1 A curve has equation $y = x + 1 + \dfrac{3}{x-1}$. Using differentiation, find the set of values of x where the curve is strictly decreasing. Give your answers in exact form. [4]

2 A sequence of real numbers, x_n, satisfies the recurrence relation

$$x_{n+1} = a(-3)^n + bn^2 + cx_n, \text{ for } n, a, b, c \in \mathbb{Z} \text{ and } n \geq 1.$$

Given that $x_1 = 1$, $x_2 = -8$, $x_3 = 70$, and $x_4 = -377$, find the values of a, b and c. [3]

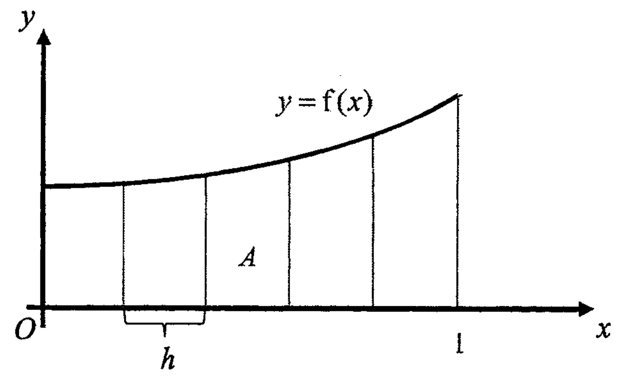
Hence, find the values of x_{10} and x_{11}. [2]

3 It is given that $\left(2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} + y^2\right)\cos x = \dfrac{1}{xy^2}$, where $0 < x < \dfrac{\pi}{2}$ and $y \neq 0$.

(a) Using the substitution $u = xy^2$, show that the differential equation can be reduced to $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{\sec x}{u}$. [3]

(b) Hence find the general solution to the differential equation. [3]

4 The diagram shows a sketch of the function $\mathrm{f}(x) = \mathrm{e}^{3x} + 1$, for $0 \leq x \leq 1$. The region bounded by the curve and the lines $y = 0$, $x = 0$ and $x = 1$ is A. The region A is split into 5 vertical strips of equal width h, as shown in the diagram.

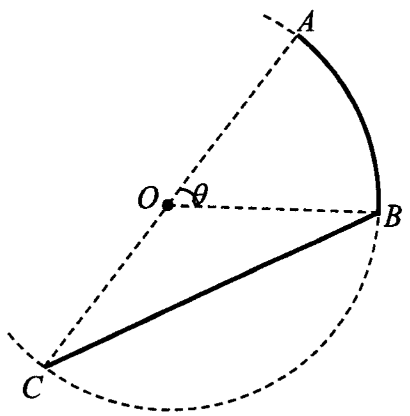


(a) State the value of h and using a suitable sketch, explain whether $\sum_{k=1}^{5}\left(h\,\mathrm{f}(kh)\right)$ is less or more than the area of A. [3]

(b) A is now split into n vertical strips of equal width. Using calculus, find the exact value of

$$\lim_{n\to\infty}\frac{1}{n}\left(\mathrm{e}^{\frac{3}{n}} + \mathrm{e}^{\frac{6}{n}} + \mathrm{e}^{\frac{9}{n}} + \ldots + \mathrm{e}^{\frac{3n-3}{n}} + \mathrm{e}^3 + n\right).$$ [3]

5



The tourism board plans to construct a tram track around a tourist attraction. The diagram above shows part of the blueprint of the track, where the tram will run along a circular path with centre O and radius 450 m, from a fixed point A to a variable point B, and then straight across to a fixed point C. The angle AOB is denoted as θ, where $0 \le \theta \le \pi$. The speed of the tram along the circular path will be maintained at 8 m/s and the speed along the straight path BC will be maintained at $\sqrt{6}$ m/s.

(a) Show that the time taken for the tram to travel from point A to C is $\frac{225}{4}\theta + 150\sqrt{6}\cos\frac{\theta}{2}$. [3]

(b) Use calculus to find the maximum time taken for the tram to travel from point A to point C.
(You need not show that your answer gives a maximum.) [3]

6 Given that b is a real constant such that $0 < b < 4$, describe fully a sequence of transformations that transforms the curve $y = x^2$ to the curve $y = 4x^2 + bx + 1$. [4]

Sketch the curve $y = \frac{1}{4x^2 + bx + 1}$. Give the equation of any asymptotes and the coordinates of any axial intercepts and turning points, in terms of b where appropriate. [3]

7 **(a)** Use the formula listed in the List of Formulae (MF27) to explain why $\sum_{r=0}^{\infty} \frac{1}{r!} = \mathrm{e}$. [1]

(b) Show that $\sum_{r=0}^{\infty} \frac{r+1}{r!} = 2\mathrm{e}$. [3]

(c) Find the value, in terms of e, of $\sum_{r=6}^{\infty} \frac{r+1}{r!}$. [3]

8 The function f is defined by

$$f : x \mapsto \frac{ax-3}{5x-4}, \text{ for } x \in \mathbb{R}, \; x \neq \frac{4}{5}, \text{ and } a \in \mathbb{R}.$$

(a) Find the range of f in terms of a. [1]

The function f is such that $f(x) = f^{-1}(x)$ for all x in the domain of f.

(b) Find the value of a. [2]

(c) Hence find $f^{2025}(2)$. [2]

Another function g is defined by

$$g : x \mapsto \left(x - \frac{1}{5}\right)^2 + \frac{2}{5}, \text{ for } x \in \mathbb{R}, \; 0 < x < \frac{3}{5}.$$

(d) Find the range of fg. [2]

9 **(a)** Find $\int (2x-1)\cos x \, dx$. [3]

(b) Hence find the value of $\int_0^{\frac{\pi}{2}} |2x-1| \cos x \, dx$. Give your answer in the form $A - 4\cos B$, where A and B are exact constants to be determined. [4]

10 **(a)** Express $\frac{1}{Q(500-Q)}$ in partial fractions. [2]

(b) A group of scientists is monitoring the population of a particular species of birds on an island. At time t years after the start of the monitoring, the number of birds on the island is Q. The scientists observe that the birth rate is proportional to the bird's population and the birds are dying at a rate proportional to the square of the bird's population. There were 1000 birds on the island when the scientists first started monitoring the population. They discover that the population remains unchanged when there are 500 birds.

(i) Show that the differential equation relating Q and t is given by

$$\frac{dQ}{dt} = kQ(500-Q),$$

where k is a positive constant. [2]

(ii) Hence, solve the differential equation, expressing Q in terms of k and t. [5]

11 The points A and B have coordinates $(15,3,0)$ and $(5,9,5)$ respectively. Two lines l_1 and l_2, which are perpendicular to each other, have the following equations.

$$l_1 : \mathbf{r} = (15-\lambda)\mathbf{i} + (3+2\lambda)\mathbf{j} + 4\lambda\mathbf{k},$$

$$l_2 : \mathbf{r} = (5+8\mu)\mathbf{i} + (9-2\mu)\mathbf{j} + (5+m\mu)\mathbf{k},$$

where λ and μ are parameters and m is a constant.

(a) Find the value of m. [2]

(b) Given that l_1 and l_2 intersect, find the coordinates of the point of intersection E. [2]

(c) Find a cartesian equation of the plane Π which contains the points A, B and E. [3]

(d) The point D has coordinates $(-1,-3,2)$. Find the position vector of the point F, the foot of perpendicular of D to Π. [3]

(e) Find the exact area of the circle that passes through A, D and F. [2]

12 **Do not use a graphing calculator for this question.**

It is given that $\mathrm{f}(z) = z^4 - 6z^2 + k$, where k is a non-zero constant.

(a) If k is a purely imaginary number, determine, with justification, whether $\mathrm{f}(z) = 0$ can have real roots. [1]

(b) Show that $\mathrm{f}(-z) = \mathrm{f}(z)$. [1]

(c) Given that $2+\mathrm{i}$ is a root of the equation $\mathrm{f}(z) = 0$, determine k. Hence, or otherwise, find the remaining roots, showing your workings clearly. [6]

Use the value of k found in part **(c)** for the rest of this question.

(d) Given that the product of all the roots of $\mathrm{f}(z) = 0$ is D, find the value of D, showing your workings clearly. [2]

(e) A complex number w_1 satisfies the equation $kw^4 - 6w^2 + 1 = 0$. Given that w_1 can be obtained from $2+\mathrm{i}$, find w_1. [2]

13 Frederick, a social media influencer, is starting a new online account at the start of a month. From the second month onwards:

- His organic followers at the end of each month will be a times the total followers he had at the end of the previous month, where a is a positive constant.
- A company he engaged in will provide him with 1000 additional followers in the middle of each month.

Let $F(n)$ denote the total number of organic and additional followers Frederick had at the end of n months after he started his new online account, for $n \in \mathbb{Z}^+$.

(a) Write down a recurrence relationship between $F(n)$ and $F(n+1)$ for $n \in \mathbb{Z}^+$, giving your answer in terms of a. [2]

It is found that Frederick has 3500 followers at the end of the first month.

(b) Show that $F(3) = 3500a^2 + 1000a + 1000$. [2]

(c) Find an expression for $F(n)$. Hence, find the number of months required for Frederick's followers to exceed one million if $a = 1.5$. [4]

(d) Given that the number of followers exceeds 20000 by the end of the first year, determine the range of values of a. [2]

Frederick finds out that, instead of the additional 1000 followers in the middle of every month, the company can only provide him with additional b followers in the middle of every month.

(e) Given that the number of followers remains constant since the end of the first month, find the relationship between a and b. [2]

Section A: Pure Mathematics [40 marks]

1 The region A is bounded by the curves $y=\sqrt{x+1}$, $y=\sqrt{7-2x}$, the x-axis and the y-axis.

(a) Find the exact area of A. [4]

(b) Find the volume of the solid obtained when A is rotated through 2π radians about the y-axis. [3]

2 It is given that $y=\ln\left[\sin\left(x+\frac{\pi}{4}\right)\right]$, where $-\frac{\pi}{4}<x<\frac{3\pi}{4}$.

(a) Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1=0$. Hence find the first four non-zero terms of the Maclaurin expansion of y, leaving your answer in exact form. [6]

(b) Verify the result obtained in part **(a)** using standard series from the List of Formulae (MF27). [5]

3 The parametric equations of the curve C are

$$x=1-3\operatorname{cosec}\theta \text{ and } y=2\cot\theta-3, \text{ where } 0\leq\theta\leq\pi.$$

(a) Show $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{3}\sec\theta$. Hence find the equation of the normal to C at the point where $\theta=\frac{\pi}{4}$. Give the equation in the form $y=Ax+B$, where A and B are exact constants to be found. [4]

(b) Show that the normal found in part **(a)** will cut C again. [2]

(c) Find the Cartesian equation of C. [2]

(d) Sketch C, indicating clearly its key features. [3]

(e) Find the range of values of m such that there is no intersection between the line $y=m(x-1)-3$ and C. [2]

4 Let A, B and C be the points on the same plane with position vectors **a**, **b** and **c** respectively and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$. It is given that vectors **a**, **b** and **c** are unit vectors.

(a) (i) By considering $\mathbf{c}\cdot\mathbf{c}$, find the value of $\mathbf{a}\cdot\mathbf{b}$. [3]

(ii) Find the angle $\angle AOB$. [2]

(iii) Draw the position vectors **a**, **b** and **c** on a single diagram. Using your diagram, identify the type of triangle ABC. [2]

(b) The point D has position vector $\mathbf{a}+\mathbf{b}$. Find the area of the quadrilateral $ACBD$. [2]

Section B: Probability and Statistics [60 marks]

5 The eleven letters in the word INSPIRATION are each printed on separate, identical cards.

(a) Find the number of ways in which the cards can be arranged in a row if,

(i) there are no restrictions, [1]

(ii) the letters N are together or the letters I must all be separated, but not both. [3]

(b) Three of the eleven cards are removed at random. Find the probability that the letters on the eight cards left behind are all distinct. [2]

6 A basketball free throw game involves a team of two players, Ben and John. The game consists of at most three throws and the moment a player makes a successful shot, the team wins, and the game will end.

The game uses the following rules.

- Only one player is selected for each game and the probability that Ben is selected in a game is 0.7.
- The probabilities that Ben and John make a successful shot in any single attempt are 0.1 and 0.07 respectively.
- The shots are independent of each other.

(a) Find the probability that the team wins the game. [3]

(b) The team did not win the game. Find the probability that John was the one who was selected to shoot. [3]

(c) The team attempts the game repeatedly until the first game is won. Find the least number of attempts required such that the probability of winning within n games is at least 0.95. [2]

7 An ice-cream seller records the monthly ice cream sales, s thousands dollars for different temperature, t degrees Celsius during the winter season. The recorded values are shown in the table below.

t	1	4	5	6	7	8	9
s	14	15	15	16	18	21	23

(a) It is given that the regression line of s on t is $s = 1.125t + 11$. Using this regression line, find the sum of the squares of the residuals. [1]

(b) State the coordinates of an additional data point such that, with all 8 data points, the regression line remains the same as in part **(a)**. [1]

(c) Sketch a scatter diagram of s against t for the data given in the table. [1]

The following three models are proposed, where a, b, c, d, f and h are positive constants.

(A) $s = at^2 + b$

(B) $s = -ce^t + d$

(C) $s = f\ln(t+h)$

(d) Explain which of these models give the best fit to the data. State the values of the constants for the chosen model. [2]

A temperature of F degrees Fahrenheit is equivalent to a temperature of C degrees Celsius, where $F = \frac{9}{5}C + 32$.

(e) Using the model you chose in part **(d)**, re-write the equation so that it can be used to estimate the monthly sales when the temperature, T, is given in degrees Fahrenheit. [2]

8 A food producer claims that the mean mass of a can of beans it produces is 425 g. Following customer feedback, the production manager wishes to test if the mean mass of a can of beans is indeed 425 g.

The production manager took a random sample of size 50 and the mass of each can, in x g, is recorded and the results are shown below:

$$\sum x = 21209, \ \sum (x - 424.18)^2 = 522$$

(a) State what it means for a sample to be random in this context. [1]

(b) Find the unbiased estimates for the population mean and variance. [2]

(c) State the hypotheses for the manager's test, defining any parameters you use. Carry out the test at the 5% level of significance, giving your conclusion in the context of the question. [5]

The production manager wishes to test whether the mean mass of a can of beans has increased using the alternative manufacturing process. He finds that the mean mass of 55 randomly chosen cans is 426.5 g. He carries out a hypothesis test at 10% level of significance.

(d) Explain, with justification, how the population standard deviation of the mass of a can produced under the alternative process will affect the conclusion made by the production manager. [3]

9 In each round of a treasure hunt game, a player randomly selects a spot from a large number of predefined treasure locations on an island. Each spot uncovers one outcome, and a score x is awarded based on the outcome. The table below shows all the possible outcomes in a round and the corresponding scores. Each round is independent and the game resets after every round, so that the probabilities remain unchanged.

Outcome	Cursed trap	Small trap	Mystery box	Gold chest
Score, x	-3	-0.3	p	5
$\mathrm{P}(X=x)$	0.2	p	q	0.1

(a) Show that $\mathrm{E}(X)=-0.1+0.4p-p^2$. [2]

(b) Hence find the maximum and minimum possible values of $\mathrm{E}(X)$. [2]

(c) The treasure hunt game is played for 30 rounds. If $p=0.4$, find the probability that the player's mean score exceeds 0. [3]

(d) The treasure hunt game is played for 10 rounds. Given that the probability of finding more than 3 Small traps in these 10 rounds is 10%, find the value of p. [2]

(e) Over a long period of time, it is observed that the number of Small traps found in 10 rounds of the game follows a bimodal distribution, with one of the modes being 3. Find the two exact possible values of p, showing your working clearly. [3]

10 **In this question you should state the parameters of any distributions you use.**

At a burger shop, the wait time, W (in minutes), is defined to be the time from when a customer places an order at the counter until the food is collected. It was proposed that W is modelled by $\mathrm{N}(2, 1.5^2)$.

(a) Give a reason why this model is not suitable. [1]

A new model for W is given by $\mathrm{N}(5, 1^2)$.

(b) Find the range of values of k such that at least 90% of customers experience a wait time longer than k minutes. [2]

A customer bought burgers from the shop on three independent occasions, with wait times denoted by W_1, W_2 and W_3 respectively. Let $\bar{W} = \frac{W_1 + W_2 + W_3}{3}$.

(c) Find the values of $\mathrm{Var}(\bar{W} - W_1)$ and $\mathrm{Var}(\bar{W}) + \mathrm{Var}(W_1)$ and hence, determine whether $\mathrm{Var}(\bar{W} - W_1) = \mathrm{Var}(\bar{W}) + \mathrm{Var}(W_1)$. [2]

(d) Find the probability that the mean wait time is within one minute of the wait time on the first occasion. [3]

On the 4th occasion, the customer went to the restroom immediately after the order was placed. The time spent in the restroom, in T minutes, is modelled by $T \sim N(7, 1.5^2)$. Assume that the time taken to walk to the restroom is negligible and W and T follow independent normal distributions.

(e) The customer is in a rush and will not wait for more than 3 minutes after leaving the restroom. Given that the burger is not ready for collection after the customer leaves the restroom, find the probability that the customer will leave the shop without collecting the burger. [4]

To reduce the wait time for customers, the shop later installs self-order kiosks. The wait time from when a customer places an order at the kiosk until the food is collected is modelled as being reduced by 20% compared to that at the counter.

(f) 8 customers who ordered at the counter and 8 customers who ordered at the kiosk are randomly selected and their wait times are recorded. The wait time of each customer is independent of the others. Find the probability that exactly 2 of these 16 customers have a wait time of at least 5 minutes. [4]