

2025 Year 6 H2 Math Prelim Exam P1
Suggested Solutions

Qn	Suggested Solutions	Comments
1	$u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u \dots (1)$ $2xy \frac{dy}{dx} = x^2 - y^2$ $\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} = \frac{x^2 - u^2x^2}{2ux^2} = \frac{1 - u^2}{2u} \dots (2)$ <p>Equate (1) with (2):</p> $\frac{du}{dx}x + u = \frac{1 - u^2}{2u}$ $\Rightarrow \frac{du}{dx}x = \frac{1 - u^2}{2u} - u = \frac{1 - 3u^2}{2u}$ $\frac{2u}{1 - 3u^2} \frac{du}{dx} = \frac{1}{x} \text{ (shown)}$ $\int \frac{2u}{1 - 3u^2} du = \int \frac{1}{x} dx$ $-\frac{1}{3} \int \frac{-6u}{1 - 3u^2} du = \int \frac{1}{x} dx$ $-\frac{1}{3} \ln 1 - 3u^2 = \ln x + c \quad (\because x > 0)$ $\ln 1 - 3u^2 = -3 \ln x - 3c$ $1 - 3u^2 = \pm e^{-3c} \cdot e^{\ln \frac{1}{x^3}} = \frac{A}{x^3} \text{ (where } A = \pm e^{-3c}\text{)}$ $3u^2 = 1 - \frac{A}{x^3}$ $\left(\frac{y}{x}\right)^2 = \frac{1}{3} \left(1 - \frac{A}{x^3}\right)$ $y = -\sqrt{\frac{x^2}{3} \left(1 - \frac{A}{x^3}\right)} = -x \sqrt{\frac{1}{3} \left(1 - \frac{A}{x^3}\right)} \quad (\because y < 0)$	<p>Use $y = ux$ or $u = \frac{y}{x}$ to find $\frac{dy}{dx}$ or $\frac{du}{dx}$ (by implicit differentiation) before showing $\frac{2u}{1 - 3u^2} \frac{du}{dx} = \frac{1}{x}$</p> <p>Note: u, x, y are variables</p> <p>Separate variables u & x to find the general solution of y in terms of x by integration</p> <p>Take note that $x > 0$ & $y < 0$</p>
Total marks: 6		

Qn	Suggested Solutions	Comments
2(a)	$u_n = 2(4-3x)^n$ $u_{n+1} = 2(4-3x)^{n+1}$ $\frac{u_{n+1}}{u_n} = 4-3x \text{ (constant)}$ <p>Since $4-3x$ is a constant common ratio, the series is a geometric series.</p> <p>For the sum to infinity to exist,</p> $ 4-3x < 1$ $-1 < 4-3x < 1$ $-5 < -3x < -3$ $1 < x < \frac{5}{3}$	<p>As it is given that the series is geometric, the general form for common ratio $\frac{u_{n+1}}{u_n} =$ constant must be used to justify it.</p> <p>For the sum to infinity to exist, $\text{common ratio} < 1$.</p>
(b)	$\sum_{r=0}^{n-1} [2(4-3x)^r + (r+1)(2r+5)]$ $= \sum_{r=0}^{n-1} 2(4-3x)^r + \sum_{r=0}^{n-1} (r+1)(2r+5)$ $= \sum_{r=0}^{n-1} 2 + \sum_{r=0}^{n-1} (r+1)(2r+5) \text{ (since } x=1)$ $= 2n + \sum_{r=1}^n ((r-1)+1)(2(r-1)+5)$ $= 2n + \sum_{r=1}^n r(2r+3)$ $= 2n + 2 \sum_{r=1}^n r^2 + 3 \sum_{r=1}^n r$ $= 2n + 2 \left(\frac{n}{6}\right)(n+1)(2n+1) + 3 \left(\frac{n}{2}\right)(1+n)$ $= n \left(\frac{2}{3}n^2 + \frac{5}{2}n + \frac{23}{6}\right)$ $a = \frac{2}{3}, b = \frac{5}{2}, c = \frac{23}{6}$	<p>Note that $\sum_{r=0}^{n-1} f(r)$ has n terms</p> <p>Replace r as $(r-1)$ to change the limits for $\sum_{r=0}^{n-1} f(r)$</p> <p>$\sum_{r=1}^n c = cn$ where c is a constant</p> <p>$\sum_{r=1}^n r = \frac{n(n+1)}{2}$ (AP)</p> <p>$\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ is applicable for lower limit '1', & the upper limit 'n'</p>
		Total marks: 7

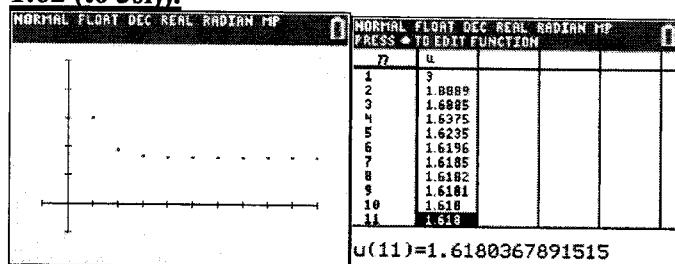
Qn	Suggested Solutions	Comments
3(a)	$e^x \sin(x + \pi) = e^x [\sin x \cos \pi + \cos x \sin \pi]$ $= -e^x \sin x$ $= - \left(1 + x + \frac{x^2}{2!} + \dots \right) \left(x - \frac{x^3}{3!} + \dots \right)$ $= -x - x^2 - \frac{x^3}{3} + \dots$ <p>Alternative (using repeated differentiation)</p> $f(x) = e^x \sin(x + \pi)$ $f'(x) = e^x \sin(x + \pi) + e^x \cos(x + \pi)$ $= y + e^x \cos(x + \pi)$ $f''(x) = \frac{dy}{dx} - e^x \sin(x + \pi) + e^x \cos(x + \pi)$ $= \frac{dy}{dx} - y + e^x \cos(x + \pi) \quad \text{OR} \quad 2e^x \cos(x + \pi)$ $f'''(x) = \frac{d^2y}{dx^2} - \frac{dy}{dx} - e^x \sin(x + \pi) + e^x \cos(x + \pi)$ $= \frac{d^2y}{dx^2} - \frac{dy}{dx} - y + e^x \cos(x + \pi)$ $\text{OR} \quad 2e^x [\cos(x + \pi) - \sin(x + \pi)]$ <p>When $x = 0$,</p> $f(0) = 0$ $f'(0) = -1$ $f''(0) = -2$ $f'''(0) = -2$ $f(x) = -x + \frac{-2}{2!}x^2 + \frac{-2}{3!}x^3 + \dots$ $= -x - x^2 - \frac{x^3}{3} + \dots$	<p>Common mistake:</p> <p>When expanding $\sin(x + \pi)$ using Maclaurin series, some students wrongly substitute $(x + \pi)$ directly into the sine expansion:</p> $\sin(x + \pi) =$ $(x + \pi) - \frac{(x + \pi)^3}{3!} + \frac{(x + \pi)^5}{5!} - \dots$ <p>Note that expansion of $(x + \pi)^3$ and $(x + \pi)^5$ both contain constant term, x term and x^2 terms. This means every term in the infinite expansion contributes to the first few powers of x. You cannot cleanly stop at the x^2 term, so this defeats the purpose of using Maclaurin series.</p> <p>→ Do not substitute constants (like π) directly into Maclaurin standard series expansions. Always simplify first.</p> <p>Students should use addition formula for $\sin(a + b)$ or trigonometric identity $\sin(x + \pi) = -\sin x$ before applying standard Maclaurin series</p>
(b)	$ax(1 + bx)^c = ax \left(1 + cbx + \frac{c(c-1)}{2!}(bx)^2 + \dots \right)$ $= ax + abcx^2 + \frac{ab^2c(c-1)}{2}x^3 + \dots$ <p>Comparing with answer in (a),</p> $a = -1, abc = -1, \frac{ab^2c(c-1)}{2} = -\frac{1}{3}$ <p>Solving,</p>	<p>Students should apply Maclaurin standard series expansion of $(1 + x)^n$ which is more efficient than using repeated differentiation</p> <p>Students should apply result from comparing coefficient of x and x^2 term $a = -1$ and $bc = 1$ to simplify the equation for the x^3 coefficient</p> $\frac{ab^2c(c-1)}{2} = \frac{-b(c-1)}{2} \text{ before}$

$bc = 1 \Rightarrow b = \frac{1}{c} \quad \text{and} \quad \frac{b(c-1)}{2} = \frac{1}{3}$ $\frac{c-1}{2c} = \frac{1}{3}$ $3c - 3 = 2c$ $c = 3$ <p>Therefore, $b = \frac{1}{3}$</p> $ax(1+bx)^c = -x \left(1 + \frac{1}{3}x\right)^3$ <p>The x^4 term = $-x \left[\binom{3}{3} (1)^{3-3} \left(\frac{1}{3}x\right)^3 \right] = -\frac{1}{27}x^4$</p> <p>The coefficient of x^4 is $-\frac{1}{27}$.</p>	<p>solving. This will avoid many careless mistakes made.</p>
Total marks: 8	

Qn	Suggested Solutions	Comments
4(a)	$x_{n+1} = \frac{5x_n + 2}{2x_n + 3}$ <p>As $n \rightarrow \infty$, $x_n \rightarrow l$, $x_{n+1} \rightarrow l$.</p> $x_{n+1} = \frac{5x_n + 2}{2x_n + 3}$ $l = \frac{5l + 2}{2l + 3}$ $2l^2 + 3l = 5l + 2$ $2l^2 - 2l - 2 = 0$ $l^2 - l - 1 = 0$ $l = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$ $\therefore l = \frac{1 + \sqrt{5}}{2} \quad \text{or} \quad l = \frac{1 - \sqrt{5}}{2}$	<p>Given that a sequence (generated by a recurrence relation) converges, letting $x_n \rightarrow l$ and $x_{n+1} \rightarrow l$ as $n \rightarrow \infty$ is the standard method to find the limit l of the sequence – do familiarise yourself with this technique.</p> <p>Some students assumed that $n \rightarrow \infty$ implies that $x_n \rightarrow \infty$ and got confused with the method of finding the limit of a sequence defined by a formula for the nth term (in terms of n, i.e. $u_n = f(n)$ (e.g. $u_n = \frac{3n-5}{n-2}$)) – do note the difference.</p> <p>A few students erroneously assumed that the sequence is an AP/GP from the start – not every sequence given is an AP/GP.</p>

(b)

The sequence is **decreasing** and **converges to** $\frac{1+\sqrt{5}}{2}$ (or **1.62 (to 3sf)**).



OR

HISTORY	HISTORY
5Ans+2 2Ans+3 1.888888889	5Ans+2 2Ans+3 1.637532134
5Ans+2 2Ans+3 1.68852459	5Ans+2 2Ans+3 1.623514953
5Ans+2 2Ans+3 1.637532134	5Ans+2 2Ans+3 1.619581612
5Ans+2 2Ans+3 1.618471532	5Ans+2 2Ans+3 1.618471532
5Ans+2 2Ans+3 1.618157735	
5Ans+2 2Ans+3 1.61806899	
5Ans+2 2Ans+3 1.618043889	

This part could have been done without (a) – students can use the GC to observe the behaviour of the sequence. In general, there are two parts to describing the behaviour of a sequence:

- Trends (increasing / decreasing, constant, alternating)
- Long-run behaviour of an infinite sequence (convergent or divergent)

For example: The sequence is a _____ sequence and diverges/converges to _____.

(c)

$$x_{n+1} = \frac{5x_n + 2}{2x_n + 3}$$

By making x_n the subject,

$$\begin{aligned} x_{n+1}(2x_n + 3) &= 5x_n + 2 \\ (2x_{n+1} - 5)x_n &= 2 - 3x_{n+1} \\ x_n &= \frac{2 - 3x_{n+1}}{2x_{n+1} - 5} \end{aligned}$$

Starting with $x_5 = \frac{3503}{2158}$, we can use the GC to compute

$$x_4 = \frac{2 - 3x_5}{2x_5 - 5} = \frac{2 - 3\left(\frac{3503}{2158}\right)}{2\left(\frac{3503}{2158}\right) - 5} = \dots = \frac{6193}{3784} = \frac{563}{344} = 1.63663$$

	HISTORY
x_5	3503 2158 1.62326228
x_4	2-3Ans 2Ans-5 1.636627907
x_3	2-3Ans 2Ans-5 1.685185185

$$\therefore x_1 = 2.9$$

	NORMAL FLOAT DEC REAL RADIAN MP
	1.636627907
x_2	2-3Ans 2Ans-5 1.685185185
x_1	2-3Ans 2Ans-5 2.9

Students can try to use the recurrence relation (subbing in $n = 4$) to first find x_4 using $x_5 = \frac{5x_4 + 2}{2x_4 + 3}$; from this, it can be observed that we need to make x_4 the subject (and hence in general, x_n the subject). This will allow us to use the calculator to help compute x_3, x_2 and finally x_1 as seen from the solution. Quite a number of students did this on their own, using the recurrence relation again and again (repeating the process and making x_3 the subject then x_2 the subject, etc) and managed to obtain x_1 successfully – this is ok too for this question. But do note that this method may not be feasible if say x_{50} was given instead of x_5 and we need to obtain x_1 - we will need the help of the calculator to do this then.

Alternative

Make use of the SEQ function in the GC, using the recurrence relation,

$$x_n = \frac{2-3x_{n+1}}{2x_{n+1}-5} \text{ and } x_5 = \frac{3503}{2158}$$

we can key in x_n as u_n , x_{n+1} as u_{n-1} and $x_5 = \frac{3503}{2158}$ as u_1

(we are just making use of the GC to help us compute “backwards”, the values using the recurrence relationship – in the GC, we cannot input x_n in terms of x_{n+1} .)

NORMAL FLOAT DEC REAL RADIAN MP

INITIAL CONDITION

Plot1 Plot2 Plot3

TYPE: SEQ(n) SEQ(n+1) SEQ(n+2)

nMin=1

%u(n+1) = (2-3u(n))/(2u(n)-5)

u(1) = 3503/2158

u(2) =

%v(n+1) =

v(1) =

v(2) =



















n	u(n)			
1	1.6233			
2	1.6366			
3	1.6852			
4	1.875			
5	2.9			
6	-8.375			
7	-1.247			

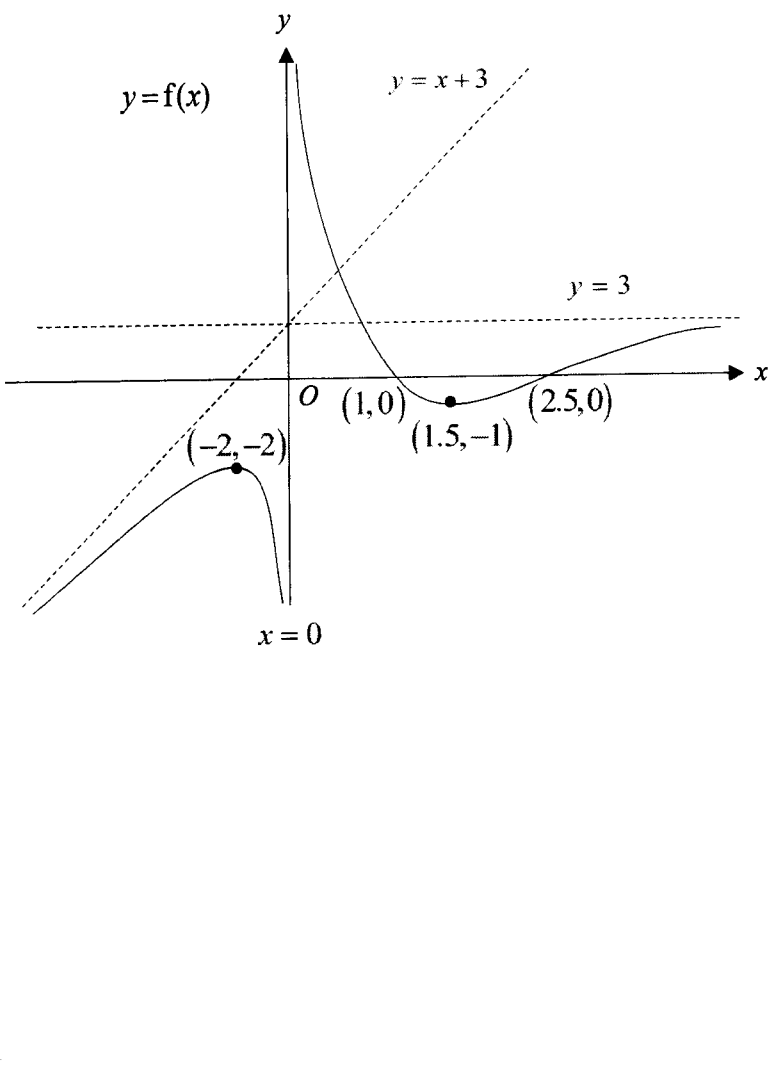
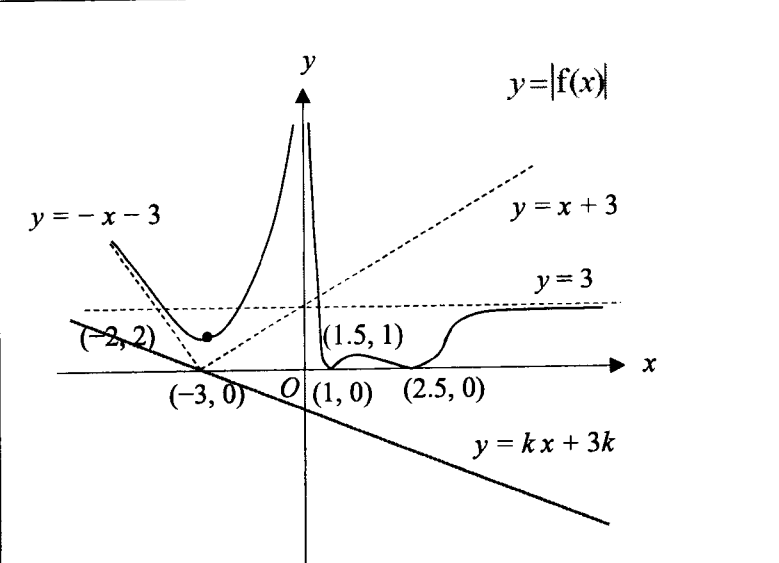
n=1

n	u(n)	
1	1.6233	=x ₅
2	1.6366	=x ₄
3	1.6852	=x ₃
4	1.875	=x ₂
5	2.9	=x ₁
6	-8.375	
7	-1.247	

∴ $x_1 = 2.9$

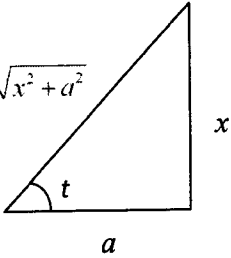
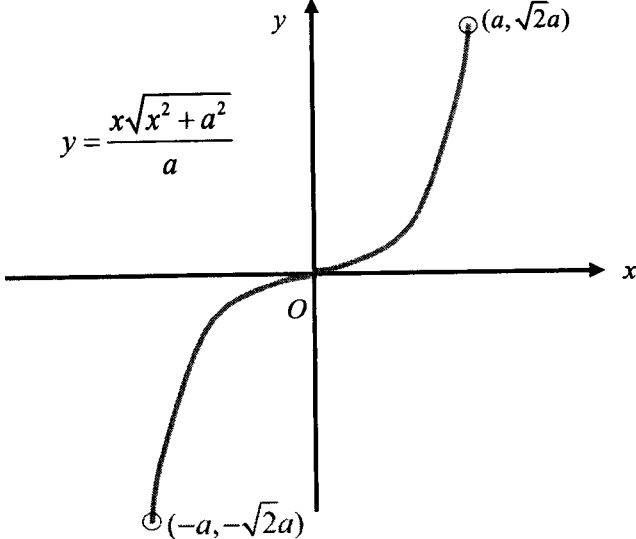
Total marks: 7

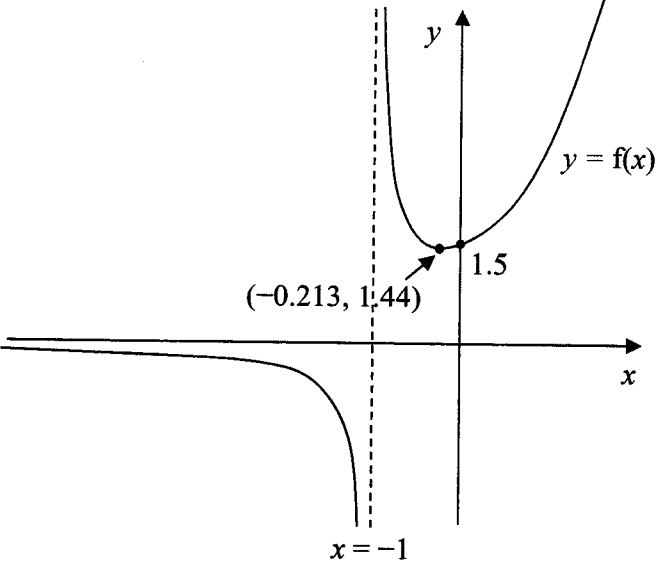
Qn	Suggested Solution	Comments																								
5(a)	<p>Max pt at $x = -2$, Min pt at $x = 1.5$.</p> <p>(some explanation below; no need to show) By looking at the graph of $y = f'(x)$, we can use the usual first or second derivative test to determine the nature of the stationary points on the graph of $y = f(x)$:</p> <p>First Derivative Test</p> <table border="1" data-bbox="288 533 1062 775"> <thead> <tr> <th>x</th> <th>-2^-</th> <th>-2</th> <th>-2^+</th> </tr> </thead> <tbody> <tr> <td>$\frac{dy}{dx}$ (value of $f'(x)$)</td> <td>+ve (f' graph above x-axis)</td> <td>0</td> <td>-ve (f' graph below x-axis)</td> </tr> <tr> <td>Shape</td> <td></td> <td></td> <td></td> </tr> </tbody> </table> <p>\therefore Maximum point at $x = -2$</p> <table border="1" data-bbox="288 842 1062 1084"> <thead> <tr> <th>x</th> <th>1.5^-</th> <th>1.5</th> <th>1.5^+</th> </tr> </thead> <tbody> <tr> <td>$\frac{dy}{dx}$ (value of $f'(x)$)</td> <td>-ve (f' graph below x-axis)</td> <td>0</td> <td>+ve (f' graph above x-axis)</td> </tr> <tr> <td>Shape</td> <td></td> <td></td> <td></td> </tr> </tbody> </table> <p>\therefore Minimum point at $x = 1.5$</p> <p>Second Derivative Test</p> <p>At $x = -2$, gradient of f' graph is negative $\Rightarrow \frac{d^2y}{dx^2} < 0$ \therefore Maximum point at $x = -2$</p> <p>At $x = 1.5$, gradient of f' graph is positive $\Rightarrow \frac{d^2y}{dx^2} > 0$ \therefore Minimum point at $x = 1.5$</p>	x	-2^-	-2	-2^+	$\frac{dy}{dx}$ (value of $f'(x)$)	+ve (f' graph above x -axis)	0	-ve (f' graph below x -axis)	Shape				x	1.5^-	1.5	1.5^+	$\frac{dy}{dx}$ (value of $f'(x)$)	-ve (f' graph below x -axis)	0	+ve (f' graph above x -axis)	Shape				<p>Students are expected to identify clearly which are the maximum/minimum points.</p>
x	-2^-	-2	-2^+																							
$\frac{dy}{dx}$ (value of $f'(x)$)	+ve (f' graph above x -axis)	0	-ve (f' graph below x -axis)																							
Shape																										
x	1.5^-	1.5	1.5^+																							
$\frac{dy}{dx}$ (value of $f'(x)$)	-ve (f' graph below x -axis)	0	+ve (f' graph above x -axis)																							
Shape																										
(b)	<p>f is decreasing: $-2 \leq x < 0$ or $0 < x \leq 1.5$</p>	<p>Many students forgot to exclude the point at which $x = 0$ (where the graph is undefined).</p>																								

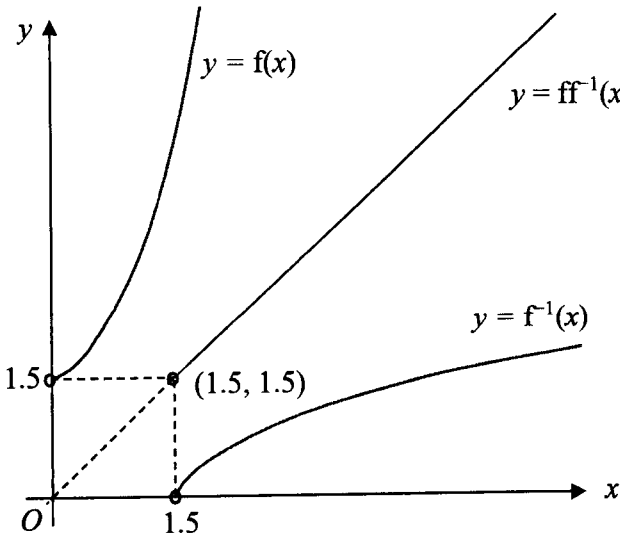
<p>(c)</p>	 <p>Graph of $y=f(x)$ showing asymptotes $y=x+3$ and $y=3$. The graph has x-intercepts at $(1,0)$ and $(2.5,0)$, and a local minimum at $(1.5,-1)$. A point $(-2,-2)$ is marked on the curve. The vertical asymptote is $x=0$.</p>	<p>Few students managed to get full marks for this question. There are a few important points to note:</p> <ul style="list-style-type: none"> • The minimum point (with y-coord -1) should be “above”/ “higher” than the maximum point (with y-coord -2) • The oblique asymptote $y=x+3$ and horizontal asymptote $y=3$ should both intersect at the y-axis; both will pass through the point $(0,3)$ • Some students still left the line $y=-x-3$ but it should no longer be drawn since it is not an asymptote to the graph of $y=f(x)$ • Some students were careless in labelling the coordinates of the turning points/axial intercepts • Some graphs were not well drawn, especially how they tend to the asymptotes – do take note
<p>(d)</p>	 <p>Graph of $y= f(x)$ showing asymptotes $y=x+3$ and $y=3$. The graph has x-intercepts at $(-3,0)$, $(1,0)$, and $(2.5,0)$, and a local minimum at $(-2,2)$. A line $y=kx+3k$ is shown intersecting the graph at $(-3,0)$. The vertical asymptote is $x=0$.</p> <p>Range of values of x: $-1 \leq k < 0$.</p>	<p>The line $y=kx+3k$ should pass through the point $(-3,0)$; it is the point independent of k on the line $y=kx+3k$. We can “pivot” the line about this point in order to vary the gradient of the line, k, to observe and hence determine the range of values of k for which there is no intersection between the line $y=kx+3k$ and the graph of $y= f(x)$. This is equivalent to having no real solution to the equation $f(x) =kx+3k$, as required by the question.</p> <p>Few students could get the range of values of k accurately, either missing</p>

		out the value of 1, or the upper bound of 0, or carelessly “reversing” the bounds.
		Total marks: 9

Qn	Suggested Solutions	Comments
6(a)	$\text{LHS} = \frac{\tan t}{\sqrt{1 + \tan^2 t}} = \left(\frac{\sin t}{\cos t}\right) \left(\frac{1}{\sec t}\right)$ $= \left(\frac{\sin t}{\cos t}\right) (\cos t) = \sin t = \text{RHS (shown)}$	<p>As this is a show question, all steps should be shown correctly. That is,</p> $\tan t = \frac{\sin t}{\cos t}$ $1 + \tan^2 t = \sec^2 t$ $\sec t = \frac{1}{\cos t}$ <p>Do note that</p> $\sqrt{\sec^2 t} = \sec t = \sec t$ <p>$\therefore \sec t > 0$ for $-\frac{\pi}{4} < t < \frac{\pi}{4}$</p>
(b)	$x = a \tan t \Rightarrow \tan t = \frac{x}{a}$ $y = a \sec^2 t \sin t$ $= a(\tan^2 t + 1)(\sin t)$ $= a(\tan^2 t + 1) \left(\frac{\tan t}{\sqrt{1 + \tan^2 t}}\right)$ $= a \left(\left(\frac{x}{a}\right)^2 + 1\right) \left(\frac{\frac{x}{a}}{\sqrt{1 + \left(\frac{x}{a}\right)^2}}\right)$ $= x \sqrt{1 + \left(\frac{x}{a}\right)^2}$ $= \frac{x\sqrt{x^2 + a^2}}{a}, \text{ where } -a < x < a$	<p>Cartesian equation \rightarrow only has x and y. Parametric has trigo, likely trigo identities have to be used.</p> <p>We should simplify this as we are given in the question that $a > 0$.</p> $\sqrt{a^2} = a = a$ <p>The cartesian equation comes with the domain since</p> $-\frac{\pi}{4} < t < \frac{\pi}{4} \Rightarrow -1 < \tan t < 1$

	<p>Otherwise</p> $x = a \tan t \Rightarrow \tan t = \frac{x}{a} \Rightarrow \cos t = \frac{a}{\sqrt{x^2 + a^2}}$ $y = a \sec^2 t \sin t$ $= a \left(\frac{1}{\cos^2 t} \right) (\sin t)$ $= a \left(\frac{1}{\cos t} \right) (\tan t)$ $= a \left(\frac{\sqrt{x^2 + a^2}}{a} \right) \left(\frac{x}{a} \right)$ $= \frac{x\sqrt{x^2 + a^2}}{a}, \text{ where } -a < x < a$ 	See this with a right-angled triangle.
(c)	$f(-x) = \frac{-x\sqrt{(-x)^2 + a^2}}{a} = -\frac{x\sqrt{x^2 + a^2}}{a} = -f(x) \text{ (shown)}$ 	As $-a < x < a$, there should be end-points for this graph. Since this is an odd function, the graph should also be symmetrical about the origin. Can see the shape from the GC too.
(d)	$\text{Area} = 2 \int_0^A \frac{x\sqrt{x^2 + a^2}}{a} dx \text{ (by symmetry)}$ $= \frac{1}{a} \int_0^A 2x\sqrt{x^2 + a^2} dx \text{ (using } \int f'(x)[f(x)]^n dx)$ $= \left(\frac{1}{a} \right) \left[\frac{2}{3} (x^2 + a^2)^{\frac{3}{2}} \right]_0^A$ $= \frac{2}{3a} \left((A^2 + a^2)^{\frac{3}{2}} - a^3 \right)$	Recall: $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$
Total marks: 8		

Qn	Suggested Solution	Comments
7		While it is not necessary to present the graph of f , it will be useful in visualising part of the problem.
(a)	<p>Since $g'(x) > 0$ for all real x, and $g(1) = -0.5$, the min. value of $g(x)$ is -0.5 and it increases as x increases. Depending on the function, g either increases to $+\infty$ or an asymptotic value greater than -0.5, i.e. $R_g = [-0.5, \infty)$ or $[-0.5, a)$, where $a \in \mathbb{R}$, $a > -0.5$. Either way, $R_g \subseteq D_f$, where $D_f = \mathbb{R} \setminus \{-1\}$. Hence, fg exists.</p> <p><i>The second part of this question is not assessed due to its unintended complexity – depending on the function g, the answer to this part will be quite different. ** Everyone was awarded the 1 mark originally assigned to this part.</i></p>	<p>Writing D_f in set notation.</p> <ul style="list-style-type: none"> • $\mathbb{R} \setminus \{-1\}$ • $(-\infty, -1) \cup (-1, \infty)$
(b)	<p>Given $f(x) = e^x + \frac{1}{2x+2}$,</p> $fg(x) = f[g(x)] = e^{g(x)} + \frac{1}{2g(x)+2}.$ <p>Hence, $e^{g(x)} + \frac{1}{2g(x)+2} = \frac{x}{\sqrt{e}} + \frac{1}{2\ln x + 1}$.</p> <p>Comparing in terms,</p> $2g(x) + 2 = 2\ln x + 1$ $g(x) = \ln x - \frac{1}{2}$ <p>Check: $e^{g(x)} = e^{\ln x - \frac{1}{2}} = \frac{x}{\sqrt{e}}$</p>	

	$\therefore g(x) = \ln x - \frac{1}{2}. \text{ (Equivalently } g(x) = \ln \frac{x}{\sqrt{e}} \text{.)}$ <p>Alternative</p> $fg(x) = \frac{x}{\sqrt{e}} + \frac{1}{2 \ln x + 1} = xe^{-\frac{1}{2}} + \frac{1}{2 \ln x + 1}$ $= e^{\ln x} \cdot e^{-\frac{1}{2}} + \frac{1}{2 \left(\ln x - \frac{1}{2} \right) + 2}$ $= e^{\left(\ln x - \frac{1}{2} \right)} + \frac{1}{2 \left(\ln x - \frac{1}{2} \right) + 2}$ <p>Thus, $g(x) = \ln x - \frac{1}{2}$.</p> <p>Note: It is possible to do this part by observation.</p>	
(c)	<p>For f^{-1} to exist, the function must be 1-1. From graph, min. point on f is $(-0.231, 1.44)$. Since k needs to be an integer, the least integer value of k is 0.</p>	
(d)(i)		<ul style="list-style-type: none"> • $ff^{-1}(x) = x$ • $D_{ff^{-1}} = D_{f^{-1}}$ $= R_f$ $= (1.5, \infty)$
(d)(ii)	<p>Observe that $e + \frac{1}{4} = e^1 + \frac{1}{2(1)+2}$. This corresponds to the point $\left(1, e + \frac{1}{4}\right)$ on f. Hence, on the graph of f^{-1}, at $x = e + \frac{1}{4}$, $y = 1$. i.e. $\left(e + \frac{1}{4}, 1\right)$.</p> <p>Let $y = e^x + \frac{1}{2x+2}$ (expression for f).</p>	

An expression for f^{-1} is $x = e^y + \frac{1}{2y+2}$.

(Note: From here, y refers to the graph of f^{-1} .)

Differentiate wrt x ,

$$1 = e^y \cdot \frac{dy}{dx} + (-1)(2y+2)^{-2} (2) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{e^y - \frac{2}{(2y+2)^2}}$$

At point $(e + \frac{1}{4}, 1)$,

$$\frac{dy}{dx} = \frac{1}{e^1 - \frac{2}{(2[1]+2)^2}} = \frac{1}{e - \frac{1}{8}} = \frac{8}{8e-1} = 0.386 \text{ (to 3 s.f.)}$$

Alternative (Quicker if you know the relationship)

Gradient of $y = f^{-1}(x)$ at $x = e + \frac{1}{4}$

$$= \frac{1}{\text{Gradient of } f \text{ at } x=1}$$

$$= \frac{1}{f'(1)} = \frac{1}{2.5933} = 0.386 \text{ (to 3 s.f.)}$$

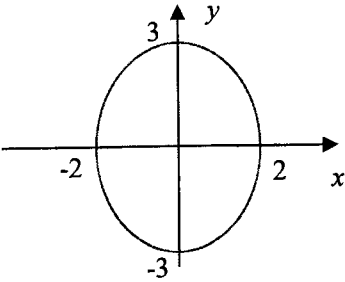
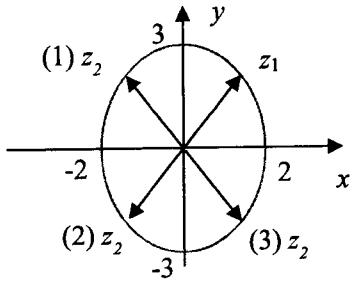
Notice that this is similar to

$$f(x) = e^x + \frac{1}{2x+2}$$

$$f'(x) = e^x - \frac{1}{(2x+2)^2}$$

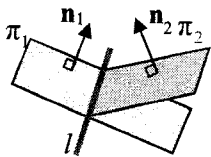
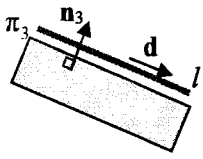
This is the same as evaluating $f'(1)$.

Total marks: 10

Qn	Suggested solutions	Comments
8(a)	<p>Since $x = \text{Re}(z)$ and $y = \text{Im}(z)$,</p>  <p> $x = 2 \cos(t), \quad y = 3 \sin(t)$ $\cos(t) = \frac{x}{2}, \quad \sin(t) = \frac{y}{3}$ $\cos^2(t) + \sin^2(t) = 1$ $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ </p>	<p>Concepts:</p> <ol style="list-style-type: none"> 1. Relate real and imaginary component of a complex number to a set of parametric eqns. 2. Forming cartesian eqn from parametric (trigo identity) 3. Recognition of standard curves (in this case, ellipse). For this qn, be aware that the possible positions of points representing the complex number z is an ellipse. Do not draw a circle.
(b)	 <p>From the argand diagram, we know that there are 3 possible positions for z_2.</p> <ol style="list-style-type: none"> (1) $\arg(z_1 + z_2) = \frac{\pi}{2}$ (2) $\arg(z_1 + z_2)$ is undefined as $z_1 = -z_2$ (3) $\arg(z_1 + z_2) = 0$ <p>$\therefore \arg(z_1 + z_2)$ is either undefined, 0 or $\frac{\pi}{2}$.</p>	<p>Concepts:</p> <ol style="list-style-type: none"> 1. z means the distance of the point representing z from the origin. 2. Since $z_1 = z_2$ and both points representing the complex numbers must be on the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$, If (a,b) represents z_1, then z_2 must be either of these 3 options: <ul style="list-style-type: none"> • (a, -b) • (-a, b) • (-a, -b) 3. View $z_1 + z_2$ as a vector addition, then find its argument. 4. $\arg(0)$ is undefined.
(c)	<p>From the diagram, if z_2 is at position (1), α will not be real.</p> $(z - z_1)(z - z_2) = 0 \Rightarrow z^2 - z(z_1 + z_2) + z_1 z_2$ <p>Compare with $z^2 + \alpha z + \beta = 0$, coefficient of the z term is $\alpha = -(z_1 + z_2)$.</p>	<p>Concepts:</p> <ol style="list-style-type: none"> 1. Write linear factors to get the coefficient of z. 2. Check the coefficient for each of the 3 options to support your explanation.

	<p>If $z_1 = a + ib$, then $z_2 = -a + ib$ $\alpha = -(z_1 + z_2) = -i2b$</p> <p>Hence, α need not be real.</p>	
(d)	$ z_1 = \frac{\sqrt{26}}{2}$ $(2 \cos t)^2 + (3 \sin t)^2 = \left(\frac{\sqrt{26}}{2}\right)^2 = \frac{13}{2}$ $4 \cos^2 t + 9 \sin^2 t = \frac{13}{2}$ $4(1 - \sin^2 t) + 9 \sin^2 t = \frac{13}{2}$ $\sin^2 t = \frac{1}{2}$ $t = \frac{\pi}{4} \quad (\because z_1 \text{ is in 1st quad.)}$ <p>Alternatively,</p> $ z_1 = \sqrt{a^2 + b^2} = \frac{\sqrt{26}}{2} \Rightarrow a^2 + b^2 = \frac{13}{2}$ <p>Since pt representing z_1 is on the ellipse.</p> $\left(\frac{a}{2}\right)^2 + \left(\frac{b}{3}\right)^2 = 1$ <p>Solving the 2 equations, $a = \sqrt{2}$, $b = \frac{3}{2}\sqrt{2}$ $(z_1 \text{ is in 1st quadrant, so } a \text{ and } b \text{ must be +ve)}$</p> $z_1 = \sqrt{2} + \left(\frac{3}{2}\sqrt{2}\right)i, \quad z_2 = -\sqrt{2} + \left(\frac{3}{2}\sqrt{2}\right)i$ <p>From (c), coefficients : $\alpha = -(z_1 + z_2)$ and $\beta = z_1 z_2$</p> $\alpha = -3\sqrt{2}i$ $\beta = \left(\sqrt{2} + \left(\frac{3}{2}\sqrt{2}\right)i\right)\left(-\sqrt{2} + \left(\frac{3}{2}\sqrt{2}\right)i\right)$ $= 2\left(1 + \frac{3}{2}i\right)\left(-1 + \frac{3}{2}i\right)$ $= 2\left(-1 - \frac{9}{4}\right) = -\frac{13}{2}$	<p>Concepts:</p> <ol style="list-style-type: none"> $z = \sqrt{x^2 + y^2}$ Form 1 equation using parametric to solve for t. From (b), you have an idea that z_2 must come from the 2nd quadrant. Use symmetry of ellipse to get z_2 from z_1. Use z_2 and z_1 to form linear factors to get the value of α and β.
		Total marks: 11

Qn	Suggested Solution	Comments
9(a)	<p>Shortest distance = $\frac{\left 5 - \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right }{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{7}{3}$ units</p> <p>Alternative By observation, $(0, -5, 0)$ is a point on the plane.</p> <p>Shortest distance = $\frac{\left \begin{bmatrix} \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \end{bmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right }{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{7}{3}$ units</p> <p>Alternative Let foot of perpendicular be F. Since F is on the line passing through $(1, -2, 4)$ and perpendicular to the π_1,</p> $\vec{OF} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ <p>Solving simultaneously,</p> $\left[\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 5$ $\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 5$ $(2 + 2 + 8) + \mu(4 + 1 + 4) = 5$ $\mu = -\frac{7}{9}$ <p>Shortest distance = $\left \left[\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \frac{7}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right] - \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \right$</p> $= \left -\frac{7}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right $ $= \frac{7}{9} \sqrt{9}$ $= \frac{7}{3} \text{ units}$	<p>Formula</p> <p>Deduce any point on the plane (which satisfy $2x - y + 2z = 5$, then use length of projection onto the normal vector of π_1.</p> <p>This method is not efficient in finding shortest distance between a point and a line.</p> <p>However, if in the case you need to find the coordinates of the foot of perpendicular, you should learn how to do this more efficiently by grouping the terms.</p> <p>Note that there is no need to evaluate $\left[\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \frac{7}{9} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right]$.</p>

<p>(b)</p>	$2x - y + 2z = 5$ $x + 3y - az = 3$ <p>Direction vector of l:</p> $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -a \end{pmatrix} = \begin{pmatrix} a-6 \\ 2a+2 \\ 7 \end{pmatrix}$ <p>Let $z = 0$,</p> $2x - y = 5 \text{ --- (1)}$ $x + 3y = 3 \text{ --- (2)}$ $x = \frac{18}{7}, y = \frac{1}{7} \text{ (Using GC)}$ $\therefore l: \mathbf{r} = \begin{pmatrix} \frac{18}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a-6 \\ 2a+2 \\ 7 \end{pmatrix}, \lambda \in \mathbb{R}$ <p>Alternative</p> $2x - y + 2z = 5 \text{ --- (1)}$ $x + 3y - az = 3 \text{ --- (2)}$ $2 \times (2) - 1: 7y - (2a+2)z = 1$ $y = \frac{1}{7} + \frac{2a+2}{7}z$ $(2) + 3 \times (1): 7x + (-a+6)z = 18$ $x = \frac{18}{7} + \frac{a-6}{7}z$ <p>Let $z = \lambda'$,</p> $l: \mathbf{r} = \begin{pmatrix} \frac{18}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} + \lambda' \begin{pmatrix} \frac{a-6}{7} \\ \frac{2a+2}{7} \\ 1 \end{pmatrix}, \lambda' \in \mathbb{R}$ $\therefore l: \mathbf{r} = \begin{pmatrix} \frac{18}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a-6 \\ 2a+2 \\ 7 \end{pmatrix}, \lambda \in \mathbb{R} \quad (\lambda = \frac{1}{7} \lambda')$	 <p>l is a common line on both planes. This means that $l \perp \mathbf{n}_1$ and $l \perp \mathbf{n}_2$.</p> <p>Thus l is parallel to $(\mathbf{n}_1 \times \mathbf{n}_2)$.</p> <p>To find a point on the line, the easiest way is to let one of the coordinates be zero so that we can solve 2 unknowns with 2 equations. Since the unknown a is the coefficient of z, we will find the common point of the two planes where its z-coordinate = 0.</p> <p>To solve this algebraically, we need to make x and y in terms of z respectively.</p> <p>Then the numbers independent of z will give the point on the plane, and the coefficients of z will give the direction vectors of the line of intersection.</p> <p>You may also make y and z in terms of x, or make x and z in terms of y.</p>
<p>(c)</p>	<p>For l to be parallel to π_3,</p> $\begin{pmatrix} a-6 \\ 2a+2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} b \\ -2 \\ 4 \end{pmatrix} = 0$ $ab - 6b - 4a - 4 + 28 = 0$ $ab - 6b - 4a + 24 = 0$ $b(a-6) - 4(a-6) = 0$ $(a-6)(b-4) = 0 \text{ (shown)}$	<p>When a line is parallel to a plane, its direction vector is perpendicular to the normal vector of the plane.</p> 

(d)

$$\pi_1: \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 5; \quad \pi_2: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ -a \end{pmatrix} = 3; \quad \pi_3: \mathbf{r} \cdot \begin{pmatrix} b \\ -2 \\ 4 \end{pmatrix} = 3$$

For the three planes form a triangular prism,

(1) The line of intersection between π_1 and π_2 , l , is parallel to $\pi_3 \Rightarrow a=6$ or $b=4$ (from (c))

(2) l is not on π_3 regardless of b .

$$\begin{pmatrix} \frac{18}{7} \\ \frac{1}{7} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ -2 \\ 4 \end{pmatrix} \neq 3$$

$$b\left(\frac{18}{7}\right) - 2\left(\frac{1}{7}\right) + 4(0) \neq 3 \Rightarrow b \neq \frac{23}{18}$$

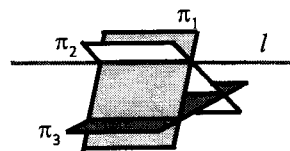
(3) π_1 is not parallel to π_3

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \neq m \begin{pmatrix} b \\ -2 \\ 4 \end{pmatrix} \Rightarrow m \neq \frac{1}{2} \Rightarrow b \neq 4$$

(4) π_2 is not parallel to π_3

$$\begin{pmatrix} 1 \\ 3 \\ -a \end{pmatrix} \neq k \begin{pmatrix} b \\ -2 \\ 4 \end{pmatrix} \Rightarrow k \neq -\frac{3}{2} \Rightarrow b \neq -\frac{2}{3} \text{ or } a \neq 6$$

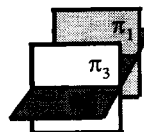
$$a = 6 \text{ and } b \in \mathbb{R} \setminus \left\{ -\frac{2}{3}, \frac{23}{18}, 4 \right\}$$



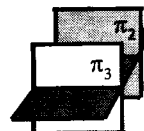
The condition “or” means that we only need to make sure **either** $a=6$ **or** $b=4$, **not both**.

To ensure this, we must make sure any point on l is not a point on π_3 , i.e. does not satisfy the equation of π_3 .

To exclude this case:

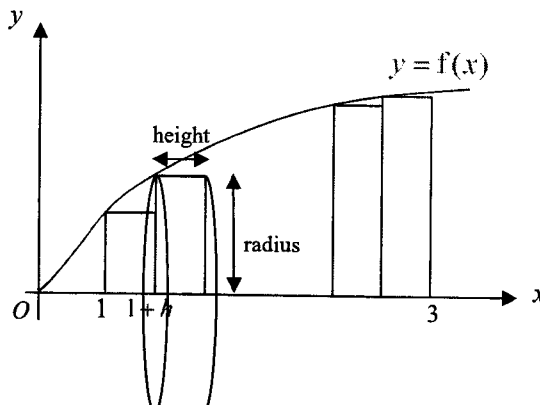
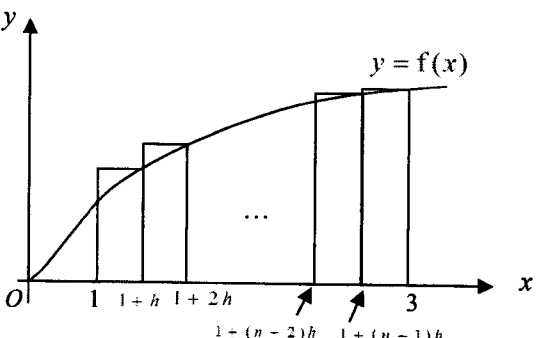


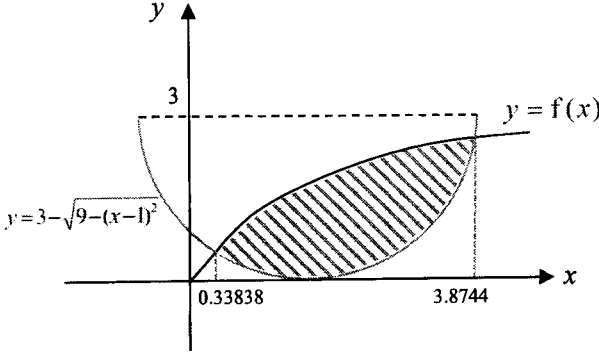
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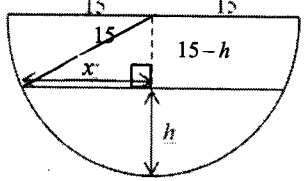
The condition “or” means that we only need to make sure **either** $b \neq -\frac{2}{3}$ **or** $a \neq 6$, **not both**.

Total marks: 12

Qn	Suggested Solution	Comments
10(a)	 $V = \pi h [f(1)]^2 + \pi h [f(1+h)]^2 + \dots + \pi h [f(1+(n-1)h)]^2$ $= \pi h \left([f(1)]^2 + [f(1+h)]^2 + \dots + [f(1+(n-1)h)]^2 \right)$ $= \pi h \sum_{r=0}^{n-1} [f(1+rh)]^2 \quad (\text{shown})$ $h = \frac{3-1}{n} = \frac{2}{n}$	<ul style="list-style-type: none"> • Total volume of n cylindrical discs = $\sum [\pi(\text{radius})^2(\text{height})]$. In this qn, radius = $f(x)$, height = h. It's incorrect to use sum area of n rectangles to become sum volume of n cylindrical discs. • As this is a "show" question, working must be clearly presented to get full credit. • Read the question carefully, as many students seem to have forgotten to find h.
(b)	$V_2 = \pi h [f(1+h)]^2 + \pi h [f(1+2h)]^2 + \dots + \pi h [f(1+nh)]^2$ $= \pi h \left([f(1+h)]^2 + [f(1+2h)]^2 + \dots + [f(1+nh)]^2 \right)$ $= \pi h \sum_{r=1}^n [f(1+rh)]^2$ 	
(c)	<p>From GC:</p> $\lim_{n \rightarrow \infty} V_1 = V = \pi \int_1^3 \left(\frac{x^2}{\sqrt{1+e^x}} \right)^2 dx = 12.291 = 12.3 \text{ unit}^3 \quad (3 \text{ sf})$	<ul style="list-style-type: none"> • In general, volume of revolution = limiting sum of volume of cylindrical discs, $\lim_{n \rightarrow \infty} V_1 = \pi \int y^2 dx$. • Use GC to evaluate the numerical volume. The question didn't ask for "exact" volume, so no need to perform tedious integration. The low marks given indicate the effort/time needed.

<p>(d)</p>	 <p> $(x-1)^2 + (y-3)^2 = 9.$ $y = 3 \pm \sqrt{9 - (x-1)^2}$ The required eqn is $y = 3 - \sqrt{9 - (x-1)^2}$ </p> <p>From the GC, points of intersection are $x = 0.33838$ or 3.8744</p> $\text{Area} = \int_{0.33838}^{3.8744} \left(\frac{x^2}{\sqrt{1+e^x}} \right) - \left(3 - \sqrt{9 - (x-1)^2} \right) dx$ $= 3.01 \text{ unit}^2 \quad (3 \text{ sf})$	<ul style="list-style-type: none"> In general, when doing area/volume questions, sketch a diagram first to see the relationship between the curves involved. Once the required region is identified, proceed to find other relevant info e.g. intersection points in this case. Use 5sf intermediate answers for better accuracy before presenting in 3sf for the final answer. $f(x)$ is defined for $x \geq 0$ so we should focus on the correct region. The question asked for the “area bounded” by the two curves only. Hence other “non-relevant” areas should not be included e.g. $\int_0^{0.33838} \left(3 - \sqrt{9 - (x-1)^2} \right) - (f(x)) dx$ is bounded by the two curves + y-axis.
		Total marks: 10

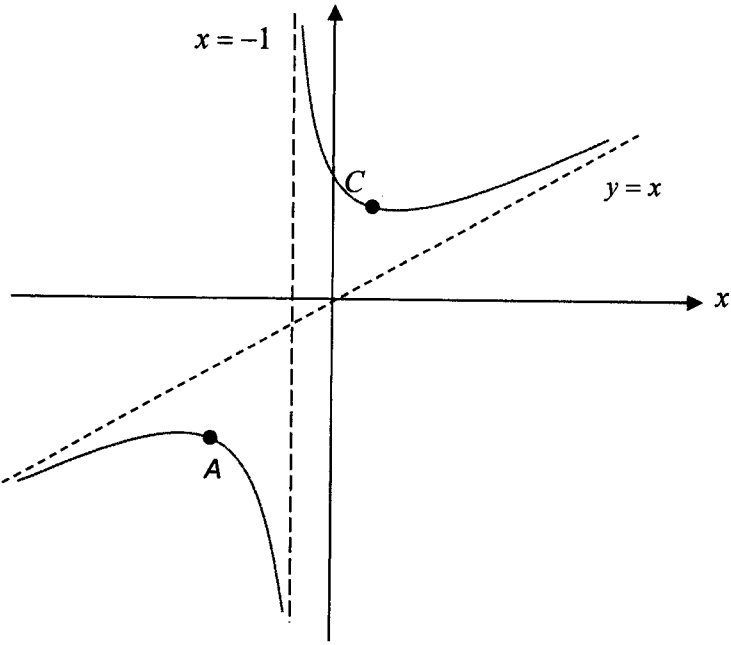
Qn	Suggested Solutions	Comments
11(a)	$k = 2\pi r^2(30) + 2\pi rL(10) + \pi r^2(20)$ $k = 80\pi r^2 + 20\pi rL$ $k = 20\pi r(4r + L)$ $\Rightarrow L = \frac{k}{20\pi r} - 4r$ $V = \frac{2}{3}\pi r^3 + \pi r^2 L$ $= \frac{2}{3}\pi r^3 + \pi r^2 \left(\frac{k}{20\pi r} - 4r \right)$ $= \frac{2}{3}\pi r^3 + \frac{kr}{20} - 4\pi r^3$ $= \frac{k}{20}r - \frac{10}{3}\pi r^3 \text{ (shown)}$	<ul style="list-style-type: none"> This is a ‘show’ question so clear steps must be shown as the answer is already given. Students need to expand and simplify $\pi r^2 \left(\frac{k}{20\pi r} - 4r \right)$ by expanding to get $\frac{kr}{20} - 4\pi r^3$. Note that k is a fixed constant.

<p>(b)</p>	<p>For maximum V as r varies,</p> $\frac{dV}{dr} = 0$ $\frac{dV}{dr} = \frac{k}{20} - 10\pi r^2$ $r = \sqrt{\frac{k}{200\pi}} \quad (\text{reject } r < 0)$ $\frac{d^2V}{dr^2} = -20\pi r < 0$ <p>Therefore, V is maximum.</p> <p>Cost of material for the hemispherical base</p> $= 2\pi r^2 (30) = 2\pi \left(\sqrt{\frac{k}{200\pi}} \right)^2 (30) = \frac{3k}{10} \text{ dollars}$	<ul style="list-style-type: none"> • Writing 'it is a maximum' is vague, students need to write V is a maximum instead. • Cost of the material is requested so we need to multiply 30 to $2\pi \left(\sqrt{\frac{k}{200\pi}} \right)^2$.
<p>(c)(i)</p>	 <p>By Pythagoras' Theorem,</p> $x^2 = 15^2 - (15-h)^2$ $x^2 = (15 - (15-h))(15 + (15-h))$ $x^2 = h(30-h)$ $W = \frac{\pi h(3x^2 + h^2)}{6} = \frac{\pi h(3(h(30-h)) + h^2)}{6}$ $W = \frac{\pi h(90h - 2h^2)}{6} = \frac{\pi(45h^2 - h^3)}{3}$ $\frac{dW}{dh} = \frac{\pi(90h - 3h^2)}{3}$ $\frac{dW}{dt} = \frac{dW}{dh} \times \frac{dh}{dt} = \frac{\pi(90h - 3h^2)}{3} \times \frac{dh}{dt}$ <p>At $h = 3$,</p> $\frac{dh}{dt} = \frac{\frac{dW}{dt}}{\frac{dW}{dh}} = \frac{-20}{\frac{\pi(90(3) - 3(3)^2)}{3}}$ $= -0.078595 = -0.0786 \text{ cm/min (3 s.f.)}$	<ul style="list-style-type: none"> • Key idea is to form a right-angle triangle by finding the height, $15-h$ and apply pythagoras theorem to link up x and h. • We are finding the rate of change related to the water in the tank and the question has already denoted it by W. Don't use another notation. • As a general guide: what is given and what do you need to find? Thereafter, form the connected rate of change equation: $\frac{dh}{dt}$ (find) $= \frac{dW}{dt}$ (given) \times ? (to form) $\therefore ? = \frac{dh}{dW}$ • Also, $\frac{dW}{dt}$ is negative 20 as the question states water is leaking \Rightarrow volume of water is decreasing as time increases. • It is easier to differentiate W wrt to h than wrt x.

(c)(ii)	$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ $x^2 = h(30-h) \Rightarrow x = \sqrt{h(30-h)} = \sqrt{30h-h^2} \quad (x > 0)$ $\frac{dx}{dh} = \frac{30-2h}{2\sqrt{30h-h^2}} = \frac{15-h}{\sqrt{30h-h^2}}$ <p>At $h=3$,</p> $\frac{dx}{dt} = \left(\frac{15-3}{\sqrt{30(3)-(3)^2}} \right) \times (-0.078595)$ $= -0.10479 = -0.105 \text{ cm/min (3sf)}$	<ul style="list-style-type: none"> Some students use an alternative method to differentiate W instead but it is more tedious. We cannot sub. in $h=3$ into $W = \frac{\pi h(3x^2+h^2)}{6}$ then differentiate $W = \frac{3\pi(3x^2+9)}{6}$ wrt x. Doing so, we are treating h as a constant which is not true. ❖ Always differentiate first then sub. the given value.
		Total marks: 12

2025 Year 6 H2 Math Prelim Exam P2
Suggested Solutions

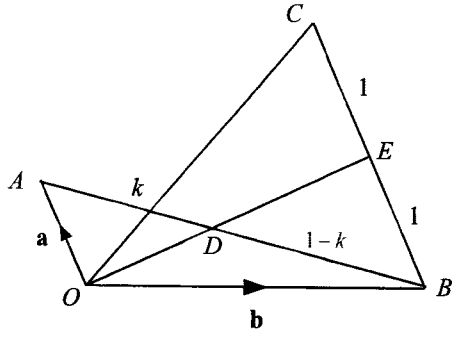
Section A: Pure Mathematics [40 marks]

Qn	Suggested Solutions	Comments
1(a)	$y = x + \frac{9}{x+1}$ $\frac{dy}{dx} = 1 - \frac{9}{(x+1)^2}$ $\frac{d^2y}{dx^2} = \frac{18}{(x+1)^3}$ <p>C concaves upwards $\Rightarrow \frac{d^2y}{dx^2} = \frac{18}{(x+1)^3} > 0$</p> $(x+1)^3 > 0 \Rightarrow x > -1$	<p>When a function $y = f(x)$ concaves upwards, the gradient of its gradient function is positive; i.e. $\frac{d^2y}{dx^2} > 0$.</p>
(b)	 <p>Stationary points: $A(-4, -7)$, $B(2, 5)$ y-intercept: $C(0, 9)$</p>	<p>For $y = x + \frac{a}{x+b}$, where a & b are constants, there will be a vertical asymptote ($x = -b$) & an oblique asymptote ($y = x$).</p> <p>For any axial intercept(s), they can be found by letting x & y be zero respectively.</p> <p>For any stationary point(s), they can be found from $\frac{dy}{dx} = 0$ or using the GC.</p>
		Total marks: 5

Qn	Suggested Solution	Comments
2(a)	$\int e^{2x} \tan^{-1}(e^x) dx$ $= \int t^2 \tan^{-1}(t) \frac{dt}{t}$ $= \int t \tan^{-1}(t) dt \text{ (shown)}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $t = e^x \Rightarrow dt = e^x dx$ $\therefore dx = \frac{dt}{e^x} = \frac{dt}{t}$ </div>	<ul style="list-style-type: none"> • Clear working needs to be shown to get full credit.
(b)	$\int t \tan^{-1}(t) dt$ $= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \frac{t^2}{1+t^2} dt$ $= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2} \right) dt$ $= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} \int \left(1 - \frac{1}{1+t^2} \right) dt$ $= \frac{t^2}{2} \tan^{-1} t - \frac{1}{2} (t - \tan^{-1} t) + C$ $= \frac{\tan^{-1} t}{2} (t^2 + 1) - \frac{t}{2} + C$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $u = \tan^{-1} t \quad \frac{dv}{dt} = t$ $\frac{du}{dt} = \frac{1}{1+t^2} \quad v = \frac{t^2}{2}$ </div> $\int_0^1 e^{2x} \tan^{-1}(e^x) dx$ $= \left[\frac{\tan^{-1} t}{2} (t^2 + 1) - \frac{t}{2} \right]_1^e$ $= \left(\frac{\tan^{-1} e}{2} (e^2 + 1) - \frac{e}{2} \right) - \left(\tan^{-1} 1 - \frac{1}{2} \right)$ $= \frac{\tan^{-1} e}{2} (e^2 + 1) - \frac{e}{2} - \frac{\pi}{4} + \frac{1}{2}$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $x = 0, t = e^0 = 1$ $x = 1, t = e$ </div>	<ul style="list-style-type: none"> • Use LIATE to decide on the choice of term to differentiate. • $\frac{t^2}{1+t^2}$ can be made proper by splitting the numerator or by long div • Note that $\tan^{-1} 1 = \frac{\pi}{4}$
		Total marks: 7

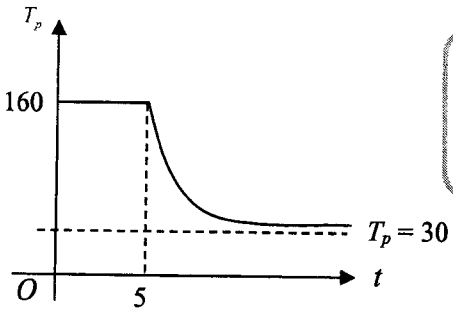
	Suggested Solution	Comments
3(a)	$x^3 + y^3 - 5xy = 0$ Differentiate wrt x : $3x^2 + 3y^2 \frac{dy}{dx} - 5 \left(x \frac{dy}{dx} + y \right) = 0$ $(3y^2 - 5x) \frac{dy}{dx} = 5y - 3x^2$ $\frac{dy}{dx} = \frac{5y - 3x^2}{3y^2 - 5x}$	Note that product rule have to be applied $\frac{d}{dx}(xy) = x \frac{dy}{dx} + y(1)$
(b)	<p>Gradient of normal =</p> $\frac{3y^2 - 5x}{5y - 3x^2}$ <p>For normal to be parallel to the y-axis, gradient of normal is undefined. Hence, $5y - 3x^2 = 0$</p> $y = \frac{3}{5}x^2 \text{ ---- (1)}$ <p>Subst (1) into equation of curve,</p> $x^3 + \left(\frac{3}{5}x^2\right)^3 - 5x\left(\frac{3}{5}x^2\right) = 0$ $\frac{27}{125}x^6 - 2x^3 = 0$ <p>Using GC, $x = 2.0999 = 2.10$ (3s.f) or $x = 0$ (rejected $\because x > 0$) $y = 2.6457 = 2.65$ (3s.f) Coordinates of the point is (2.10, 2.65).</p> <p>Alternative (if exact answers needed)</p> $x^3 + \frac{27}{125}x^6 - 3x^3 = 0$ $\frac{27}{125}x^6 - 2x^3 = 0$ $\frac{x^3}{125}(27x^3 - 250) = 0$ <p>Since $x > 0$, $x = \frac{\sqrt[3]{250}}{3} = \frac{5}{3}\sqrt[3]{2}$.</p> $y = \frac{3}{5} \left(\frac{\sqrt[3]{250}}{3} \right)^2 = \frac{250^{\frac{2}{3}}}{15} = \frac{5}{3}\sqrt[3]{4}$	<p>Some students had the misconception of $3y^2 - 5x = 0$ but notice gradient of normal would be 0 not undefined</p> <p>Since the question do not require exact answer, students can solve the equation using GC directly</p>

	Coordinates of the point is $\left(\frac{5\sqrt[3]{2}}{3}, \frac{5\sqrt[3]{4}}{3}\right)$.	
(c)	<p>For stationary point, $\frac{dy}{dx} = 0$</p> <p>From part (a), $3x^2 + 3y^2 \frac{dy}{dx} - 5\left(x \frac{dy}{dx} + y\right) = 0$.</p> <p>Differentiate wrt x :</p> <div style="text-align: center; border: 1px solid black; padding: 2px; width: fit-content; margin: 0 auto;">Apply product rule</div> $6x + 3y^2 \frac{d^2y}{dx^2} + 6y \left(\frac{dy}{dx}\right)^2 - 5\left(x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}\right) = 0$ $6x + 3y^2 \frac{d^2y}{dx^2} + 6y \left(\frac{dy}{dx}\right)^2 - 5x \frac{d^2y}{dx^2} - 10 \frac{dy}{dx} = 0$ <p>At the stationary point $x = \frac{5\sqrt[3]{2}}{3}$, $y = \frac{5\sqrt[3]{4}}{3}$, $\frac{dy}{dx} = 0$.</p> $\frac{d^2y}{dx^2} = \frac{-6x}{3y^2 - 5x} = -1.2 < 0$ <p>Hence, maximum point.</p>	<p>Note that finding 2nd derivative is much more efficient than 1st derivative method</p> <p>Note</p> $\frac{d}{dx}(3y^2) = 6y \frac{dy}{dx}$ <p>Ensure clear working is shown (substitute values of x, y and $\frac{dy}{dx}$ OR calculate correct value of 2nd derivative) to determine the sign of 2nd derivative and hence nature of stationary point</p> <p style="text-align: right;">Total marks: 7</p>

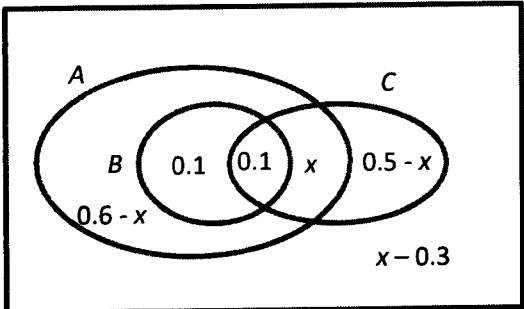
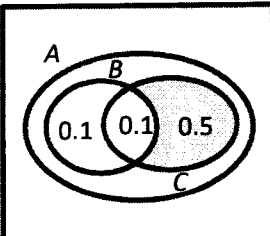
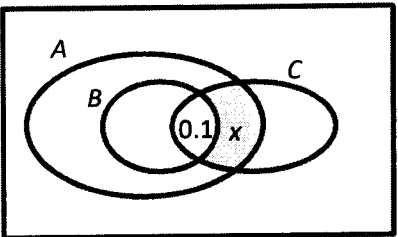
Qn	Suggested Solution	Comments
		
<p>4(a)</p>	<p>Let $\overline{AD} = k\overline{AB}$.</p> $\overline{OD} = \frac{k\mathbf{b} + (1-k)\mathbf{a}}{k + (1-k)} = k\mathbf{b} + (1-k)\mathbf{a}, \quad k \in \mathbb{R}$ $l_{OD} : \mathbf{r} = \lambda(k\mathbf{b} + (1-k)\mathbf{a}), \quad \lambda \in \mathbb{R}$	<ul style="list-style-type: none"> Qn is asking for vector eqn of line OD. Line OD \neq \overline{OD}. Eqn of line OD is of the form: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where \mathbf{a} is the position vector of a fixed point on the line (choose vector O or \overline{OD}) and \mathbf{b} is the direction vector of the line (choose \overline{OD}).
<p>(b)</p>	$\overline{OE} = \frac{1}{2}(\overline{OC} + \overline{OB})$ $= \frac{1}{2}(3\mathbf{a} + \mathbf{b} + \mathbf{b})$ $= \frac{1}{2}(3\mathbf{a} + 2\mathbf{b})$ <p>Since O, D and E are collinear, point E lies on l_{OD}.</p> <p>Method 1</p> $\overline{OE} = \lambda(k\mathbf{b} + (1-k)\mathbf{a})$ $\frac{1}{2}(3\mathbf{a} + 2\mathbf{b}) = \lambda(k\mathbf{b} + (1-k)\mathbf{a})$ $\frac{3}{2}\mathbf{a} + \mathbf{b} = \lambda(1-k)\mathbf{a} + \lambda k\mathbf{b}$ <p>Since \mathbf{a} and \mathbf{b} are non-zero and non-parallel vectors,</p> $\lambda k = 1 \quad \dots (1)$ $\lambda(1-k) = \frac{3}{2} \Rightarrow \lambda - \lambda k = \frac{3}{2} \quad \dots (2)$ <p>Subst (1) into (2): $\lambda - 1 = \frac{3}{2} \Rightarrow \lambda = \frac{5}{2}$</p> <p>From (1): $\lambda k = 1 \Rightarrow k = \frac{2}{5}$</p> <p>Method 2</p> <p>Let $\overline{OD} = \mu\overline{OE}$</p> $k\mathbf{b} + (1-k)\mathbf{a} = \frac{\mu}{2}(3\mathbf{a} + 2\mathbf{b}) = \frac{3\mu}{2}\mathbf{a} + \mu\mathbf{b}$	<ul style="list-style-type: none"> $\overline{OE} \neq \frac{1}{2}\overline{BC}$ as both \overline{OE} and \overline{BC} are not even parallel. $\overline{OE} = \frac{1}{2}(\overline{OC} + \overline{OB})$ using Ratio Thm

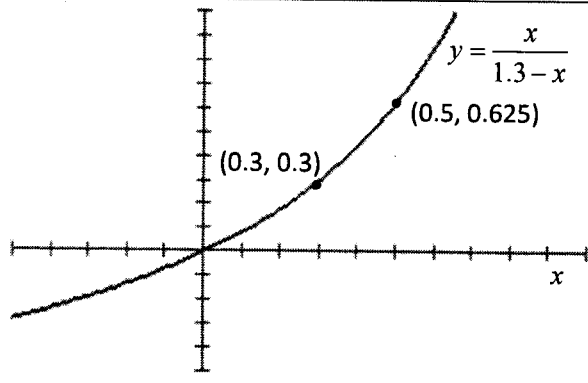
	<p>Since \mathbf{a} and \mathbf{b} are non-zero and non-parallel vectors, $k = \mu \dots (1)$ $1 - k = \frac{3\mu}{2} \dots (2)$ Subst (1) into (2): $\Rightarrow 1 - \mu = \frac{3\mu}{2} \Rightarrow \mu = \frac{2}{5}$ From (1): $k = \frac{2}{5}$</p>	
(c)	<p>$\mathbf{a} = 1, \mathbf{b} = 2$ and $3\mathbf{a} - 2\mathbf{b} = \sqrt{31}$.</p> $ 3\mathbf{a} - 2\mathbf{b} = \sqrt{31}$ $ 3\mathbf{a} - 2\mathbf{b} ^2 = 31$ $(3\mathbf{a} - 2\mathbf{b}) \cdot (3\mathbf{a} - 2\mathbf{b}) = 31$ $9\mathbf{a} \cdot \mathbf{a} - 12\mathbf{a} \cdot \mathbf{b} + 4\mathbf{b} \cdot \mathbf{b} = 31$ $9 \mathbf{a} ^2 - 12\mathbf{a} \cdot \mathbf{b} + 4 \mathbf{b} ^2 = 31$ $9(1)^2 - 12\mathbf{a} \cdot \mathbf{b} + 4(2)^2 = 31$ $\mathbf{a} \cdot \mathbf{b} = -\frac{6}{12} = -\frac{1}{2}$ <p>Let θ the angle between \mathbf{a} and \mathbf{b}.</p> $\therefore \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{ \mathbf{a} \mathbf{b} } = \frac{-\frac{1}{2}}{1 \times 2} = -\frac{1}{4}$ $\therefore \theta = 104.5^\circ \text{ (1.82 rad)}$	<ul style="list-style-type: none"> • Concept (derived from scalar product formula): $(3\mathbf{a} - 2\mathbf{b}) \cdot (3\mathbf{a} - 2\mathbf{b})$ $= 3\mathbf{a} - 2\mathbf{b} ^2$ $\neq (3\mathbf{a} - 2\mathbf{b})^2$ <p>or $(3\mathbf{a} - 2\mathbf{b})^2$</p> <div style="border: 1px solid black; padding: 5px; margin-top: 10px;"> <p>Qn did not ask for acute angle hence we cannot put modulus for $\mathbf{a} \cdot \mathbf{b}$</p> </div>
(d)	<p>$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \hat{\mathbf{a}}$ is the length of projection of \mathbf{b} onto \mathbf{a}. (note that $\hat{\mathbf{a}}$ is the unit vector)</p>	<ul style="list-style-type: none"> • Note that $\hat{\mathbf{a}}$ is the unit vector. So, it should be length of projection of \mathbf{b} onto $\hat{\mathbf{a}}$ and not the other way round.
Total marks: 11		

Qn	Suggested Solution	Comments
5(a)	$\frac{dT}{dt} = -k(T - 30)$ $\int \frac{1}{T - 30} dT = -k \int dt$ $\ln T - 30 = -kt + c$ $T - 30 = \pm e^c \cdot e^{-kt} = Ae^{-kt} \quad (A = \pm e^c)$ $T = 30 + Ae^{-kt}$ <p>Subst $t = 0, T = 160,$ $160 = 30 + A \Rightarrow A = 130$ $\therefore T = 30 + 130e^{-kt}$</p>	<div style="border: 1px solid black; padding: 5px; margin-bottom: 10px;"> <p>k is given as a positive constant. Since T is decreasing, a negative sign must be applied.</p> </div> <div style="border: 1px solid black; padding: 5px;"> <p>You will need to "remove" the modulus before applying the initial conditions</p> </div>

(b)	As the actual cooling starts 5 minutes later, we need to shift the cooling function 5 minutes to the right. Replace t with $t-5 \Rightarrow a = -5$.	
(c)	<p>When $t \geq 5$,</p> $T_p(t) = T(t-5) = 30 + 130e^{-k(t-5)}$ <p>Subst $t = 15$, $T_p = 100$</p> $100 = 30 + 130e^{-10k}$ $e^{-10k} = \frac{70}{130}$ $\therefore k = -\frac{1}{10} \ln \frac{7}{13} = 0.061904$ $T(t-5) = 30 + 130e^{-0.061904(t-5)}$ $= 30 + 177.16e^{-0.061904t}$ $= 30 + 177e^{-0.0619t} \text{ (shown)}$	<p>Alternatively, you can think of using part (a), can think of it as 10 mins has lapsed to find k.</p> <p>Subst $t = 10$, $T = 100$</p> $100 = 30 + 130e^{-10k}$ $e^{-10k} = \frac{70}{130}$ $\therefore k = -\frac{1}{10} \ln \frac{7}{13} = 0.061904$
(d)	$60 = 30 + 177.16e^{-0.061904t}$ $e^{-0.061904t} = \frac{30}{177.16}$ $t = 28.687 = 28.7 \text{ min}$	<p>Please be aware that you can do part (c) and (d) without attempting the earlier parts.</p>
(e)	 <p style="text-align: center;">If there is a horizontal asymptote, you will need draw and label it.</p>	<p style="text-align: right;">Total marks: 10</p>

Section B: Probability and Statistics [60 marks]

Qn	Suggested solutions	Comments
6(ai)	 <p>Note: $P(B \cap C) = P(A \cap B \cap C) = 0.1$ since $B \subset A$.</p> <p>Max x occurs when the part in C not in B (0.5) is entirely in A. Hence, max $x = 0.5$.</p> <p>Min x occurs when as much of the part in C not in B is outside of A as possible.</p> $P(A \cup C) \leq 1$ $P(A) + P(C) - P(A \cap C) \leq 1$ $0.8 + 0.6 - (0.1 + x) \leq 1$ $x \geq 0.3$ <p>Hence, min $x = 0.3$.</p> <p>From the diagram, min $x = 0.3$, max $x = 0.5$</p> <p>Alternative</p> $\left. \begin{array}{l} 0.6 - x \geq 0 \Rightarrow x \leq 0.6 \\ x - 0.3 \geq 0 \Rightarrow x \geq 0.3 \\ 0.5 - x \geq 0 \Rightarrow x \leq 0.5 \end{array} \right\} \xrightarrow{\text{Taking Intersection}} 0.3 \leq x \leq 0.5$ <p>Hence, min $x = 0.3$, max $x = 0.5$.</p>	<p>Venn diagrams are useful for this type of question.</p> <p>Maximum $P(A \cap B' \cap C)$:</p>  <p>Minimum $P(A \cap B' \cap C)$:</p> 
(ii)	$P(A \cap B' \cap C A \cup C) = \frac{P((A \cap B' \cap C) \cap (A \cup C))}{P(A \cup C)}$ $= \frac{P(A \cap B' \cap C)}{P(A \cup C)} = \frac{x}{1.3 - x}$	<p>Important</p> <ol style="list-style-type: none"> 1) Include brackets where appropriate 2) $(A \cap B' \cap C)$ is a subset of $(A \cup C)$ 3) $P((A \cap B' \cap C) \cap (A \cup C)) \neq P(A \cap B' \cap C) \times P(A \cup C)$ as they may not be independent 4) $P((A \cap B' \cap C) \cap (A \cup C)) \neq P(A \cap B' \cap C) \cap P(A \cup C)$ (Notation error)

	 <p>For $0.3 \leq x \leq 0.5 \Rightarrow 0.3 \leq P(A \cap B' \cap C A \cup C) \leq 0.625$ Min $P(A \cap B' \cap C A \cup C) = 0.3$ Max $P(A \cap B' \cap C A \cup C) = 0.625$</p> <p>Alternative</p> $P(A \cap B' \cap C A \cup C) = \frac{x}{1.3-x} = \frac{1.3}{1.3-x} - 1$ <p>For $0.3 \leq x \leq 0.5$, $\frac{1.3}{1.3-x} - 1$ is increasing.</p> Min $P(A \cap B' \cap C A \cup C) = \frac{0.3}{1.3-0.3} = 0.3$ Max $P(A \cap B' \cap C A \cup C) = \frac{0.5}{1.3-0.5} = 0.625$	<p>From the graph, we can find the maximum and minimum y value for $0.3 \leq x \leq 0.5$.</p> <p>Note that $(A \cap B' \cap C)$ is a subset of $(A \cup C)$</p> <p>Use reduced sample space to obtain the fraction. We can only just evaluate the situation at the end points as the function is increasing.</p>
(b)	$P(A \cap C) = P(A) \cdot P(C)$ $0.1 + x = (0.8)(0.6)$ $P(A \cap B' \cap C) = x = 0.38$	<p>Note that only A and C are independent events.</p>
Total marks: 7		

	Suggested Solution	Comments
7(a)	<p>There are 13 seats for the remaining 11 people to choose from.</p> $\text{Number of ways} = \binom{13}{11} 11! = 3113510400$ <p>Alternative</p> $\text{Number of ways} = \frac{13!}{2!} = 3113510400$	<p>It's a particular student; there is no need to choose the student. There is only one way to sit him/her. Arrangement is needed here.</p> <p>The reason for $2!$ is due to the 2 empty seats.</p>
(b)	<p>Stage 1: Seat the 2 particular students: $\binom{7}{2} \times 2!$</p>	<p>Note that there is a total of 7 non-aisle seats. We settle these 2</p>

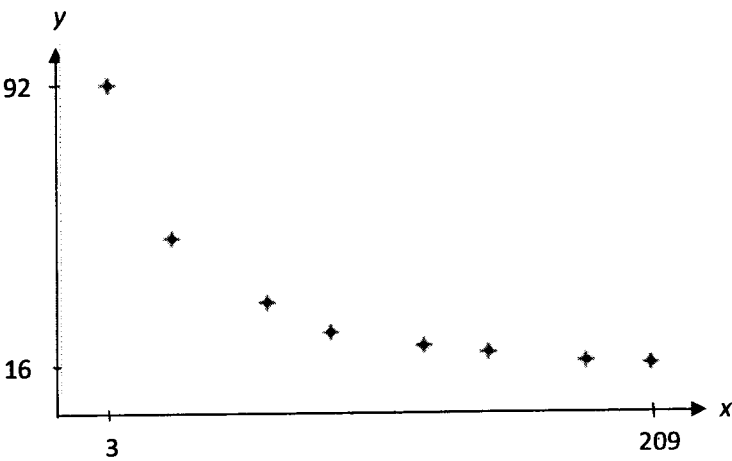
	<p>Stage 2: There are 12 remaining seats to be arranged for 10 students. $\binom{12}{10} \times 10!$ or $\frac{12!}{2}$</p> <p>Number of ways $= \binom{7}{2} \times 2! \times \frac{12!}{2} = 1.01 \times 10^{10}$ or 10059033600</p>	<p>'particular' students to deal with the restrictions.</p> <p>As we are doing the arrangement in stages, we need to use the multiplication principle.</p>																																																											
	<p>(c) The 5 from tennis must be in the middle segment: No of ways to arrange them = $6!$ or $\binom{6}{5} \times 5! = 720$</p> <p>For the bowling and softball students: Number of ways = $5 \times 4 \times 3! = 720$</p> <div style="display: flex; justify-content: space-around; border: 1px solid black; padding: 5px;"> Arrange the 4 from bowling Arrange the 3 from softball </div> <p>5: (from the perspective of bowling first)</p> <div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p>①</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>G</th><th>H</th><th>I</th></tr> </thead> <tbody> <tr><td>B</td><td>B</td><td>S</td></tr> <tr><td>B</td><td>S</td><td>S</td></tr> <tr><td>B</td><td>⊗</td><td>⊗</td></tr> </tbody> </table> </div> <div style="text-align: center;"> <p>②</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>G</th><th>H</th><th>I</th></tr> </thead> <tbody> <tr><td>B</td><td>B</td><td>B</td></tr> <tr><td>S</td><td>S</td><td>B</td></tr> <tr><td>S</td><td>⊗</td><td>⊗</td></tr> </tbody> </table> </div> </div> <div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p>③</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>G</th><th>H</th><th>I</th></tr> </thead> <tbody> <tr><td>S</td><td>B</td><td>B</td></tr> <tr><td>S</td><td>B</td><td>B</td></tr> <tr><td>S</td><td>⊗</td><td>⊗</td></tr> </tbody> </table> </div> <div style="text-align: center;"> <p>④</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>G</th><th>H</th><th>I</th></tr> </thead> <tbody> <tr><td>S</td><td>S</td><td>S</td></tr> <tr><td>B</td><td>B</td><td>B</td></tr> <tr><td>B</td><td>⊗</td><td>⊗</td></tr> </tbody> </table> </div> </div> <div style="text-align: center;"> <p>⑤</p> <table border="1" style="border-collapse: collapse; text-align: center;"> <thead> <tr><th>G</th><th>H</th><th>I</th></tr> </thead> <tbody> <tr><td>B</td><td>S</td><td>S</td></tr> <tr><td>B</td><td>B</td><td>S</td></tr> <tr><td>B</td><td>⊗</td><td>⊗</td></tr> </tbody> </table> </div>	G	H	I	B	B	S	B	S	S	B	⊗	⊗	G	H	I	B	B	B	S	S	B	S	⊗	⊗	G	H	I	S	B	B	S	B	B	S	⊗	⊗	G	H	I	S	S	S	B	B	B	B	⊗	⊗	G	H	I	B	S	S	B	B	S	B	⊗	⊗
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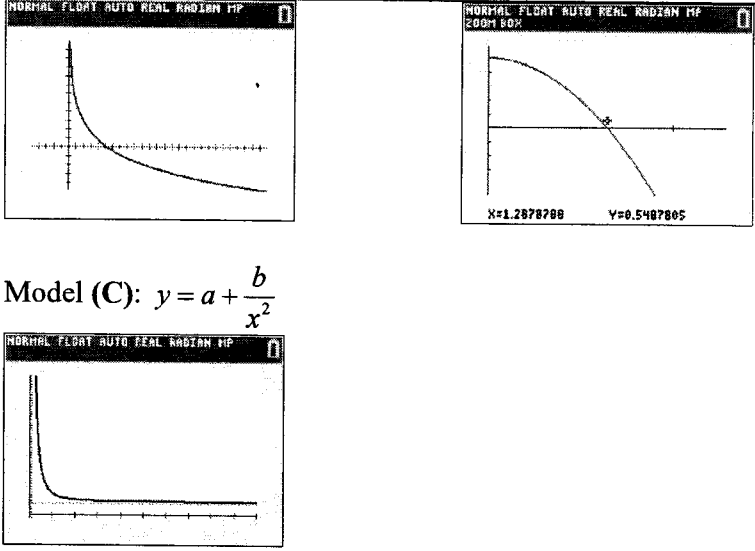
Qn	Suggested soln	Comments
8(a)	<p> $P(X = 2)$ $= P(2 \text{ diff. colours among the 4 discs})$ $= P(2 \text{ discs each of two diff. colours})$ $+ P(3 \text{ discs of one colour} + 1 \text{ disc of diff. colour})$ </p> <p> $= \frac{\binom{4}{2} \binom{6}{2} \binom{6}{2} + \binom{4}{2} \binom{6}{1} \binom{6}{3} \times 2}{\binom{24}{4}}$ </p> <p> $= \frac{465}{1771} \text{ (shown)}$ </p> <p>Alternative</p> <p> $P(X = 2)$ </p> <p> $= \binom{4}{2} \left[\frac{6}{24} \cdot \frac{5}{23} \cdot \frac{6}{22} \cdot \frac{5}{21} \cdot \frac{4!}{2!2!} \right]$ </p> <p> $+ \binom{4}{2} \binom{2}{1} \left[\frac{6}{24} \cdot \frac{5}{23} \cdot \frac{4}{22} \cdot \frac{6}{21} \cdot \frac{4!}{3!} \right]$ </p> <p> $= \frac{465}{1771} \text{ (shown)}$ </p>	<p>There are other ways to do this. Presented here are the recommended approaches as they are likely easier to understand by most.</p> <p>It is a good idea to indicate the type of cases that you are considering in a systemic, neat manner to make it easier to understand your approach.</p> <p>The combination approach is probably the most efficient.</p> <p>For those that uses the one-by-one probability approach (see alternative), you must remember to consider the different permutations that arises.</p>
(b)	<p> $P(X = 1) = P(4 \text{ discs all same colour})$ </p> <p> $= \frac{\binom{4}{1} \binom{6}{4}}{\binom{24}{4}}$ </p> <p> $= \frac{10}{1771}$ </p>	<p>Recall that the last probability can be obtained using 1 minus all the remaining probability values obtained. You do not need to directly compute the last probability. In this question, the case of $P(X = 3)$ is best done this way.</p> <p>A probability distribution is a listing of all the different $P(X = x)$ for each x</p>

<p> $P(X = 4) = P(\text{1 disc each of 4 diff. colours})$ $= \frac{\binom{6}{1}\binom{6}{1}\binom{6}{1}\binom{6}{1}}{\binom{24}{4}}$ $= \frac{216}{1771}$ $P(X = 3) = 1 - P(X = 1) - P(X = 2) - P(X = 4)$ $= \frac{1080}{1771}$ <p>or direct</p> $P(X = 3) = \frac{\binom{4}{3}\binom{3}{1}\binom{6}{2}\binom{6}{1}\binom{6}{1}}{\binom{24}{4}} = \frac{1080}{1771}$ <table border="1" data-bbox="319 940 1005 1052"> <thead> <tr> <th>x</th> <th>1</th> <th>2</th> <th>3</th> <th>4</th> </tr> </thead> <tbody> <tr> <td>$P(X = x)$</td> <td>$\frac{10}{1771}$</td> <td>$\frac{465}{1771}$</td> <td>$\frac{1080}{1771}$</td> <td>$\frac{216}{1771}$</td> </tr> </tbody> </table> <p>Alternative</p> $P(X = 1) = \binom{4}{1} \left[\frac{6}{24} \cdot \frac{5}{23} \cdot \frac{4}{22} \cdot \frac{3}{21} \right] = \frac{10}{1771}$ $P(X = 4) = \left[\frac{6}{24} \cdot \frac{6}{23} \cdot \frac{6}{22} \cdot \frac{6}{21} \cdot 4! \right] = \frac{216}{1771}$ $P(X = 3) = \binom{4}{3} \binom{3}{1} \left[\frac{6}{24} \cdot \frac{6}{23} \cdot \frac{6}{22} \cdot \frac{5}{21} \cdot \frac{4!}{2!} \right] = \frac{1080}{1771}$ </p>	x	1	2	3	4	$P(X = x)$	$\frac{10}{1771}$	$\frac{465}{1771}$	$\frac{1080}{1771}$	$\frac{216}{1771}$	<p>value. Do include the case of $P(X = 2)$ even if it is done in previous part.</p> <p>There is no such case for $X = 0$. X has to be at least be 1, the case where all the discs of 1 (same) colour.</p>
x	1	2	3	4							
$P(X = x)$	$\frac{10}{1771}$	$\frac{465}{1771}$	$\frac{1080}{1771}$	$\frac{216}{1771}$							
<p>(c) Let Y be the number of rounds, out of 8, with at most 2 different colour discs. i.e. where Joe wins \$10.</p> $Y \sim B(8, P(X \leq 2)) \text{ i.e. } Y \sim B\left(8, \frac{475}{1771}\right)$ <p>For Joe to win some money at the end, he must win \$10 in at least 3 rounds.</p> $P(Y \geq 3) = 1 - P(Y \leq 2)$ $= 0.3672845593$ $= 0.367 \text{ (to 3 s.f.)}$ <p>Alternative 1</p> <p>Let V be the number of rounds, out of 8, with more than 2 different colour discs. i.e. where Joe loses \$5.</p>	<p>Define any additional random variable(s) used clearly.</p> <p>DO NOT use X again which has already been used in the question.</p>										

	<p>$V \sim B(8, P(X > 2))$ i.e. $V \sim B\left(8, \frac{1296}{1771}\right)$</p> <p>For Joe to win some money at the end, he can at most lose 5 rounds.</p> <p>$P(V \leq 5) = 0.3672845593 = 0.367$ (to 3 s.f.)</p> <p>Alternative 2</p> <p>$P(X = 2) = \frac{475}{1771}$</p> <p>For Joe to win some money at the end, he must win \$10 in at least 3 rounds.</p> <p>$P(\text{Joe wins least 3 rounds})$</p> <p>$= 1 - P(\text{Joe wins 2 or fewer rounds})$</p> <p>$= 1 - \left[\left(\frac{1296}{1771}\right)^8 + \binom{8}{1} \left(\frac{475}{1771}\right) \left(\frac{1296}{1771}\right)^7 + \binom{8}{2} \left(\frac{475}{1771}\right)^2 \left(\frac{1296}{1771}\right)^6 \right]$</p> <p>$= 0.3672845593 = 0.367$ (to 3 s.f.)</p>	
(d)	<p>Expected "profit" from 1 round</p> <p>$= \left(10 \times \frac{475}{1771}\right) + \left(-5 \times \frac{1296}{1771}\right) = -\frac{1730}{1771} = -0.9768492377$</p> <p>As Joe is expected to lose \$0.98 per round, and each round is independent, he will be expected to lose money at the end of the game. It is not worthwhile for him to play the game as he will likely lose money.</p> <p>Alternative 1</p> <p>Expected "profit" from 8 rounds (1 game)</p> <p>$= \left[\left(10 \times \frac{475}{1771}\right) + \left(-5 \times \frac{1296}{1771}\right) \right] \times 8 = -7.814793902 = -7.81$</p> <p>Joe will be expected to lose \$7.81 in each complete game. It is not worthwhile for him to play the game as he will likely lose money.</p> <p>Alternative 2</p> <p>Using Y defined in (c),</p> <p>$E(Y) = 8 \left(\frac{475}{1771}\right) = \frac{3800}{1771} = 2.145680407 = 2.14 < 3$</p> <p>This means Joe is expected to win in less than 3 rounds in 1 game. Since he needs to win at least 3 rounds to win money, it is not worthwhile for him to play the game as he will likely lose money.</p> <p>Alternative 3</p>	<p>Need to account for both the probability of winning/losing AND the value of the win/lose.</p> <p>Cannot conclude by simply comparing probability values of X or computing the expectation of X only. (These does not account for the value of the win/lose.)</p>

	<p>Let W be the amount he wins in 1 game (8 rounds).</p> $W = 10Y - 5(8 - Y) = 15Y - 40$ $E(W) = E(15Y - 40) = 15E(Y) - 40$ $= 15 \left[8 \times \frac{475}{1771} \right] - 40$ $= -7.814793902 = -7.81$ <p>Joe will be expected to lose \$7.81 in each complete game. It is not worthwhile for him to play the game as he will likely lose money.</p>	
		Total marks: 10

Qn	Suggested Solution	Comments
9(a)	Scatter Diagram I: 0 Scatter Diagram II: 1 Scatter Diagram III: - 0.7	
(b)(i)	 <p>From the scatter diagram in (i), it can be observed that as x increases, y decreases at a decreasing rate. Thus, a linear model would not be appropriate.</p> <p>OR</p> <p>Inappropriate to be modelled by a linear model because the data points in the scatter diagram seem to fit a curve that is decreasing at a decreasing rate / decreasing and concave upwards, rather than a line.</p>	<p>For this scatter diagram, there must be exactly 8 points.</p> <p>The axes must be labelled with minimum & maximum values shown.</p> <p>The comment must be explained with reference to the scatter diagram.</p>
(ii)	Model (A): $y = a - b \ln x$ Model (B): $y = a - bx^2$	

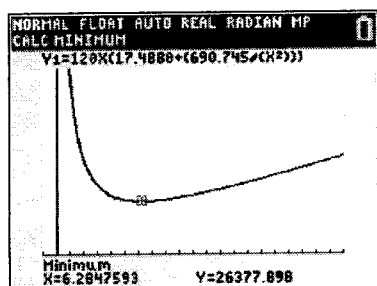
	 <p>Model (C): $y = a + \frac{b}{x^2}$</p> <p>From the scatter diagram in (b)(i), it can be observed that as x increases, y decreases at a decreasing rate which is consistent with Models (A) and (C). However, based on the context of the question, there should be a positive lower limit to the number of working days required for the renovation works to be completed even if infinitely many workers are hired. Model (A) predicts that y will decrease indefinitely, while Model (C) has a limit to the value y can take ($y = a$ is a horizontal asymptote for Model (C)).</p> <p>Thus Model (C) is the most appropriate.</p>	<p>The choice & explanation must take reference from the scatter diagram & the context of this question.</p> <p>Note that it's necessary to mention why C is the most appropriate; not just why A & B are not.</p>
<p>(iii)</p>	<p>Equation of regression line for Model (C):</p> $y = 17.4880 + \frac{690.745}{x^2} = 17.5 + \frac{691}{x^2}$ <p>$r = 0.992$ (to 3sf)</p>	<p>The final answer should be in 3 s.f. though the intermediate steps give more s.f.</p>
<p>(iv)</p>	$y = 17.4880 + \frac{690.745}{9^2} = 26.0 = 26 \text{ (to nearest whole no.)}$ <p>Since $x = 9$ is within the data range ($3 \leq x \leq 20$) and r-value of 0.992 (which is close to 1) indicates that the model used for the estimation is appropriate, the estimate is reliable.</p>	<p>The answer is to be rounded off (NOT up as this is an estimation based on the best-fit equation) to an integer.</p> <p>Both reasons must be given to justify its reliability.</p>
<p>(v)</p>	<p>Let T dollars be the total amount of workers' wages to be paid.</p> $T = 120xy = 120x \left(17.4880 + \frac{690.745}{x^2} \right)$ <p>Using GC Table:</p>	<p><i>Key Concept:</i> To minimise $120xy$ or xy</p>

NORMAL FLOAT AUTO REAL RADIAN MP					NORMAL FLOAT AUTO REAL RADIAN MP				
PRESS + FOR Δ Tbl					PRESS + FOR Δ Tbl				
X	Y2				X	Y2			
1	34988				10	29275			
2	45642				11	30620			
3	33926				12	32090			
4	29117				13	33657			
5	27074				14	35301			
6	26406				15	37004			
7	28531				16	38758			
8	27150				17	40551			
9	28097				18	42378			
10	29275				19	44235			
11	30620				20	46116			

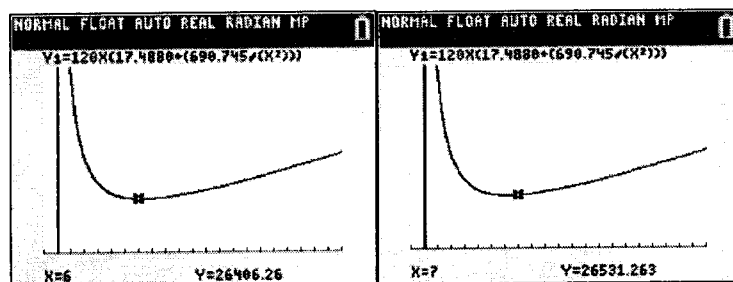
X=1 X=10

The developer should hire 6 workers to minimise the total amount of workers' wages to be paid out.

Alternative (Use Graph)

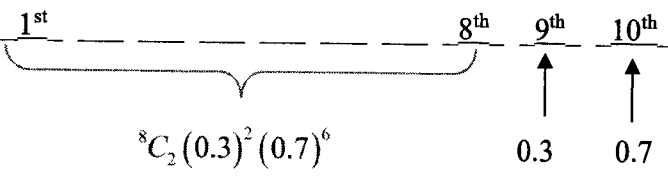
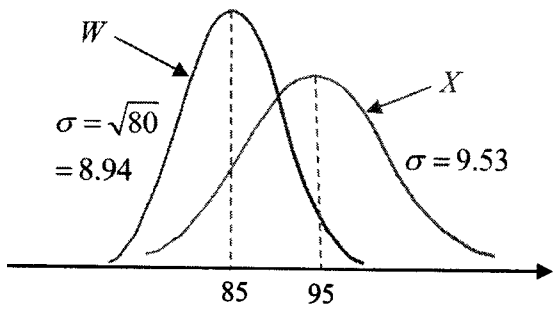


The minimum value should be either 6 or 7.



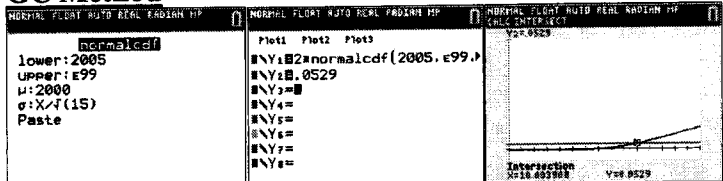
The developer should hire 6 workers to minimise the total amount of workers' wages to be paid out.

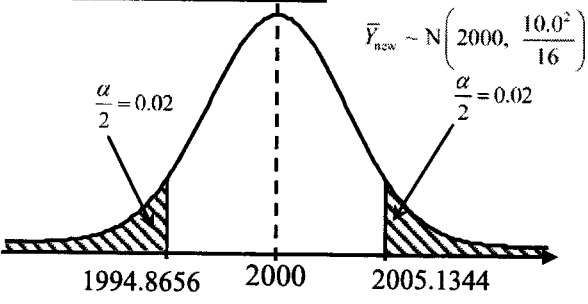
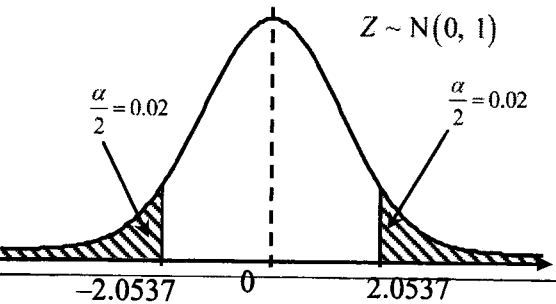
Total = 11 marks

Qn	Suggested Solution	Comments
10(a)	$X \sim N(95, \sigma^2)$ $P(X \leq 90) = 0.3$ $P\left(Z \leq \frac{90-95}{\sigma}\right) = 0.3$ $\frac{-5}{\sigma} = -0.52440$ $\therefore \sigma = 9.5347 = 9.53 \text{ (3 s.f.) (shown)}$	<ul style="list-style-type: none"> • Clear working needs to be shown to get full credit. Use of table is more useful to find unknown integers and not rational numbers like 9.53. Methods like standardising or graphical are preferred in this case.
(b)	9.15 – 11.00 pm = 105 min $P(X > 105) = 0.147 \text{ (3 s.f.)}$	
(c)	Prob that member adhere = $P(X \leq 90) = 0.3$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $2 \text{ members adhere} = (0.3)^2$ $6 \text{ members didn't adhere} = (0.7)^6$ $\text{Cases} = {}^8C_2$ </div>  <p>Required probability</p> $= {}^8C_2 (0.3)^2 (0.7)^6 (0.3)(0.7)$ $= 0.0623 \text{ (3 s.f.)}$	
(d)		These are the key features : <ul style="list-style-type: none"> • pop means 85, 95 • diff height $h_x < h_w$ • spread $\sigma_x > \sigma_w$ • bell shape (asymptotic feature) • label X & W clearly
(e)	Let W denote usage time of a trial member (in min). $W \sim N(85, 80)$ $E(X_1 + X_2 - 2W) = 2E(X) - 2E(W)$ $= 2(95) - 2(85) = 20$ $\text{Var}(X_1 + X_2 - 2W) = 2\text{Var}(X) + 4\text{Var}(W)$ $= 2(9.53^2) + 4(80) = 501.64$ $X_1 + X_2 - 2W \sim N(20, 501.64)$	

	$P(X_1 + X_2 - 2W \geq 15)$ $= 1 - P(-15 < X_1 + X_2 - 2W < 15)$ $= 0.647 \text{ (3 s.f.)}$	<ul style="list-style-type: none"> • “differs from..” \Rightarrow modulus
(f)	The usage time of all paying and trial members are independent of each other.	<ul style="list-style-type: none"> • Insufficient to say - usage time of paying members are independent of trial members • The “independence” applies for both “within” paying members & trial members also
		Total marks: 12

Qn	Suggested soln	Comments
11(a)	Using GC, unbiased estimate of the population variance, $s^2 = 11.07672684^2 = 122.6938776 \text{ g}^2 = 123 \text{ g}^2$ (3 s.f.) <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p>*Impt Note: unbiased estimate of the population variance is defined as s^2, we can only get an “estimate” from sample data; this is not the same as the actual population variance σ^2 thus, it is not right to say that $\sigma^2 = \dots = 123 \text{ g}^2$</p> <p>Similarly, $E(X) = \mu$ (population mean) $\neq \bar{x}$ (sample mean)</p> </div>	Students can make use of the GC (GC Lists) to do this – refer to Example 8(a)(ii) of Lecture Notes Chapter 18 Sampling Method 2-GC. Many students tried to compute s^2 on their own and could not get the right answer.
(b)	A one-tail (lower-tail) test should be conducted as he suspects that his machines now pack Regular-sized packets that are underweight OR A one-tail (lower-tail) test should be conducted as he suspects that his machines now pack Regular-sized packets which have <u>mean mass</u> less than the intended 1kg.	We are doing a test for “means”, thus if students mention “mass”, it should specifically be referred to as “ <u>mean mass</u> ”. Words should be used to explain why – not just math symbols without any definitions.
(c)	Let μ be the population mean mass of a Regular-sized packet of flour in g. $H_0: \mu = 1000$ $H_1: \mu < 1000$ Perform 1-tail test at 5% level of significance Under H_0 , $\bar{X} \sim N\left(1000, \frac{122.69}{50}\right)$ approximately by Central Limit Theorem, since $n = 50$ is large. $\bar{x} = 997.4$ (exact), $n = 50$ $p\text{-value} = 0.0484807326 = 0.0485$ (3 s.f.) Since $p\text{-value} < 0.05$, we reject H_0 . There is sufficient evidence at 5% level of significance to conclude that the <u>mean mass</u> of a Regular-sized packet of flour is less than 1000 g. or There is sufficient evidence at 5% level of significance to conclude that his machines now pack Regular-sized packets that are underweight.	μ was often not defined. Some students still made errors in writing the test statistic, $\bar{X} \sim N\left(1000, \frac{122.69}{50}\right),$ sometimes writing X instead of \bar{X} or using $\bar{x} = 997.4$ as the mean instead of 1000 or not dividing s^2 by n . Some students indicated the standardised test statistic instead, but this was often not well done. The correct standardised test statistic should be: $Z = \frac{\bar{X} - 1000}{\sqrt{\frac{122.69}{50}}} \sim N(0,1)$ Many who did this substituted the value of \bar{x} in and wrote

	<p>It is not necessary to assume that the mass of Regular-sized packets of flour, X, follow a normal distribution as the sample size of 50 is large enough for us to use Central Limit Theorem to approximate the distribution of the sample mean mass, \bar{X}, to a normal distribution.</p> <p>Students needed to state 3 main points on whether the assumption was necessary:</p> <ul style="list-style-type: none"> • not necessary • n large • Central Limit Theorem (<i>spell out instead of just stating as "CLT"</i>) for the approximation for \bar{X} (sample mean mass) and not X (mass) <p><i>*Note that Central Limit Theorem (CLT) can only be used to approximate the distributions of $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$ (e.g. sample mean mass) or $X_1 + X_2 + \dots + X_n$ (total sample mass) for a sufficiently large n; it cannot be used to approximate X (mass); CLT tells us nothing about the distribution of X.</i></p>	<p>$\frac{997.4 - 1000}{\sqrt{\frac{122.69}{50}}} \sim N(0,1)$;</p> <p>a number/value cannot follow a distribution (it is not a random variable).</p> <p>Explanation on whether the assumption was necessary was not well done and sometimes missed out.</p> <p>Note that the p-value needs to be accurate to obtain credit for the conclusion. Also, the conclusion was sometimes not well written with students missing out the mention of the significance level and "<u>mean</u> mass".</p>
<p>(d)</p>	<p>Let Y g be the mass of a Large-sized packet of flour, and μ_y be the population mean.</p> <p>$H_0: \mu_y = 2000$</p> <p>$H_1: \mu_y \neq 2000$</p> <p>Perform 2-tail test at $\alpha\%$ level of significance</p> <p>Under H_0, $\bar{Y} \sim N\left(2000, \frac{\sigma^2}{15}\right)$.</p> <p>$\bar{y} = \frac{\sum y}{15} = 2005$</p> <p>Since $2005 > 2000$, $p\text{-value} = 2P(\bar{Y} \geq 2005) = 0.0529$</p> <p>GC Method</p>  <p>Using GC, $\sigma = 10.004 = 10.0$ (to 3 s.f.) (Shown)</p>	<p>The p-value was often not well written and there were some "double-errors" that led to the given $\sigma = 10.0$ value.</p> <p>There should be clear working to show how the value of σ was obtained since its value is given and this is a "show" qns – students can show evidence of this by leaving σ to a higher sf value (e.g. $\sigma = 10.004$) first if the GC method was used.</p>

	<p>Standardisation Method</p> $z_{\text{cal}} = \frac{2005 - 2000}{\frac{\sigma}{\sqrt{15}}} = \frac{5\sqrt{15}}{\sigma}$ $2P(Z \geq z_{\text{cal}}) = 0.0529$ $P\left(Z \geq \frac{5\sqrt{15}}{\sigma}\right) = \frac{0.0529}{2}$ $\frac{5\sqrt{15}}{\sigma} = 1.935736352$ $\sigma = 10.004 = 10.0 \text{ (to 3.s.f.)}$	
(e)	<p>$H_0: \mu_Y = 2000$ $H_1: \mu_Y \neq 2000$ Perform 2-tail test at 4% level of significance</p> <p>Under H_0, $\bar{Y}_{\text{new}} \sim N\left(2000, \frac{10.0^2}{16}\right)$</p> $\bar{y}_{\text{new}} = \frac{\sum y + h}{16} = \frac{30075 + h}{16}$ <p>Method 1 (critical-value)</p>  <p>Since H_0 is rejected: \bar{y}_{new} lies in critical region</p> $\bar{y}_{\text{new}} \leq 1994.8656 \text{ or } \bar{y}_{\text{new}} \geq 2005.1344$ $\frac{30075 + h}{16} \leq 1994.8656 \text{ or } \frac{30075 + h}{16} \geq 2005.1344$ $h \leq 1842.85 \text{ or } h \geq 2007.15$ $\therefore h \leq 1842 \text{ (nearest g) or } h \geq 2008 \text{ (nearest g)}$ <p>Method 2 (standardised critical-value)</p> 	<p>The critical/rejection region/criteria should not be a strict inequality and should include the "equal sign", i.e. "\leq, \geq", instead of "$<$, $>$".</p> <p>Some students carelessly still took the sample size, n, as 15.</p> <p>Method 1 (critical value method) is strongly recommended for such questions. As there are no unknowns in the distribution parameters, Method 1 is preferred over Method 2.</p> <p>Method 3 (p-value method) was generally not well done as it proved to be challenging for most students to write out the p-value accurately and to consider the two required cases, thus the critical value method is strongly advised/preferred for such questions.</p>

Since H_0 is rejected: z -value lies in critical region

z -value ≤ -2.0537 or z -value ≥ 2.0537

$$\frac{30075+h}{16} - 2000 \leq -2.0537 \quad \text{or} \quad \frac{30075+h}{16} - 2000 \geq 2.0537$$

$$\frac{\sqrt{10.0^2/16}}{\sqrt{10.0^2/16}} \leq -2.0537 \quad \text{or} \quad \frac{\sqrt{10.0^2/16}}{\sqrt{10.0^2/16}} \geq 2.0537$$

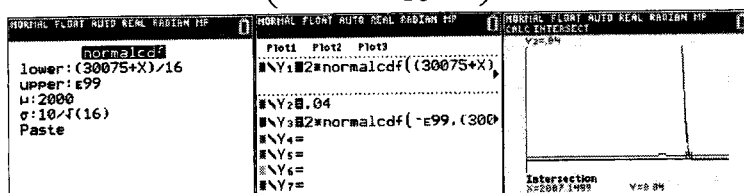
$$h \leq 1842.85 \quad \text{or} \quad h \geq 2007.15$$

$$\therefore h \leq 1842 \text{ (nearest g)} \quad \text{or} \quad h \geq 2008 \text{ (nearest g)}$$

Method 3 (p-value)

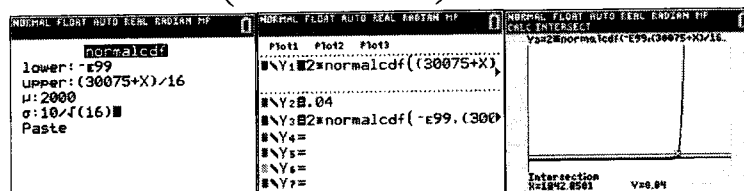
Since H_0 is rejected,

$$\text{Either } p\text{-value} = 2P\left(\bar{Y}_{\text{new}} \geq \frac{30075+h}{16}\right) \leq 0.04$$



From GC, $h \geq 2008$ (nearest g)

$$\text{Or } p\text{-value} = 2P\left(\bar{Y}_{\text{new}} \leq \frac{30075+h}{16}\right) \leq 0.04$$



From GC, $h \leq 1842$ (nearest g)

Total marks: 12