (to 1 d.p.) or } 0.985 \text{ (to 3 s.f.)}$$

Method ②:

Let the acute angle between l and p be θ.

$$\theta = \sin^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$$

$$= \sin^{-1} \frac{5}{\sqrt{6}\sqrt{6}}$$

$$= 56.44269024°$$

$$\approx 56.4° \text{ (to 1 d.p.)}$$

Examiners Feedback:

Common mistakes:

1) Erroneously thinking that the (acute) angle α, between the line l and the plane p is $\alpha = \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$ when it should have been $\alpha = \sin^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$ instead.

2) Wrong claiming that the (acute) angle α, between the line l and the plane p is $\alpha = \pi - \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$ when it should have been $\alpha = \frac{\pi}{2} - \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right|}$ instead.

It is given that a variable point R lies on p and is at a distance of $\sqrt{22}$ from the point Q with coordinates $(3,4,-2)$.

(c) Find the foot of perpendicular from Q to p. [4]

Let F be the foot of perpendicular from Q to p.

Method ①:

$$l_{QF}: \mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \ \mu \in \mathbb{R}$$

Since F lies on l_{QF},

$$\overrightarrow{OF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \text{ for some } \mu \in \mathbb{R}.$$

Since F also lies on p, $\overrightarrow{OF} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$.

$$\left[\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = -3$$

$$1(3+\mu) - (4-\mu) - 2(-2-2\mu) = -3$$

$$3 + 6\mu = -3$$

$$\Rightarrow \mu = -1$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$$

The foot of perpendicular from Q to p is $(2,5,0)$.

Method ②:

$\overrightarrow{QF} = \left(\overrightarrow{QA} \cdot \hat{\mathbf{n}}\right)\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a normal vector of p

$$= \left(\begin{pmatrix} 1-3 \\ 0-4 \\ 2+2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= \frac{1}{6} \left(\begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \right) \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$\therefore \overrightarrow{OF} = \overrightarrow{OQ} + \overrightarrow{QF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$$

The foot of perpendicular from Q to p is $(2,5,0)$.

Examiners Feedback:

Common mistakes:

1) Quite aplenty of students failed to recognize the foot of perpendicular F as a **point**. Instead, they thought it to be the perpendicular (shortest) distance between the point Q to plane p, and proceeded to find that instead.

2) Quite a number of students wrongly identified $\overrightarrow{QF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$, for some $\mu \in \mathbb{R}$ when it should have been $\overrightarrow{OF} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ instead.

3) Many students claim that $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=0$ when it should have been $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3$ instead.

Students, please be aware that since the foot of perpendicular F lies on the plane p, $\overrightarrow{OF}$ must satisfy the equation of the plane p : $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3$, that is why $\overrightarrow{OF}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}=-3$.

4) Majority of the students did not write the final answer in coordinate-form. Students, please note that the foot of perpendicular is a point, not a vector. Therefore expressing the answer in coordinate-form is required.

(d) Hence describe geometrically the path traced by R. [2]

$$\overrightarrow{QF}=\begin{pmatrix}2\\5\\0\end{pmatrix}-\begin{pmatrix}3\\4\\-2\end{pmatrix}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$\left|\overrightarrow{QF}\right|=\sqrt{(-1)^2+1^2+2^2}=\sqrt{6}$$

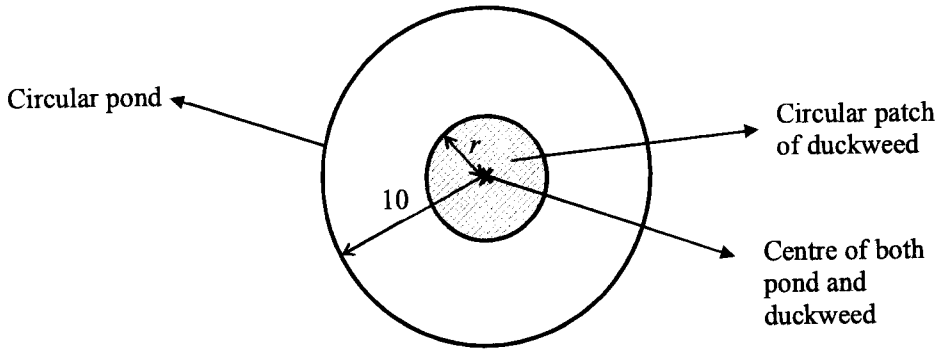
$$RF=\sqrt{22-6}=4$$

The path traced by R is a circle that lies on p with **centre** $(2,5,0)$ and **radius** 4.

Examiners Feedback:

Many students did not attempt this part of the question. Few students identified the path traced by R as a circle and even fewer managed to identify the center and the radius of the circle numerically.

9 A Koi pond farmer introduces a circular patch of duckweed at the center of his circular pond to enhance water purification and provide supplemental nutrition for his Koi, as shown in the figure below.



The duckweed grows over time, forming a continuous circular patch on the pond's surface. It is assumed that the duckweed forms a thin, floating layer with negligible thickness. The rate of increase of the area covered by duckweed is directly proportional to the area of the pond that remains uncovered by the duckweed.

(a) Given that the radius of the pond is 10 m, show that the radius, r m of the circular patch of duckweed satisfies the differential equation $\frac{dr}{dt}=\frac{k\left(100-r^2\right)}{2r}$, $k>0$. [2]

Area of pond $=\pi\left(10^2\right)=100\pi$

Area of duckweed $=\pi r^2$

Since the duckweed grows at a rate that is directly proportional to the uncovered area of the pond,

$\frac{dA}{dt}=k\left[100\pi-\pi r^2\right]$, $k>0$

But,

$A=\pi r^2$

$\therefore \frac{dA}{dt}=2\pi r\frac{dr}{dt}$

So,

$2\pi r\frac{dr}{dt}=k\pi(100-r^2)$

$\therefore \frac{dr}{dt}=\frac{k(100-r^2)}{2r}$ (shown)

Examiners Feedback:

Some tried to formulate the differential equation $\frac{dA}{dt}=k\left[100\pi-\pi r^2\right]$ but wrote $\frac{dr}{dt}$ instead of $\frac{dA}{dt}$. Most students were able to apply implicit differentiation to the area of duckweed $A=\pi r^2$ and use it in the chain rule $\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dt}$. However many students were not able to have a clear presentation to show how they obtain the differential equation $\frac{dr}{dt}=\frac{k(100-r^2)}{2r}$, $k>0$.

(b) The initial radius of the circular patch of duckweed is 1 m. Solve the differential equation, expressing r in the form $r = \sqrt{P - Qe^{-kt}}$, where P and Q are constants to be determined. [6]

$$\frac{dr}{dt} = \frac{k(100 - r^2)}{2r}$$

Separating the variables and integrating wrt t,

$$\int \frac{r}{100 - r^2}\, dr = \frac{k}{2}\int 1\, dt$$

$$-\frac{1}{2}\int \frac{-2r}{100 - r^2}\, dr = \frac{k}{2}t + C$$

$$-\frac{1}{2}\ln|100 - r^2| = \frac{k}{2}t + C$$

$$\ln|100 - r^2| = -kt - 2C$$

$$|100 - r^2| = e^{-kt-2C}$$

$$100 - r^2 = \pm e^{-2C}e^{-kt}$$

$$100 - r^2 = Qe^{-kt}, \text{ where } Q = \pm e^{-2C}$$

$$r^2 = 100 - Qe^{-kt}$$

$$r = \pm\sqrt{100 - Qe^{-kt}}$$

Since $r > 0$, $\therefore\ r = \sqrt{100 - Qe^{-kt}}$

When $t = 0$, $r = 1$,

$1 = \sqrt{100 - Qe^0}$

$1 = 100 - Q$

$Q = 100 - 1 = 99$

Hence,

$r = \sqrt{100 - 99e^{-kt}}$

Where $P = 100$ and $Q = 99$

Examiners Feedback:

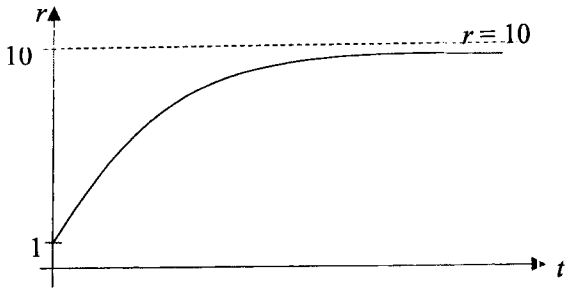
Many students were able to separate the variables correctly.

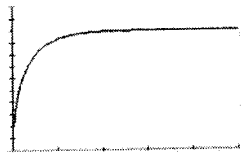
Some errors/misconceptions/presentation issues are

- forgetting the minus sign $-\frac{1}{2}\ln|100 - r^2|$
- apply wrong formula, that is, used $\ln\left(\frac{10+x}{10-x}\right)$ instead of $\ln|100 - r^2|$
- did not use modulus for $\ln|100 - r^2|$
- did not have $\pm$ when simplifying the constants
- did not have $\pm$ when square rooting both sides to find r
- did not state specifically values of P and Q
- found values of P and Q but did not state the final expression for r

(c) Sketch the graph of r against t. [3]

Since $k > 0$,





Examiners Feedback:
Most students managed to get the shape of the graph.
Some errors/presentation issues are
- there were quite a few who drew the graph touching the asymptote
- did not find the value of r when $t = 0$ and assumed $r = 0$
- did not label the equation of asymptote
- did not label r-intercept
- labeled the axes wrongly as x-axis and y-axis instead of t-axis and r-axis respectively

10 Mr Huat is saving for a car that he plans to buy in the future. He needs to save a minimum of \$150,000. A savings plan allows him to deposit \$6,000 into a savings account on the first day of every month. At the end of each month, the total amount in the savings account (including interest) is increased by 3%. Mr Huat makes an initial deposit on 1 January 2026.

(a) Show that the total amount, correct to 2 decimal places, in the savings account at the end of December 2026 is \$87,706.74. [2]

Month n	**Start of month**	**End of month**
1	6000	$6000(1.03)$
2	$6000(1.03)+6000$	$\left[6000(1.03)+6000\right](1.03)$ $=6000(1.03)^2+6000(1.03)$
3	$6000(1.03)^2+6000(1.03)+6000$	$\left[6000(1.03)^2+6000(1.03)+6000\right](1.03)$ $=6000(1.03)^3+6000(1.03)^2+6000(1.03)$
⋮	⋮	⋮
12	$6000(1.03)^{11}+6000(1.03)^{10}+\ldots+6000(1.03)+6000$	$6000(1.03)^{12}+6000(1.03)^{11}+\ldots+6000(1.03)$
⋮	⋮	⋮
n	$6000(1.03)^{n-1}+6000(1.03)^{n-2}+\ldots+6000(1.03)+6000$	$6000(1.03)^n+6000(1.03)^{n-1}+\ldots+6000(1.03)$

Total amount in the savings account at the end of December 2026

$$= \underbrace{6000(1.03)^{12} + 6000(1.03)^{11} + \ldots + 6000(1.03)}_{\text{Sum of GP: First term=6000(1.03), Common Ratio=1.03, No. of terms=12}}$$

$$= \frac{6000(1.03)(1.03^{12} - 1)}{1.03 - 1}$$

$$= 87706.74269$$

$$\approx 87706.74 \text{ (to 2 d.p.)}$$

Examiners Feedback:
Most students are able to make a list and observe the pattern of amount of savings at the end of the month. The series is a Geometric progression.
However since this is a "*Show*" question, listing out the series of terms is insufficient. The sum of GP formula needs to be applied.

There are a small number of students who used the brute force approach to list out every single term. This method may not be applicable to all questions especially if the number of terms is large.

(b) Find the month and year in which the total amount in the savings account will first exceed \$150,000. Explain whether this occurs on the first or last day of the month. [6]

Total amount in the savings account at the end of the nth month

$$= \underbrace{6000(1.03)^{n} + 6000(1.03)^{n-1} + \ldots + 6000(1.03)}_{\text{Sum of GP: First term=6000(1.03), Common Ratio=1.03, No. of terms=}n}$$

$$= \frac{6000(1.03)(1.03^{n} - 1)}{1.03 - 1}$$

$$= 206000(1.03^{n} - 1)$$

$$206000(1.03^{n} - 1) > 150000$$

$$1.03^{n} - 1 > \frac{150000}{206000}$$

$$1.03^{n} > 1.7282$$

Method ① to find least n: Algebraic

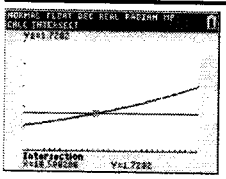
$n \ln 1.03 > \ln 1.7282$

$n > 18.507$

Least $n = 19$

Total amount in the savings account will first exceed \$150,000 on July 2027.

Method ② to find least n: Graphical



From the graph,

$n > 18.507$

Least $n = 19$

Total amount in the savings account will first exceed \$150,000 on July 2027.

When $n = 18$,
Total amount in the savings account at the end of the 18th month (i.e. June 2027)
$= 206000\left(1.03^{18} - 1\right)$
$= 144701.2106$

Total amount in the savings account at start of the 19th month (i.e. July 2027)
$= 144701.2106 + 6000$
$= 150701.2106 > 150000$

Hence, this occurs on the first day of the month.

Examiners Feedback:
Students used a variety of methods to solve the question. These included using an algebraic method to calculate the least number of months to using the GC to generate a list of values.

Regardless, students need to be careful to distinguish between the amount of savings at the start of the month and the end of the month as the GP formula is different.

Mr Huat can also choose to save regularly in a different savings account which offers no interest. He makes an initial deposit on 1 January 2026 of \$6,000. On the first day of each subsequent month, he deposits \$$d$ more than he deposited in the previous month.

(c) Find, in terms of d, the total amount in the savings account at the end of December 2026. [2]

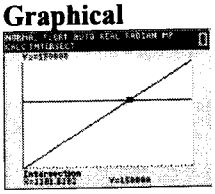
Total amount in the savings account at the end of December 2026
$= 6000 + (6000 + d) + \ldots + (6000 + 11d)$
$= \frac{12}{2}\left[6000 + (6000 + 11d)\right]$
$= 72000 + 66d$

Examiners Feedback:
Students generally made a list and observed the pattern. The series is an Arithmetic progression. However, a significant number of students calculated the amount deposited in the nth month (nth term) rather than the total sum (S_n).

(d) Hence, find the minimum value of d such that Mr Huat can buy the car by end of December 2026. Give your answer correct to the nearest integer. [2]

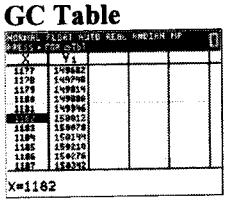
Method ① to find minimum d:
Algebraic
$72000 + 66d \geq 150000$
$66d \geq 78000$
$d \geq 1181.818182$
Hence, minimum value of
$d = 1182$ (to nearest integer).

Method ② to find minimum d:
Graphical



Hence, minimum value of
$d = 1182$ (to nearest integer).

Method ③ to find minimum d:
GC Table



Hence, minimum value of
$d = 1182$ (to nearest integer).

Examiners Feedback:
This part requires the previous part answer.
Therefore students who did not obtain the correct expression for part (c) were unable to obtain the correct value of d. They were awarded method marks for interpreting the question and formulating the inequality.

11 (a) (i) Show that $\dfrac{3x^2}{(x+1)(3x^2+x+1)}$ can be expressed as

$$\frac{3x^2}{(x+1)(3x^2+x+1)} = \frac{A}{x+1} + \frac{B}{3x^2+x+1},$$

where A and B are constants to be determined. [2]

Let $\dfrac{3x^2}{(1+x)(3x^2+x+1)} = \dfrac{A}{1+x} + \dfrac{Bx+C}{3x^2+x+1}$

By comparing numerators, $3x^2 = A(3x^2+x+1)+(Bx+C)(1+x)$

Substitute $x=-1$,

$3(-1)^2 = A[3(-1)^2+(-1)+1]$

$A=1$

Substitute $x=0$,

$3(0)^2 = (1)[3(0)^2+(0)+1]+[B(0)+C](1+0)$

$C=-1$

Substitute $x=1$,

$3(1)^2 = (1)[3(1)^2+(1)+1]+[B(1)-1](1+1)$

$B=0$

Hence, $\dfrac{3x^2}{(1+x)(3x^2+x+1)} = \dfrac{1}{1+x} - \dfrac{1}{3x^2+x+1}$

Examiners Feedback:
Well attempted. Most students are able to get this part correct. Those who didn't is due to carelessness.

(ii) Hence, find $\int \frac{3x^2}{(x+1)(3x^2+x+1)}\,dx$. [4]

$$\int \frac{3x^2}{(x+1)(3x^2+x+1)}\,dx$$

$$= \int \left(\frac{1}{x+1} - \frac{1}{3x^2+x+1}\right)dx$$

$$= \ln|x+1| - \frac{1}{3}\int \frac{1}{x^2+\frac{x}{3}+\frac{1}{3}}\,dx$$

$$= \ln|x+1| - \frac{1}{3}\int \frac{1}{x^2+\frac{x}{3}+\left(\frac{1}{6}\right)^2-\left(\frac{1}{6}\right)^2+\frac{1}{3}}\,dx$$

$$= \ln|x+1| - \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{6}\right)^2+\frac{11}{36}}\,dx$$

$$= \ln|x+1| - \frac{1}{3}\left(\frac{1}{\sqrt{11/36}}\right)\tan^{-1}\left(\frac{x+1/6}{\sqrt{11/36}}\right)+C$$

$$= \ln|x+1| - \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)+C$$

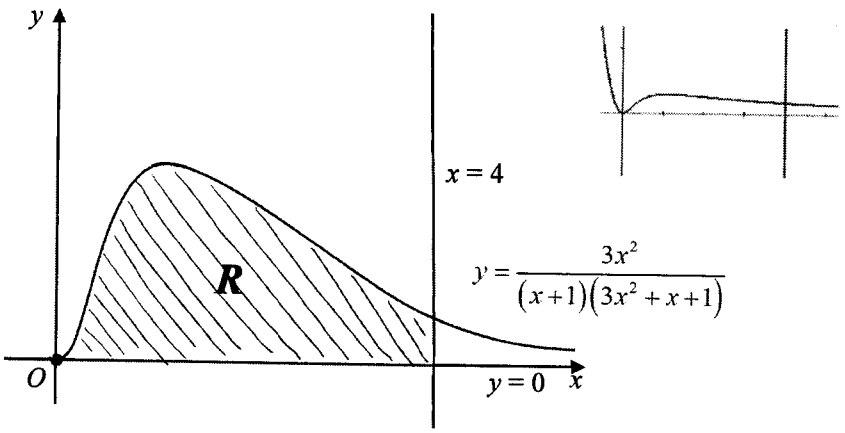
Examiners Feedback:

A few details to note when performing integration:

- If the result has a "ln", one must remember to include the modulus, i.e $\ln|x+1|$
- When we have $\int \frac{1}{3x^2+1}\,dx$, it is more advisable to work with $\frac{1}{3}\int \frac{1}{x^2+(1/3)}\,dx$ than $\int \frac{1}{\left(\sqrt{3}x\right)^2+1}\,dx$ because we can directly use the results in MF27
- Make sure you match to the correct form so that you can quote the correct result in the MF27.
- Remember to include the arbitrary constant " +c" in your answer
- Remember your answers must be simplified as far as possible and we should not have "fractions in fraction". For example, $\frac{6}{3\sqrt{11}}\tan^{-1}\left(\frac{x+1/6}{\sqrt{11/36}}\right)$ should become $\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)$

(b) The region R is bounded by the curve $y=\dfrac{3x^2}{(x+1)(3x^2+x+1)}$, the line $x=4$ and the x-axis.

(i) Sketch the graph of $y=\dfrac{3x^2}{(x+1)(3x^2+x+1)}$ for $x\geq 0$, giving the coordinates of any intercepts with the axes and the equations of any asymptotes. Shade the region R on your sketch. [3]



Examiners Feedback:
Generally well attempted. Those who did not obtain the full credit is due to the reason(s):
- Should only sketch graph in the region $x>0$ (right side of y-axis).
- The horizontal asymptote $y=0$ and the intercept (0,0) should be labelled
- In defining one of the boundaries of the shaded region, the value $x=4$ should be seen somewhere

(ii) Find the exact area of R. [3]

Exact area of R

$$\int_0^4 \frac{3x^2}{(x+1)(3x^2+x+1)}\,dx$$

$$=\left[\ln|x+1|-\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)\right]_0^4$$

$$=\left[\ln|4+1|-\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6(4)+1}{\sqrt{11}}\right)\right]-\left[\ln|0+1|-\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{6(0)+1}{\sqrt{11}}\right)\right]$$

$$=\ln 5-\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{25}{\sqrt{11}}\right)+\frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{1}{\sqrt{11}}\right)$$

$$=\ln 5+\frac{2}{\sqrt{11}}\left[\tan^{-1}\left(\frac{1}{\sqrt{11}}\right)-\tan^{-1}\left(\frac{25}{\sqrt{11}}\right)\right]$$

Examiners Feedback:
Generally well attempted as most candidates know that we are performing definite integration using the result from (a)(ii) with the limits from 0 to 4. Those who did not get the full credit were mostly not careful with their working of the arithmetic when putting in the limits.

(iii) Find the volume of the solid generated when R is rotated through 2π radians about the x-axis. [2]

Volume required

$$= \pi \int_0^4 \left[\frac{3x^2}{(x+1)(3x^2+x+1)} \right]^2 \, dx$$

$= 0.715$ (3s.f.)

Examiners Feedback:

Not as well attempted as expected. Mistakes were:

- Some students were unaware they can use the GC to evaluate.
- Quoted incorrectly the volume of revolution formula $\pi \int_0^4 y^2 \, dx$. Some students quoted "2π" , some did not square, some did not use the correct limits.
- A more fatal error made by some students is the misinterpretation of the formula to become $\pi \left(\int_0^4 y \, dx \right)^2$ where they simply try to square the result in (a)(ii) and then putting in the limits.

CATHOLIC JUNIOR COLLEGE
General Certificate of Education Advanced Level
Higher 2
JC2 Preliminary Examination

CANDIDATE NAME

CLASS

INDEX NUMBER

MATHEMATICS

Paper 2

9758/02
17 Sep 2025
3 hours

Additional Materials: Printed Answer Booklet
List of Formulae (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** the questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

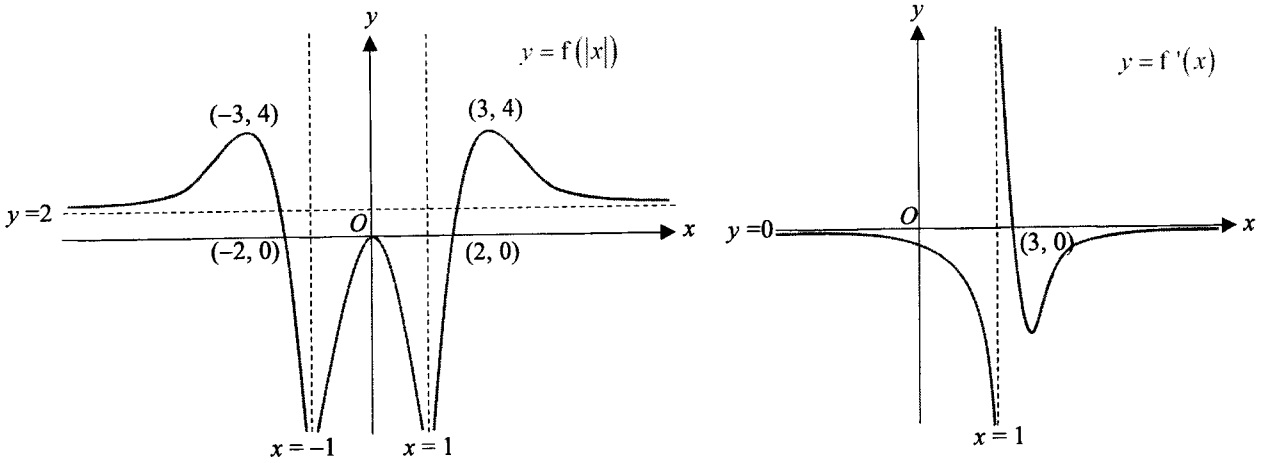
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

Section A: Pure Mathematics [40 marks]

1 The diagram shows the graphs of $y = \mathrm{f}(|x|)$ and $y = \mathrm{f}'(x)$.



The graph of $y = \mathrm{f}(|x|)$ has turning points at $(-3, 4)$ and $(3, 4)$, crosses the x-axis at $(-2, 0)$ and $(2, 0)$, and the equations of asymptotes are $x = -1$, $x = 1$ and $y = 2$. The graph of $y = \mathrm{f}'(x)$ crosses the x-axis at $(3, 0)$, and the equations of asymptotes are $x = 1$ and $y = 0$.

On separate diagrams, sketch the graphs of

(a) $y = \mathrm{f}(x)$, [2]

(b) $y = \dfrac{1}{\mathrm{f}'(x)}$, [3]

(c) $y = -\mathrm{f}(|x-1|)$, [3]

labelling clearly the equation(s) of any asymptote(s), coordinates of any axial intercept(s) and turning point(s) where applicable.

2 Functions f and g are defined by

$$\mathrm{f}: x \to \frac{4}{(x-4)^2}, \quad x \in \mathbb{R}, \ x \neq 4,$$

$$\mathrm{g}: x \to \ln\left(1 + \frac{1}{x}\right), \quad x \in \mathbb{R}, \ x > 0.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes, the coordinates of the points where it crosses the axes and the coordinates of the turning points, if any. [2]

(b) Show that gf exists. Hence find the rule, domain and range of gf. [4]

(c) If the domain of f is further restricted to $x < k$, state the largest value of k for which the function f^{-1} exists. [1]

(d) Using the restricted domain found in part **(c)**, find f^{-1} in a similar form. [3]

3

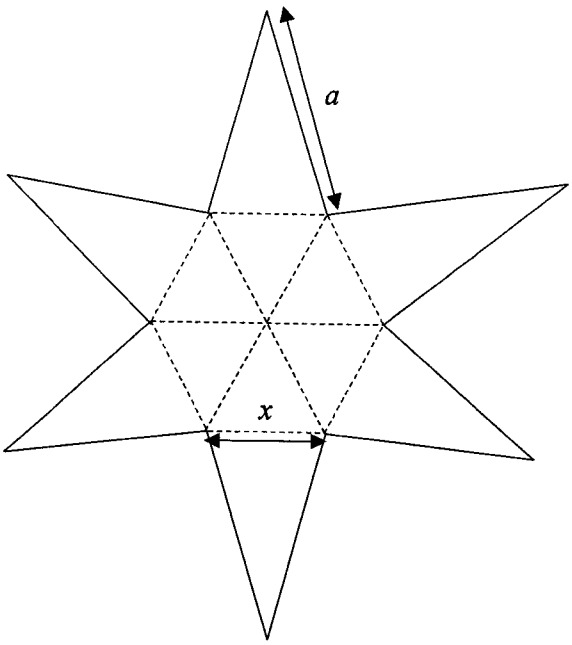


Fig. 1

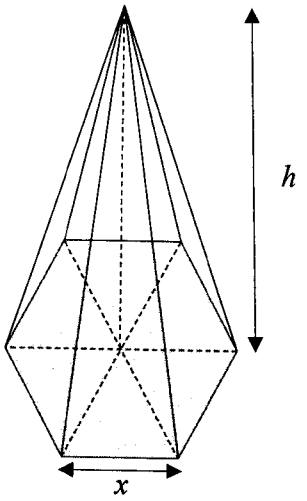


Fig. 2

Fig. 1 shows a net of a hexagonal pyramid folded from a star-shaped cardboard of equal edge length a cm. The net consists of a hexagon with equal sides of x cm and six isosceles triangles with base x cm and side a cm. The net is folded to form a right pyramid with a hexagonal base of edge length x cm and vertical height h cm, as shown in Fig. 2. The hexagonal base is made up of six equilateral triangles of side length x cm.

The volume of a right hexagonal pyramid with base edge x cm and height h cm is given by $V = \frac{\sqrt{3}}{2}x^2h$.

(a) Show that the volume of the hexagonal pyramid, V satisfies the expression given by

$$V^2 = \frac{3}{4}\left(a^2x^4 - x^6\right)$$ [2]

(b) Find, in terms of a, the maximum possible volume of the hexagonal pyramid. You need not show that this value is a maximum. [4]

(c) Find, in terms of a, the total surface area of the hexagonal pyramid when the volume is a maximum. [4]

4 One of the roots of the equation

$$2z^4 - 14z^3 + 33z^2 - 26z + p = 0, \text{ where } p \text{ is a constant}$$

is $3+\mathrm{i}$.

(a) Based on the above information only, a student claims that the equation has a root $3-\mathrm{i}$. State, with a reason, why the student's claim may not be true. [1]

(b) Show that $p = 10$. [2]

For the rest of this question, do not use a calculator.

(c) Find the roots of the equation $2z^4 - 14z^3 + 33z^2 - 26z + 10 = 0$ and mark them clearly on a single labelled Argand diagram. [7]

(d) The points of the Argand diagram in part **(c)** form the vertices of a quadrilateral. Identify the type of quadrilateral and determine its area. [2]

Section B: Probability and Statistics [60 marks]

5 The basketball club in a college has 5 centers, 8 forwards and 7 guards. With the National School Games approaching, the coach wishes to find out the opinions of members of the club about the training programme. He gives a questionnaire to all the members of the club and receives replies from everyone.

(a) Explain whether the 20 members form a sample or a population. [1]

The coach then decides to select teams to play in the matches for National School Games. A basketball team to play in a match consists of 1 center, 2 forwards and 2 guards.

(b) Explain an advantage for choosing a random sample in each category of members for the match. [1]

(c) How many different teams can be formed? [1]

In the club, one particular forward is the classmate of one particular guard. Both classmates are injured and cannot participate in a particular match. The coach decides that one of the remaining guards can play either as a guard or as a forward.

(d) How many different teams can now be formed? [3]

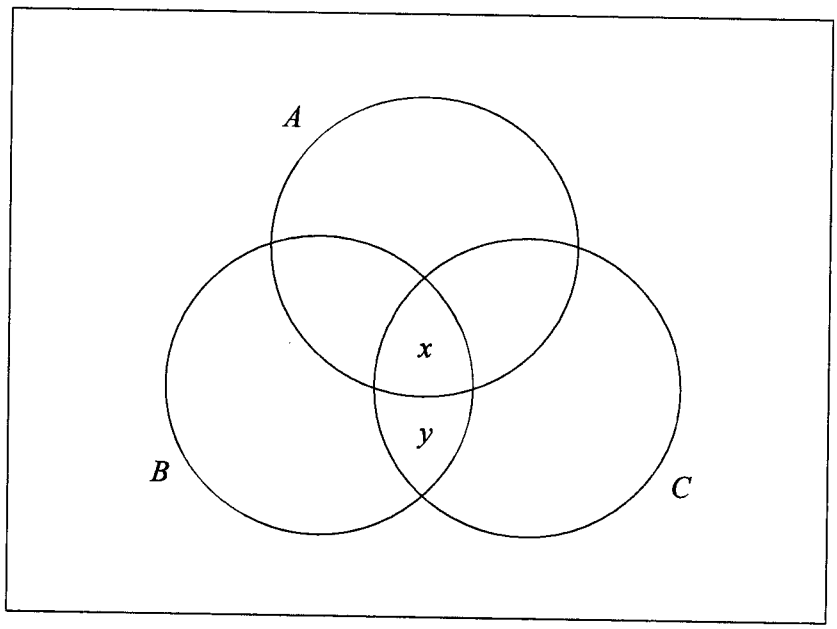
6 A group of 100 students are asked if they are student leaders, in a sports CCA, or studying in a science faculty. The number of students who are student leaders is 25, the number of students who are in a sports CCA is 30 and the number of students studying in a science faculty is 40. There are 15 student leaders who are in a sports CCA. The number of students who are student leaders, in a sports CCA and studying in a science faculty is x. The number of students who are in a sports CCA and studying in a science faculty but not a student leader is y.

One of the students is chosen at random.

A is the event that the student is a student leader.
B is the event that the student is in a sports CCA.
C is the event that the student is studying in a science faculty.

It is given that A and C are independent.

(a) Complete the Venn diagram below to represent all the above information. You are allowed to give expressions in terms of x and y. [3]



It is further given that B and C are independent.

(b) Find y in terms of x. Hence, find the greatest and least possible values of y. [4]

7 Happie, the owner of a store selling novelty items, is organizing a publicity stunt for his store. Using a stock of ultra-rare Lubaba dolls that he acquired, he sets up a game where a player opens "mystery boxes" to try to find a Lubaba doll hidden inside one of the identical boxes used for the game.

The rules of the game are:

- The player pays an initial \$5 to start the game to open one of n boxes.
- If the opened box is empty, the player can pay an additional \$3 to open a second box.
- If the second box is empty, the player can pay \$2.50 again to open a third box.
- If the third box is still empty, the player loses the game. The doll that is not won will then be donated away and the game is reset for the next player.

As the Lubaba dolls are considered rare collectors' items, it can be assumed that in every game, the players will keep opening the boxes until they win the doll and that they have the means to pay for the maximum of 3 allowable tries at opening the boxes.

Let X be the random variable denoting the number of empty boxes a player has opened in a game.

(a) Show that $\mathrm{P}(X=2)=\frac{1}{n}$. [1]

(b) Find the probability distribution of X, leaving your answers in terms of n. [2]

(c) If $n=10$, find $\mathrm{P}(X \leq \sigma)$, where σ is the standard deviation of X. [2]

(d) If the cost price of a Lubaba doll is \$10, find the number of boxes n, that Happie should prepare so that he will not make any profit nor incur any loss from organising this publicity stunt. [3]

8 In each batch of pralines that a chocolate factory produces, they are packed into boxes of 16. Due to the new health guidelines regarding sugar content, quality control tests are implemented and show that on average, the proportion of pralines that exceed the recommended sugar content is p. The number of pralines in a box that exceed the recommended sugar content is denoted by X.

(a)(i) State, in context of the question, two assumptions needed for X to be well-modelled by a binomial distribution. [2]

Assume now that X is modelled by a binomial distribution.

(ii) It is known that the most probable number of pralines that exceed the recommended sugar content is 2. Find the exact range of values that p can take. [3]

For each batch of pralines produced, the chocolate factory implements the following quality control system by testing the first random sample of 16 pralines in a box:

- If there are at most 1 praline that exceed the recommended sugar content in a box, the batch is accepted.
- If there are 3 or more pralines that exceed the recommended sugar content in a box, the batch is rejected.
- If there are 2 pralines that exceed the recommended sugar content in a box, a second box of 16 pralines is tested. The batch will be accepted if there are at most 1 praline that exceed the recommended sugar content in the second box, otherwise the batch will be rejected.

It is given that $p = 0.15$.

(b)(i) Find the probability that the batch of pralines is accepted under this quality control system. [2]

(ii) Given that the batch of pralines is accepted, find the probability that a second box of pralines is tested. [2]

9 A teacher conducts a survey on 7 students to investigate the relationship between the number of hours (h) of "screen-time" per week and the average scores (s) they obtained in a recently concluded examination She records her findings as shown in the following table.

h	8.5	14	20	27	10.5	17	23
s	68	61	44	12	α	58	31

(a) Given that the regression line of s on h is $s = -3.00799h + 100.27974$, show that $\alpha = 67.0$. [2]

(b) Draw a scatter diagram for the data. [1]

(c) Explain why $s = kh^2 + c$ is the better model compared to the one in part **(a)**, by giving appropriate reasons. State the values of k and c. [3]

(d) Using the better model, estimate the score a student can expect to obtain if he spends 7 hours of screen-time a week. Comment on the reliability of the estimate obtained. [2]

(e) The teacher observes a trend from her findings and concludes with a statement: Increased screen-time will cause the exam scores to decrease. Comment on the validity of the statement. [1]

10 Fishing Company A has two types of fishing vessels, the "Standard" vessel and the "Large" vessel. The amount of fish caught by the respective vessels on a typical fishing trip are normally distributed with means and standard deviations as shown in the table.

	Mean (kg)	Standard deviation (kg)
Standard	300	σ
Large	540	110

(a) If a Standard vessel returns with more than 400kg of fish, it is called a Bumper catch. Bumper catches occur 8% of the time. Show that $\sigma \approx 71.2$. [2]

(b) On a particular fishing trip, Company A sends out 3 Standard vessels and 2 Large vessels, with a target to catch at least 2100kg of fish. Find the probability that the fishing vessels are able to meet their target. [2]

(c) State an assumption needed in your calculation in part **(b)**. [1]

Fishing Company B operates in the same seas as Fishing Company A.
However, Company B has the improved versions of the two types of fishing vessels that Company A has: the "Premium-Standard" vessel and the "Premium-Large" vessel. In a fishing trip, the Premium-Standard vessel is capable of catching 3 times as much fish as the Standard vessel and the Premium-Large vessel is capable of catching 2 times as much fish as Large vessel.

(d) Company B sends out 1 Premium-Standard vessel and 1 Premium-Large vessel, with the same target to catch at least 2100kg of fish. Find the probability that these two fishing vessels are able to meet their target. [2]

(e) It is found that, in a sample of n randomly chosen fishing trips, the probability that the average catch of a Large vessel being less than 552kg is at least 0.7.
Find the least possible value of n. [3]

11 EastHam is a football club in a competitive league. It is found that the average time taken by EastHam to score their first goal in a match is 50 minutes. The coach trialled a new play tactic and the time taken, x minutes, to score the first goal is recorded for a random sample of 35 matches in the season. The total time taken is found to be 1650 minutes and the variance of the time is 250 minutes^2.

The coach wants to test if the average time taken to score the first goal has decreased with the implementation of the new play tactic.

(a) Explain why the coach is able to carry out a hypothesis test without any assumption about the distribution of the time taken to score the first goal. [1]

(b) Find unbiased estimates of the population mean and variance. Give your answers correct to 3 decimal places. [2]

(c) Carry out the test at 10% level of significance. You should state your hypotheses and define any symbols you use. [4]

A fan of another football club in the same league, NewPalace, claims that the average time taken by NewPalace to score their first goal in a match is 40 minutes. The time taken by NewPalace to score the first goal in a match can be assumed to be normally distributed with variance 280 minutes^2.

(d) A random sample of 50 matches is taken and a hypothesis test is carried out on whether the fan's claim is valid. Find the range of values of the mean time of this sample for which the fan's claim would be rejected at 10% level of significance. [4]

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Higher 2

JC2 Preliminary Examination

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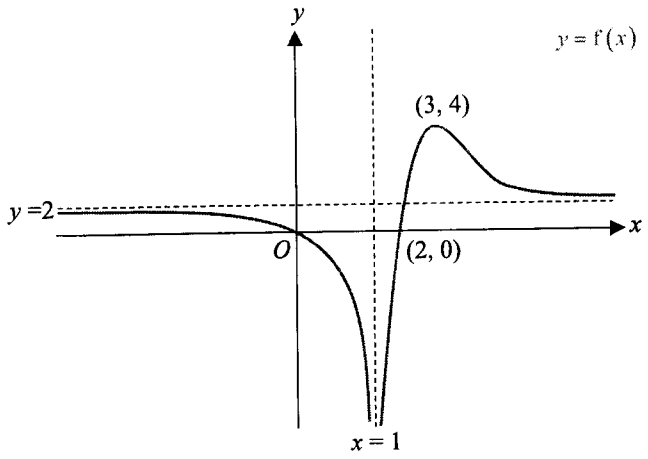
Section A: Pure Mathematics [40 marks]

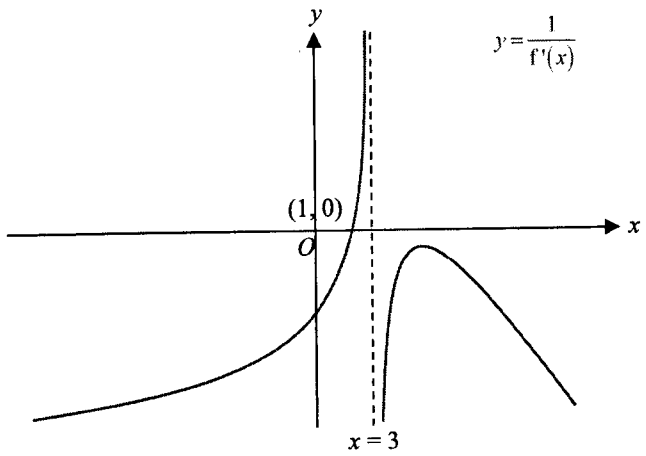
Solution:

Q1

(a)



(b)



(c)

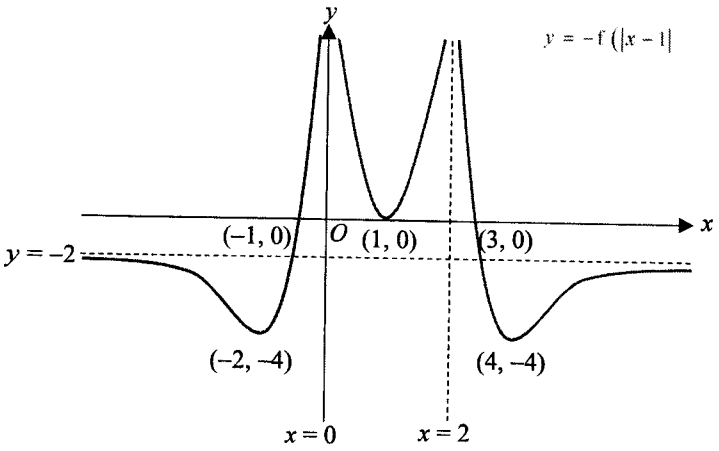
$y = \mathrm{f}(|x|)$

↓ translate 1 unit in positive x-axis direction

$y = \mathrm{f}(|x-1|)$

↓ reflect in x-axis

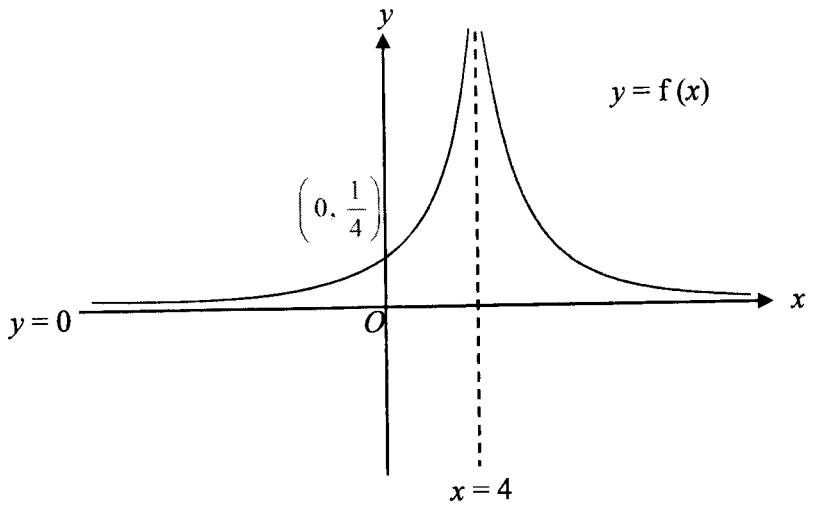
$y = -\mathrm{f}(|x-1|)$



Solution:

Q2

(a)



(b) For gf to exist, $R_f \subseteq D_g$,

$R_f = (0, \infty)$

$D_g = (0, \infty)$

Since $R_f \subseteq D_g$, gf exists (shown)

$$gf(x) = g\left[\frac{4}{(x-4)^2}\right]$$

$$= \ln\left[1 + \left(\frac{1}{\frac{4}{(x-4)^2}}\right)\right]$$

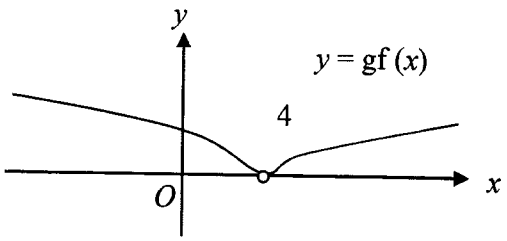
$$= \ln\left[1 + \frac{(x-4)^2}{4}\right]$$

$D_{gf} = D_f = (-\infty, 4) \cup (4, \infty)$

OR

$D_{gf} = D_f = \mathbb{R} \setminus \{4\}$

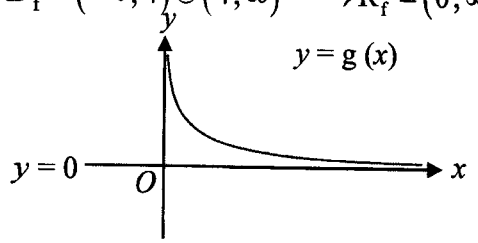
Method ①: Sketch $y = \mathrm{gf}(x)$ for given domain



$\mathrm{R_{gf}} = (0, \infty)$

Method ②: Mapping

$\mathrm{D_f} = (-\infty, 4) \cup (4, \infty) \xrightarrow{\mathrm{f}} \mathrm{R_f} = (0, \infty) \xrightarrow{\mathrm{g}} \mathrm{R_{gf}} = (0, \infty)$



(c) Largest value of k is 4.

(d)

$$y = \frac{4}{(x-4)^2}$$

$$(x-4)^2 = \frac{4}{y}$$

$$x - 4 = \pm\frac{2}{\sqrt{y}}$$

$$x = 4 + \frac{2}{\sqrt{y}} \left(\text{rej.} \because x < 4\right) \text{ or } x = 4 - \frac{2}{\sqrt{y}}$$

$\mathrm{D_{f^{-1}}} = \mathrm{R_f} = (0, \infty)$

$\mathrm{f^{-1}} : x \to 4 - \frac{2}{\sqrt{x}}, \quad x \in \mathbb{R}, x > 0$

Solution:

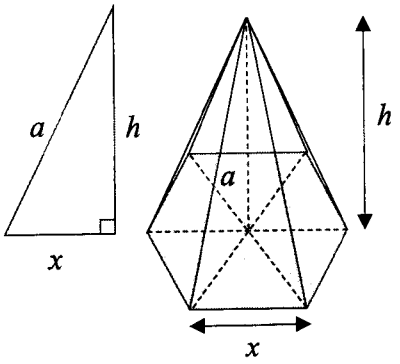
Q3

(a)

By Pythagoras Theorem,

$a^2 = x^2 + h^2$

$h = \sqrt{a^2 - x^2}$



$V = \frac{\sqrt{3}}{2}x^2 h$

$= \frac{\sqrt{3}}{2}x^2\sqrt{a^2 - x^2}$

$V^2 = \frac{3}{4}x^4\left(a^2 - x^2\right)$

$V^2 = \frac{3}{4}\left(a^2x^4 - x^6\right)$ (shown)

(b)

Method ①:

$V^2 = \frac{3}{4}\left(a^2x^4 - x^6\right)$

Differentiate implicitly with respect to x:

$2V\frac{dV}{dx} = \frac{3}{4}\left(4a^2x^3 - 6x^5\right)$

At stationary point, $\frac{dV}{dx} = 0$

$4a^2x^3 - 6x^5 = 0$

$2x^3\left(2a^2 - 3x^2\right) = 0$

Since $x \neq 0$, $2a^2 - 3x^2 = 0$

$x^2 = \frac{2}{3}a^2$

$\max V = \sqrt{\frac{3}{4}\left[a^2\left(\frac{2}{3}a^2\right)^2 - \left(\frac{2}{3}a^2\right)^3\right]}$

$= \sqrt{\frac{3}{4}\left[\frac{4}{9}a^6 - \frac{8}{27}a^6\right]}$

$= \sqrt{\frac{3}{4}\left(\frac{4}{27}a^6\right)}$

$= \sqrt{\frac{a^6}{9}}$

$= \frac{a^3}{3}$

Method ②:

$$V^2 = \frac{3}{4}\left(a^2x^4 - x^6\right)$$

$$V = \frac{\sqrt{3}}{2}\left(a^2x^4 - x^6\right)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \frac{\sqrt{3}}{2}\left[\frac{1}{2}\left(a^2x^4 - x^6\right)^{-\frac{1}{2}}\left(4a^2x^3 - 6x^5\right)\right]$$

$$= \frac{\sqrt{3}}{4}\left[\left(4a^2x^3 - 6x^5\right)\left(a^2x^4 - x^6\right)^{-\frac{1}{2}}\right]$$

At stationary point, $\frac{\mathrm{d}V}{\mathrm{d}x} = 0$

$$\frac{\sqrt{3}}{4}\left[\left(4a^2x^3 - 6x^5\right)\left(a^2x^4 - x^6\right)^{-\frac{1}{2}}\right] = 0$$

$$4a^2x^3 - 6x^5 = 0$$

$$2x^3\left(2a^2 - 3x^2\right) = 0$$

Since $x \neq 0$, $2a^2 - 3x^2 = 0$

$$x^2 = \frac{2}{3}a^2$$

$$\max V = \sqrt{\frac{3}{4}\left[a^2\left(\frac{2}{3}a^2\right)^2 - \left(\frac{2}{3}a^2\right)^3\right]}$$

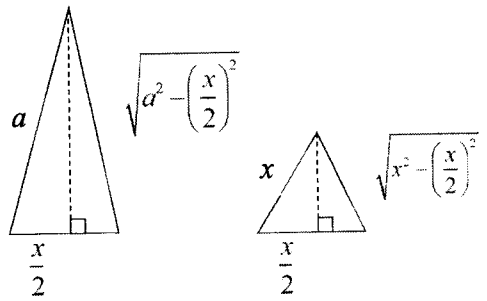
$$= \sqrt{\frac{3}{4}\left[\frac{4}{9}a^6 - \frac{8}{27}a^6\right]}$$

$$= \sqrt{\frac{3}{4}\left(\frac{4}{27}a^6\right)}$$

$$= \sqrt{\frac{a^6}{9}}$$

$$= \frac{a^3}{3}\text{ units}^3$$

(c) Total surface area, $A = 6 \times$ area of isosceles triangles $+$ area of hexagon



$$A = 6\times\frac{1}{2}\times x\times\sqrt{a^2-\left(\frac{x}{2}\right)^2}+6\times\frac{1}{2}\times x\times\sqrt{x^2-\left(\frac{x}{2}\right)^2}$$

$$= 3x\sqrt{a^2-\frac{x^2}{4}}+3x\sqrt{\frac{3x^2}{4}}$$

$$= 3\sqrt{\frac{2}{3}a^2}\sqrt{a^2-\frac{1}{4}\times\frac{2}{3}a^2}+3\sqrt{\frac{2}{3}a^2}\sqrt{\frac{3}{4}\times\frac{2}{3}a^2}$$

$$= 3\sqrt{\frac{2}{3}a^2\times\frac{5}{6}a^2}+3\sqrt{\frac{2}{3}a^2\times\frac{1}{2}a^2}$$

$$= 3\sqrt{\frac{5}{9}}a^2+\frac{3}{\sqrt{3}}a^2$$

$$= \left(\sqrt{5}+\sqrt{3}\right)a^2 \text{ units}^2$$

Solution:

Q4

(a) The student's claim may not be true as p may not be real.

(b)

$$2z^4-14z^3+33z^2-26z+p=0$$
$$2(3+\mathrm{i})^4-14(3+\mathrm{i})^3+33(3+\mathrm{i})^2-26(3+\mathrm{i})+p=0$$
$$2(8+6\mathrm{i})(8+6\mathrm{i})-14(3+\mathrm{i})(8+6\mathrm{i})+33(8+6\mathrm{i})-78-26\mathrm{i}+p=0$$
$$2(28+96\mathrm{i})-14(18+26\mathrm{i})+264+198\mathrm{i}-78-26\mathrm{i}+p=0$$
$$56+192\mathrm{i}-252-364\mathrm{i}+264+198\mathrm{i}-78-26\mathrm{i}+p=0$$
$$-10+p=0$$
$$p=10$$

(c) **Method ①:**

Since the coefficients of the polynomial are all real, $3+\mathrm{i}$ is a root, $3-\mathrm{i}$ is also a root.

$$\begin{aligned}&[z-(3+\mathrm{i})][z-(3-\mathrm{i})]\\&=[(z-3)-\mathrm{i}][(z-3)+\mathrm{i}]\\&=(z-3)^2-\mathrm{i}^2\\&=z^2-6z+9+1\\&=z^2-6z+10\end{aligned}$$

$$\begin{array}{r}2z^2-2z+1\\ z^2-6z+10\overline{)\,2z^4-14z^3+33z^2-26z+10}\\ \underline{-(2z^4-12z^3+20z^2)}\\ -2z^3+13z^2-26z\\ \underline{-(-2z^3+12z^2-20z)}\\ z^2-6z+10\\ \underline{-(z^2-6z+10)}\end{array}$$

$$\therefore\ 2z^4-14z^3+33z^2-26z+p=(z^2-6z+10)(2z^2-2z+1)$$

Method ②:

Since the coefficients of the polynomial are all real, $3+\mathrm{i}$ is a root, $3-\mathrm{i}$ is also a root.

$$\begin{aligned}2z^4-14z^3+33z^2-26z+10&=[z-(3+\mathrm{i})][z-(3-\mathrm{i})](2z^2+Az+B)\\&=[(z-3)-\mathrm{i}][(z-3)+\mathrm{i}](2z^2+Az+B)\\&=[(z-3)^2-\mathrm{i}^2](2z^2+Az+B)\\&=(z^2-6z+9+1)(2z^2+Az+B)\\&=(z^2-6z+10)(2z^2+Az+B)\end{aligned}$$

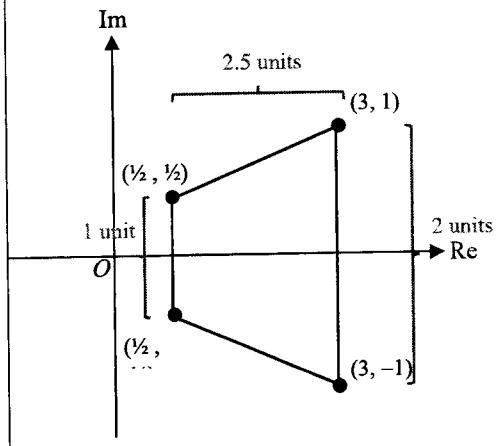
Comparing constants: $10=10B \Rightarrow B=1$

Comparing coefficients of z: $-26=-6(1)+10A \Rightarrow A=-2$

$\therefore\ 2z^4-14z^3+33z^2-26z+p=\left(z^2-6z+10\right)\left(2z^2-2z+1\right)$

$2z^2-2z+1=0$

$$z=\frac{2\pm\sqrt{4-4(2)}}{4}$$
$$=\frac{2\pm\sqrt{-4}}{4}$$
$$=\frac{2\pm 2\mathrm{i}}{4}$$
$$=\frac{1}{2}\pm\frac{1}{2}\mathrm{i}$$



(d) Trapezium

Area of qualilateral $=\frac{1}{2}(1+2)(2.5)=3.75$ units^2 (or $\frac{15}{4}$ units^2)

Section B: Probability and Statistics [60 marks]

Solution:

Q5

(a) The 20 members form a population since the coach gave questionnaire to all members in the basketball club which consists of 20 members.

(b) Having a random sample reduces biasedness, or that it would be fair, or that it would be representative of the whole membership of the club.

(c) No. of teams $= {}^{5}C_{1} \times {}^{8}C_{2} \times {}^{7}C_{2} = 2940$

(d) Let A be one of the remaining guards who can play either as a guard or as a forward.

5 centres	~~8 forwards~~	~~7 guards~~
5 centres	7 forwards	6 guards (A + 5 other guards)

Method 1:

Case ①: A plays as a guard

No. of teams $= \underbrace{{}^{5}C_{1}}_{\text{choose 1 center}} \times \underbrace{{}^{7}C_{2}}_{\text{choose 2 forwards from the remaining}} \times \underbrace{{}^{5}C_{1}}_{\text{A plays as a guard choose 1 guard from remaining 5}} = 525$

Case ②: A plays as a forward

No. of teams $= \underbrace{{}^{5}C_{1}}_{\text{choose 1 center}} \times \underbrace{{}^{7}C_{1}}_{\text{A plays as a forward choose 1 forward from the remaining 7}} \times \underbrace{{}^{5}C_{2}}_{\text{choose 2 guards from remaining 5}} = 350$

Case ③: A does not play at all

No. of teams $= \underbrace{{}^{5}C_{1}}_{\text{choose 1 center}} \times \underbrace{{}^{7}C_{2}}_{\text{choose 2 forwards from the remaining 7}} \times \underbrace{{}^{5}C_{2}}_{\text{choose 2 guards from remaining 5}} = 1050$

Total no. of teams $= 525 + 350 + 1050 = 1925$

Method 2:

Case ①: A plays as a guard, including the case that A may not be selected

No. of teams $= \underbrace{{}^{5}C_{1}}_{\text{choose 1 center}} \times \underbrace{{}^{7}C_{2}}_{\text{choose 2 forwards from the remaining}} \times \underbrace{{}^{6}C_{2}}_{\text{choose 2 guards from 6 guards including A}} = 1575$

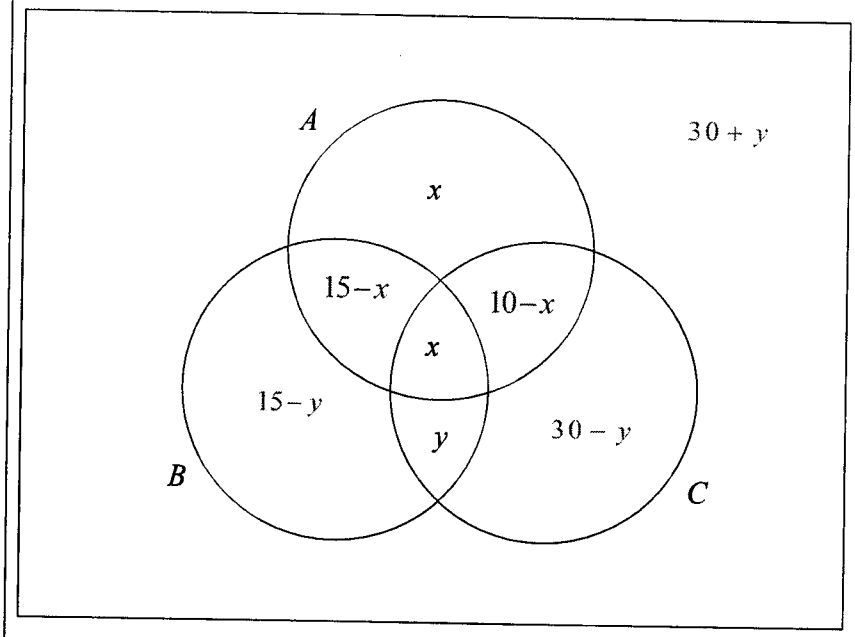
Case ②: A confirmed that he plays as a forward

	No. of teams = $\underbrace{{}^{5}C_{1}}_{\text{choose 1 center}} \times \underbrace{{}^{7}C_{1}}_{\substack{\text{A plays as a forward} \\ \text{choose 1 forward} \\ \text{from the remaining 7}}} \times \underbrace{{}^{5}C_{2}}_{\substack{\text{choose 2 guards from} \\ \text{remaining 5}}} = 350$ Total no. of teams $= 1575 + 350 = 1925$

Solution:

Q6

(a)



$\mathrm{P}(A)=\frac{25}{100}$ $\mathrm{P}(B)=\frac{30}{100}$ $\mathrm{P}(C)=\frac{40}{100}$

$\mathrm{P}(A\cap B)=\frac{15}{100}$ $\mathrm{P}(A\cap B\cap C)=\frac{x}{100}$

Since A and C are independent, $\mathrm{P}(A\cap C)=\mathrm{P}(A)\times\mathrm{P}(C)$

$$\mathrm{P}(A\cap C)=\frac{25}{100}\times\frac{40}{100}=\frac{10}{100}$$

$$\mathrm{n}(A\cap C)=10$$

(b)

Since B and C are independent, $\mathrm{P}(B\cap C)=\mathrm{P}(B)\times\mathrm{P}(C)$

$$\mathrm{P}(B\cap C)=\frac{30}{100}\times\frac{40}{100}=\frac{12}{100}$$

$$\mathrm{n}(B\cap C)=12$$

$$x+y=12$$

$$y=12-x$$

Hence, $x\geq 0$, $y\geq 0$

$y=12-x\geq 0$

Greatest value of $y=12$ occurs when $x=0$.

$\mathrm{n}(A\cap B'\cap C)=10-x\geq 0$

$x\leq 10$

Least value of $y=2$ occurs when $x=10$.

Solution:

Q7

(a)

$$P(X=2)=\frac{n-1}{n}\times\frac{n-2}{n-1}\times\frac{1}{n-2}=\frac{1}{n}$$

(b)

$$P(X=0)=\frac{1}{n}$$

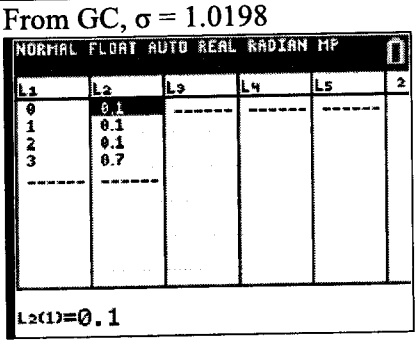
$$P(X=1)=\frac{n-1}{n}\times\frac{1}{n-1}=\frac{1}{n}$$

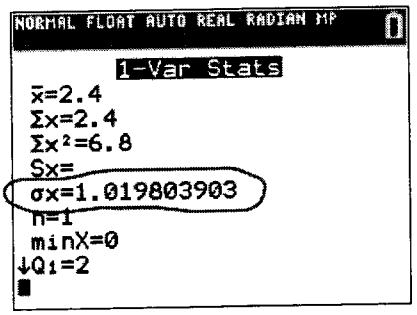
$$P(X=2)=\frac{n-1}{n}\times\frac{n-2}{n-1}\times\frac{1}{n-2}=\frac{1}{n}$$

$$P(X=3)=\frac{n-1}{n}\times\frac{n-2}{n-1}\times\frac{n-3}{n-2}=\frac{n-3}{n}$$

(c)

From GC, $\sigma = 1.0198$





Hence,

$$P(X\le 1.0198)=P(X=0)+P(X=1)=0.2$$

(d) **Method ①:**

Let Y (in dollars) be the amount obtained by Happie in a game.

x	0	1	2	3
y	5	8	10.5	10.5
$P(Y=y)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n-3}{n}$

$$E(Y)=\left(5\times\frac{1}{n}\right)+\left(8\times\frac{1}{n}\right)+\left(10.5\times\frac{1}{n}\right)+\left(10.5\times\frac{n-3}{n}\right)$$
$$=\frac{10.5n-8}{n}$$

For Happie to not have any profit/loss,

$$E(Y)-10=0$$
$$\frac{10.5n-8}{n}=10$$
$$n=16$$

Happie should prepare 16 boxes.

Method ②:

Let Y (in dollars) be the amount of "profit" Happie makes in a game.

x	0	1	2	3
y	-5	-2	0.5	0.5
$P(Y=y)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n-3}{n}$

If the doll is not won after 3 tries, it is donated away. Therefore it is still considered as a -$10 profit to Happie

$$E(Y)=\left(-5\times\frac{1}{n}\right)+\left(-2\times\frac{1}{n}\right)+\left(0.5\times\frac{1}{n}\right)+\left(0.5\times\frac{n-3}{n}\right)$$
$$=\frac{0.5n-8}{n}$$

For Happie to not have a profit,

$$E(Y)=0$$
$$\frac{0.5n-8}{n}=0$$
$$n=16$$

Happie should prepare 16 boxes.

Solution:

Q8

(a)(i)

- The probability of pralines that exceed the recommended sugar content is constant at p for every praline.
- Whether a praline has exceeded the recommended sugar content is independent of any other pralines.

(a)(ii)

$$X \sim \mathrm{B}(16, p)$$

Since mode = 2,

$\mathrm{P}(X=2) > \mathrm{P}(X=1)$ and $\mathrm{P}(X=2) > \mathrm{P}(X=3)$

$^{16}\mathrm{C}_2 p^2 (1-p)^{14} > {}^{16}\mathrm{C}_1 p (1-p)^{15}$ $\quad$ $^{16}C_2 p^2 (1-p)^{14} > {}^{16}C_3 p^3 (1-p)^{13}$

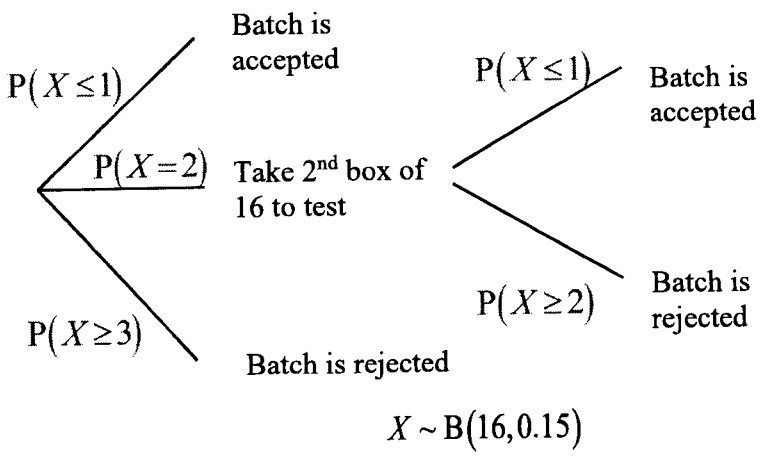
$120p^2(1-p)^{14} > 16p(1-p)^{15}$ $\quad$ $120p^2(1-p)^{14} > 56p^3(1-p)^{13}$

$15p > 2(1-p)$ $\quad$ $3(1-p) > 14p$

$p > \frac{2}{17}$ $\quad$ $p < \frac{3}{17}$

$$\therefore \frac{2}{17} < p < \frac{3}{17}$$

(b)(i)

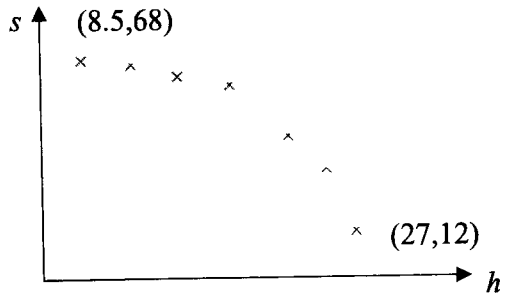


$$X \sim \mathrm{B}(16, 0.15)$$

$$\begin{aligned}
\mathrm{P}(\text{batch is accepted}) &= \mathrm{P}(X \le 1) + \mathrm{P}(X=2)\mathrm{P}(X \le 1) \\
&= \mathrm{P}(X \le 1)\left[1 + \mathrm{P}(X=2)\right] \\
&= 0.2839012136(1 + 0.2774781077) \\
&= 0.3626775851 \\
&\approx 0.363 \text{ (3 s.f.)}
\end{aligned}$$

(b)(ii)	$\mathrm{P}(\text{2nd box is tested} \mid \text{batch is accepted}) = \dfrac{\mathrm{P}(\text{2nd box is tested} \cap \text{batch is accepted})}{\mathrm{P}(\text{batch is accepted})}$ $= \dfrac{\mathrm{P}(X=2)\mathrm{P}(X \le 1)}{0.3626775851}$ $= \dfrac{0.2774781077 \times 0.2839012136}{0.3626775851}$ $= 0.2172077205$ ≈ 0.217 (3 s.f.)

Solution:

Q9	
(a)	$s = -3.00799h + 100.27974$ From GC, $\bar{h} = 17.1429$ $\bar{s} = -3.00799\bar{h} + 100.27974$ $\bar{s} = -3.00799(17.1429) + 100.27974 = 48.7140$ $\frac{68+61+44+12+\alpha+58+31}{7} = 48.7140$ $\alpha = 67.0$
(b)	
(c)	r- value between s and h is -0.961. r- value between s and h^2 is -0.991 which is closer to -1 which means that there is a stronger linear correlation between s and h^2. Also, from the scatter diagram, as h increases, s decreases at an increasing rate, which indicates that the curve does not have a linear behaviour. From G.C., $k = -0.0873$ and $c = 77.7$. Hence $s = -0.0873h^2 + 77.7$
(d)	When $h = 7$, $s = -0.087331(7)^2 + 77.727 = 73.4$ His predicted score is 73.4. Since $h = 7$ is outside the data range, the estimate is obtained via extrapolation and thus is not reliable.
(e)	Correlation is not causation. There may be other factors that contribute to this trend.

Solution:

Q10

(a) Let S and L be the random variable denoting the amount of fish caught by the Standard and Large vessels respectively.

$$S \sim \mathrm{N}(300,\ \sigma^2)$$
$$L \sim \mathrm{N}(540,\ 110^2)$$

$$\mathrm{P}(S > 400) = 0.08$$
$$\mathrm{P}\left(Z > \frac{400-300}{\sigma}\right) = 0.08$$
$$\frac{400-300}{\sigma} = 1.40507$$
$$\sigma = 71.2 \ (3 \text{ s.f.}) (\text{shown})$$

(b) Let $A = S_1 + S_2 + S_3 + L_1 + L_2$

$$A \sim \mathrm{N}\left(3\times300 + 2\times540,\ 3\times71.171^2 + 2\times110^2\right)$$
$$\sim \mathrm{N}\left(1980,\ 198.484^2\right)$$

$\mathrm{P}(A \geq 2100) = 0.273$

(c) The amount of fish caught by all the vessels are independent from one another.

(d) Let $B = 3S + 2L$

$$B \sim \mathrm{N}\left(3\times300 + 2\times540,\ 3^2\times71.171^2 + 2^2\times110^2\right)$$
$$\sim \mathrm{N}\left(1980,\ 306.574^2\right)$$

$\mathrm{P}(B \geq 2100) = 0.348$

(e)

$$\bar{L} \sim \mathrm{N}\left(540,\ \frac{110^2}{n}\right)$$

$\mathrm{P}(\bar{L} < 552) \geq 0.7$

Method ①:

n	$\mathrm{P}(\bar{L} < 552)$
23	0.6996
24	0.7035
25	0.7073

From GC, least value of n is 24

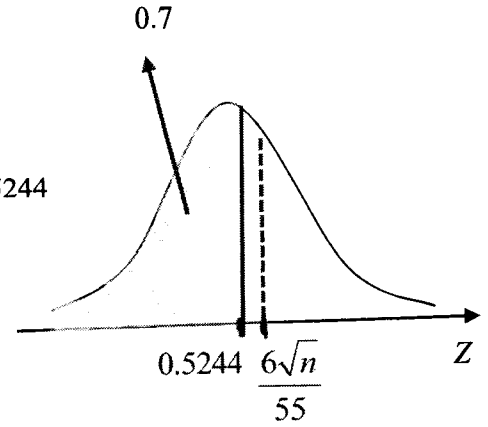
Method ②:

Standardizing,

$$\mathrm{P}\left(Z < \frac{552-540}{\sqrt{110^2/n}}\right) \geq 0.7$$

$$\mathrm{P}\left(Z < \frac{6\sqrt{n}}{55}\right) \geq 0.7$$

Consider $\mathrm{P}(Z < k) = 0.7 \quad \Rightarrow \quad k = 0.5244$



Hence $\frac{6\sqrt{n}}{55} \geq 0.5244$

$$6\sqrt{n} \geq 28.842$$

$$n \geq 23.1$$

$\therefore$ least value of n is 24.

Solution:

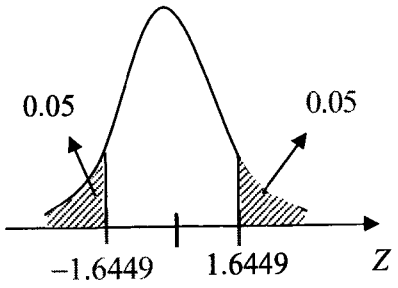
Q11	
(a)	The sample mean time for 35 matches is approximated to have a normal distribution by the Central Limit Theorem, since sample size is sufficiently large. Hence there is no need to make any assumptions about the distribution of the time taken to score the first goal.
(b)	Unbiased estimate for population mean $= \frac{1650}{35} = 47.143$ (to 3d.p.) Unbiased estimate for population variance is $s^2 = \frac{35}{34}(250) = 257.353$ (to 3d.p.)
(c)	Let X be the random variable denoting time taken by EastHam to score the first goal in a match. $H_0 : \mu = 50$ $H_1 : \mu < 50$, where μ is the population mean time taken by EastHam to score the first goal in a match Under H_0, Since n is large, $\bar{X} \sim N\left(50, \frac{257.353}{35}\right)$ approximately by CLT $Z = \frac{\bar{X} - 50}{\sqrt{s^2/n}} \sim N(0,1)$ Method ①: By GC, p-value $= 0.146 (3 \text{ s.f.})$. Since p-value > 0.1, we do not reject H_0 and conclude that there is insufficient evidence at 10% significance level to say that average time taken by EastHam to score the first goal has decreased. Method ②: $z_{test} = \frac{\left(\frac{1650}{35}\right) - 50}{\sqrt{\frac{4375/17}{35}}} = -1.05366$, $z_{test} = -1.28155$ Since $z_{test} > z_{critical}$, we do not reject H_0 and conclude that there is insufficient evidence at 10% significance level to say that average time taken by EastHam to score the first goal has decreased.

(d) Let Y be the random variable denoting time taken by NewPalace to score the first goal in a match.

$H_0 : \mu = 40$

$H_1 : \mu \neq 40$

Under H_0, $\overline{Y} \sim N\left(40, \frac{280}{50}\right)$



Since H_0 is rejected,

$$\frac{\overline{y} - 40}{\sqrt{280/50}} < -1.6449 \quad \text{or} \quad \frac{\overline{y} - 40}{\sqrt{280/50}} > 1.6449$$

$$\overline{y} < 36.1 (3 \text{ s.f.}) \quad \text{or} \quad \overline{y} > 43.9 (3 \text{ s.f.})$$

Hence, range of values of $\overline{y}$ is: $0 < \overline{y} < 36.1$ or $\overline{y} > 43.9$