Jurong Pioneer Junior College

H2 Mathematics
Preliminary Exam Paper 1 Solution

Q1

(i)

$$y=\frac{1}{4x-x^2}=\left(4x-x^2\right)^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(4x-x^2\right)^{-2}\left(4-2x\right)=\frac{2x-4}{\left(4x-x^2\right)^2}$$

At turning point,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(4x-x^2\right)^{-2}\left(4-2x\right)=0\Rightarrow x=2$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left[\left(4x-x^2\right)^{-2}\left(-2\right)-2\left(4x-x^2\right)^{-3}\left(4-2x\right)^2\right]$$

When $x=2$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left[\left(4(2)-(2)^2\right)^{-2}\left(-2\right)\right]=\frac{1}{8}>0$

Or

Using first derivative test,

x	1.9	2	2.1
$\frac{\mathrm{d}y}{\mathrm{d}x}$	−0.0126	0	0.0126
tangent	\	–	/

The turning point at $x=2$ is a minimum point.

(ii)

Area

$$=\int_1^2 \frac{1}{4x-x^2}\,\mathrm{d}x$$

$$=\int_1^2 \frac{1}{-\left(x^2-4x+2^2-2^2\right)}\,\mathrm{d}x$$

$$=\int_1^2 \frac{1}{2^2-\left(x-2\right)^2}\,\mathrm{d}x$$

$$=\left[\frac{1}{2(2)}\ln\left|\frac{2+(x-2)}{2-(x-2)}\right|\right]_1^2$$

$$=\left[\frac{1}{4}\ln\left|\frac{x}{4-x}\right|\right]_1^2$$

$$=-\frac{1}{4}\ln\frac{1}{3}$$

$$=\frac{1}{4}\ln 3$$

or

$$=\int_1^2 \frac{1}{x(4-x)}\,\mathrm{d}x$$

$$=\frac{1}{4}\int_1^2 \frac{1}{x}+\frac{1}{4-x}\,\mathrm{d}x$$

$$=\frac{1}{4}\left[\ln|x|-\ln|4-x|\right]_1^2$$

$$=\frac{1}{4}\left[\ln\left|\frac{x}{4-x}\right|\right]_1^2$$

$$=-\frac{1}{4}\ln\frac{1}{3}$$

$$=\frac{1}{4}\ln 3$$

Q2

(i)

$$\frac{1}{(x-a)^2}=|x-a|$$

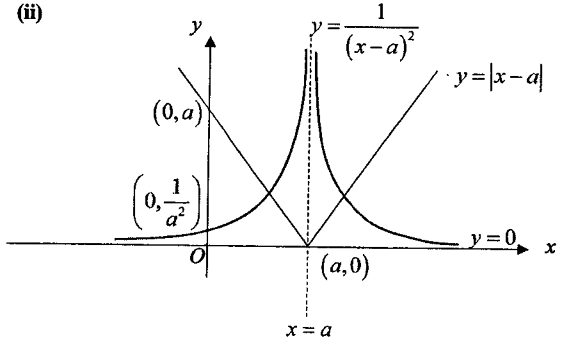
$$\frac{1}{(x-a)^2}=x-a \quad \text{or} \quad \frac{1}{(x-a)^2}=-(x-a)$$

$$(x-a)^3=1 \qquad (x-a)^3=-1$$

$$x-a=1 \qquad x-a=-1$$

$$x=a+1 \qquad x=a-1$$

(ii)



For $\frac{1}{(x-a)^2}>|x-a|$,

$a-1<x<a+1, \qquad x\neq a$

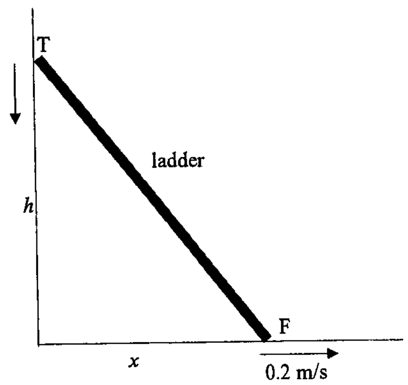
Or

$a-1<x<a \quad \text{or} \quad a<x<a+1$

Q3

Let h m be the vertical distance between the top of the ladder and the floor
Let x m be the horizontal distance between the foot of the ladder and the corner of the wall

Method 1



By Pythagoras' theorem,

$h^2 + x^2 = 3.12^2$

By implicit differentiation w.r.t. t,

$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$

When $h = 1.2$, $\frac{dx}{dt} = 0.2$ and

$x^2 = 3.12^2 - 1.2^2$

$x = 2.88 \quad (\text{Since } x > 0)$

Hence,

$2(1.2)\frac{dh}{dt} + 2(2.88)(0.2) = 0$

$\frac{dh}{dt} = -0.48$

Hence, the top of ladder is sliding down at a rate of 0.48 m/s.

Method 2

By Pythagoras' theorem,

$h^2 + x^2 = 3.12^2$

By implicit differentiation w.r.t. x, or By implicit differentiation w.r.t. h

$2h\frac{dh}{dx} + 2x = 0 \Rightarrow \frac{dh}{dx} = -\frac{x}{h}$

When $h = 1.2$, $\frac{dx}{dt} = 0.2$ and

$x^2 = 3.12^2 - 1.2^2$

$x = 2.88 \quad (\text{Since } x > 0)$

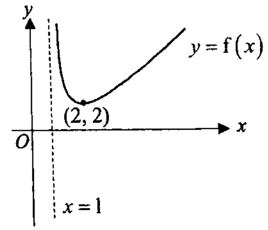
Hence, $\frac{dh}{dt} = \frac{dh}{dx} \times \frac{dx}{dt}$

$= -\frac{2.88}{1.2} \times 0.2$

$= -0.48$

Hence, the top of ladder is sliding down at a rate of 0.48 m/s.

Q4



(i)

Smallest value of $a = 2$.
(For f^{-1} to exist, f must be a one-one function.)

(ii)

Let $y = 4 + 3x - x^2$

$y = -\left[x^2 - 3x - 4\right]$ or $x^2 - 3x - 4 + y = 0$

$= -\left[\left(x - \frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 - 4\right]$

$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{25}{4}\right]$

$y = -\left(x - \frac{3}{2}\right)^2 + \frac{25}{4}$

$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-4+y)}}{2}$

$x = \frac{3 \pm \sqrt{25 - 4y}}{2}$

$x = \frac{3}{2} \pm \sqrt{\frac{25}{4} - y}$

$\left(x - \frac{3}{2}\right)^2 = \frac{25}{4} - y$

$x - \frac{3}{2} = \pm\sqrt{\frac{25}{4} - y}$

$x = \frac{3}{2} \pm \sqrt{\frac{25}{4} - y}$

since $x \le \frac{3}{2}$, $x = \frac{3}{2} - \sqrt{\frac{25}{4} - y}$

Thus, $g^{-1} : x \mapsto \frac{3}{2} - \sqrt{\frac{25}{4} - x}, \ x \le \frac{25}{4}$

$D_{g^{-1}} = R_g = \left(-\infty, \frac{25}{4}\right]$

(iii)

$R_{f^{-1}} = D_f = [2, \infty)$

$D_{g^{-1}} = R_g = \left(-\infty, \frac{25}{4}\right]$

Since $R_{f^{-1}} \not\subset D_{g^{-1}}$, $g^{-1}f^{-1}$ does not exist.

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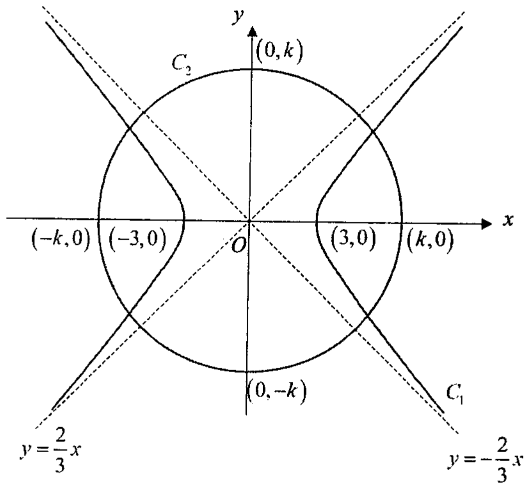
Q5

(a)

(i)



(ii)

For C_1 and C_2 to intersect, $k \geq 3$ (since $k > 0$).

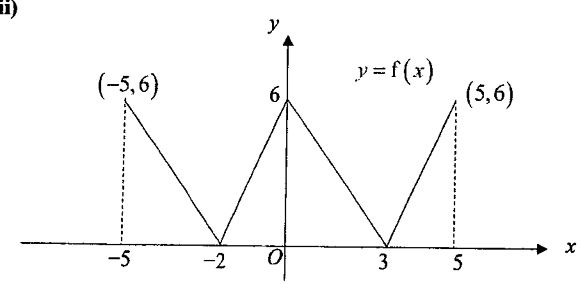
(iii)

Common lines of symmetry for both C_1 and C_2 are $x = 0$ and $y = 0$.

(b)
(i)

$\mathrm{f}(46)=\mathrm{f}(41)=\mathrm{f}(36)=......=\mathrm{f}(1)=-2(1)+6=4$

(ii)



(iii)

For $-5 \leq x \leq 5$. the roots of $\mathrm{f}(-x)=0$ are -3 and 2.

Q6

(i)

$$\int x^2 e^{-2x}\,dx$$
$$=-\frac{1}{2}x^2e^{-2x}+\int xe^{-2x}\,dx$$
$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}\,dx$$
$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C$$
$$=-\frac{1}{4}e^{-2x}\left[2x^2+2x+1\right]+C$$

$u=x^2 \qquad \frac{dv}{dx}=e^{-2x}$

$\frac{du}{dx}=2x \qquad v=-\frac{1}{2}e^{-2x}$

$u=x \qquad \frac{dv}{dx}=e^{-2x}$

$\frac{du}{dx}=1 \qquad v=-\frac{1}{2}e^{-2x}$

(ii)

Volume

$$=\pi\int_0^1\left(xe^{-x}\right)^2-\left(\frac{1}{e}x\right)^2dx \quad \text{or} \quad \pi\int_0^1\left(xe^{-x}\right)^2dx-\frac{1}{3}\pi\left(\frac{1}{e}\right)^2(1)$$
$$=\pi\int_0^1 x^2e^{-2x}-\frac{1}{e^2}x^2\,dx$$
$$=\pi\left[-\frac{1}{4}e^{-2x}\left(2x^2+2x+1\right)\right]_0^1-\frac{\pi}{3e^2}\left[x^3\right]_0^1$$
$$=\pi\left[-\frac{1}{4}e^{-2}\left(2(1)^2+2(1)+1\right)\right]+\frac{\pi}{4}-\frac{\pi}{3e^2}(1)^3$$
$$=-\frac{5\pi}{4e^2}+\frac{\pi}{4}-\frac{\pi}{3e^2}$$
$$=\frac{\pi}{4}-\frac{19\pi}{12e^2}$$
$$=\frac{\pi}{12}\left(3-19e^{-2}\right)$$

Q7

(a)(i)

$$\int\frac{x}{\sqrt{25-x^2}}\,dx$$
$$=-\frac{1}{2}\int-2x\left(25-x^2\right)^{-\frac{1}{2}}dx$$
$$=-\sqrt{25-x^2}+C$$

(a)(ii)

$$\int_{\alpha}^{4}\left|\frac{x}{\sqrt{25-x^2}}\right|\mathrm{d}x=3$$
$$-\int_{\alpha}^{0}\frac{x}{\sqrt{25-x^2}}\mathrm{d}x+\int_{0}^{4}\frac{x}{\sqrt{25-x^2}}\mathrm{d}x=3$$
$$-\left[-\sqrt{25-x^2}\right]_{\alpha}^{0}+\left[-\sqrt{25-x^2}\right]_{0}^{4}=3$$
$$\left[\sqrt{25-x^2}\right]_{\alpha}^{0}-\left[\sqrt{25-x^2}\right]_{0}^{4}=3$$
$$5-\sqrt{25-\alpha^2}-[3-5]=3$$
$$\sqrt{25-\alpha^2}=4$$
$$\alpha^2=9$$
$$\alpha=\pm 3$$
Since $\alpha<0$, $\alpha=-3$

(b)

$x=4\tan\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=4\sec^2\theta$

When $x=4$, $\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$

When $x=0$, $\tan\theta=0\Rightarrow\theta=0$

$$\int_0^4\sqrt{\frac{x^2}{16+x^2}}\,\mathrm{d}x$$
$$=\int_0^{\frac{\pi}{4}}\sqrt{\frac{16\tan^2\theta}{16+16\tan^2\theta}}\,4\sec^2\theta\,\mathrm{d}\theta$$
$$=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\tan^2\theta}{\sec^2\theta}}\,4\sec^2\theta\,\mathrm{d}\theta$$
$$=4\int_0^{\frac{\pi}{4}}\frac{\tan\theta}{\sec\theta}\sec^2\theta\,\mathrm{d}\theta$$
$$=4\int_0^{\frac{\pi}{4}}\tan\theta\sec\theta\,\mathrm{d}\theta$$
$$=4\left[\sec\theta\right]_0^{\frac{\pi}{4}}$$
$$=4\left[\sec\frac{\pi}{4}-\sec 0\right]$$
$$=4\left(\sqrt{2}-1\right)$$

or

$$=4\int_0^{\frac{\pi}{4}}\sin\theta\sec^2\theta\,\mathrm{d}\theta$$
$$=-4\int_0^{\frac{\pi}{4}}-\sin\theta\left[\cos\theta\right]^{-2}\,\mathrm{d}\theta$$
$$=-4\left[\frac{(\cos\theta)^{-1}}{-1}\right]_0^{\frac{\pi}{4}}$$
$$=4\left[\left(\cos\frac{\pi}{4}\right)^{-1}-(\cos 0)^{-1}\right]$$
$$=4\left(\sqrt{2}-1\right)$$

Q8

(a)

$u_n=an^2+bn+c$

$u_1=a+b+c=9$ ------ (1)

$u_2=4a+2b+c=27$ --- (2)

$u_3=9a+3b+c=55$ --- (3)

Using GC, $a=5, b=3, c=1$

$u_n=5n^2+3n+1$

(b)(i)

$$\sum_{r=1}^{n}\left(2r^3+5\right)=2\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}5$$
$$=2\left(\frac{n^2(n+1)^2}{4}\right)+5n$$
$$=\frac{n^2(n+1)^2}{2}+5n$$

(b)(ii)

Method 1

$$\sum_{r=2}^{n}\left(2(r+2)^3+5\right)=\left(2(2+2)^3+5\right)+\ldots+\left(2(n+2)^3+5\right)$$
$$=\sum_{r=4}^{n+2}\left(2r^3+5\right)$$
$$=\sum_{r=1}^{n+2}\left(2r^3+5\right)-\sum_{r=1}^{3}\left(2r^3+5\right)$$
$$=\frac{(n+2)^2(n+3)^2}{2}+5(n+2)-\left[\left(\frac{3^2(4)^2}{2}\right)+5(3)\right]$$
$$=\frac{(n+2)^2(n+3)^2}{2}+5n-77$$

Method 2

$$\sum_{r=2}^{n}\left(2(r+2)^3+5\right)=\left(2(2+2)^3+5\right)+\ldots+\left(2(n+2)^3+5\right)$$
$$=\sum_{r=4}^{n+2}\left(2r^3+5\right)$$
$$=2\left[\sum_{r=1}^{n+2}r^3-\sum_{r=1}^{3}r^3\right]+\sum_{r=4}^{n+2}5$$
$$=2\left[\frac{(n+2)^2(n+3)^2}{4}-\frac{3^2(4)^2}{4}\right]+5(n-1)$$
$$=\frac{(n+2)^2(n+3)^2}{2}+5n-77$$

(b)(iii)

From (i) $\sum_{r=1}^{n}u_r=\frac{n^2(n+1)^2}{2}+5n$

As $n\to\infty$, $\frac{n^2(n+1)^2}{2}\to\infty$, $5n\to\infty$

Hence, $\sum_{r=1}^{n}u_r=\frac{n^2(n+1)^2}{2}+5n\to\infty$, series $\sum_{r=1}^{\infty}u_r$ does not converge.

Q9

(i)

$$\int\frac{1}{4+9x^2}\,dx=\int\frac{1}{9\left(\frac{4}{9}+x^2\right)}\,dx=\frac{1}{6}\tan^{-1}\frac{3x}{2}+K$$

(ii)

$$f(x)=(4+9x^2)^{-1}=\frac{1}{4}\left(1+\frac{9x^2}{4}\right)^{-1}$$
$$=\frac{1}{4}\left[1+(-1)\left(\frac{9x^2}{4}\right)+\frac{-1(-2)}{2}\left(\frac{9x^2}{4}\right)^2+\ldots\right]$$
$$=\frac{1}{4}\left(1-\frac{9x^2}{4}+\frac{81}{16}x^4+\ldots\right)$$
$$\approx\frac{1}{4}-\frac{9}{16}x^2+\frac{81}{64}x^4$$

(iii)

From (i) $\tan^{-1}\frac{3x}{2}=6\int\frac{1}{4+9x^2}\,dx+C,$ where $C=-6K$

From (ii) $\tan^{-1}\frac{3x}{2}=6\int\left(\frac{1}{4}-\frac{9}{16}x^2+\frac{81}{64}x^4\right)dx+C$
$$=6\left(\frac{1}{4}x-\frac{3}{16}x^3+\frac{81}{320}x^5\right)+D$$
$$=\frac{3}{2}x-\frac{9}{8}x^3+\frac{243}{160}x^5+D$$
$x=0,\tan^{-1}0=0\Rightarrow D=0$

$$\therefore\tan^{-1}\frac{3x}{2}=\frac{3}{2}x-\frac{9}{8}x^3+\frac{243}{160}x^5$$

(iv)

$\int_0^{0.5}\tan^{-1}\frac{3x}{2}\,dx=\int_0^{0.5}\left(\frac{3}{2}x-\frac{9}{8}x^3+\frac{243}{160}x^5\right)dx=0.174$ (to 3 dp) (Using GC)

(v)

From GC, $\int_0^{0.5}\tan^{-1}\frac{3x}{2}\,dx=0.173$ (to 3 dp)

(vi)

The estimate in (iv) is accurate up to 2 decimal places but not to 3 decimal places.

To improve the estimate, we can include higher-order terms in the Maclaurin series expansion of $\tan^{-1}\frac{3x}{2}$.

Q10

(a)

$$y=\ln\left(\frac{e^2}{3x}\right)=\ln e^2-\ln 3x=2-\ln 3x$$

$$y=\ln x \xrightarrow{A} y=\ln 3x \xrightarrow{B} y=-\ln 3x \xrightarrow{C} y=2-\ln 3x$$

A: A scaling parallel to the x-axis by a factor of $\frac{1}{3}$.

B: Reflection in the x-axis.

C: A translation of 2 units in the positive direction of the y-axis.

(b)(i)

$g(x)=2f(x-1)$

$(a,b) \rightarrow (a+1,b) \rightarrow (a+1,2b)$

Hence the corresponding point R is $(a+1,2b)$.

$g'(x)=2f'(x-1)$

Given $f'(a)=5$

$f'(a+1)=5$ [gradient remains the same after translation]

At $(a+1,2b)$,

$g'(x)=2f'(a+1)=2(5)=10$

> **Alternative**
> When $x=a+1$
> $g'(x)=2f'(a+1-1)=2f'(a)$
> Given $f'(a)=5$,
> $g'(x)=2(5)=10$

(b)(ii)

$g(x)=\frac{1}{f(x)}$

The corresponding point R is $\left(a,\frac{1}{b}\right)$.

$g'(x)=-\frac{f'(x)}{[f(x)]^2}$

Given $f'(a)=5$, at $\left(a,\frac{1}{b}\right)$,

$g'(x)=-\frac{f'(a)}{[f(a)]^2}=-\frac{5}{b^2}$

Q11

(i)

$u_{n+1}=(1+k)u_n$

$\frac{u_{n+1}}{u_n}=1+k=$ constant independent of n (since k is a constant),

$\therefore \{u_n\}$ is a geometric progression.

(ii)

Method 1

$u_1=311 \qquad r=1+k$

$S_3=\frac{311((1+k)^3-1)}{1+k-1}=4043$ or Using GC Table

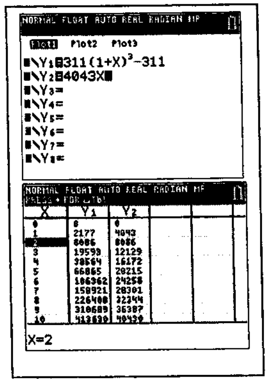
$311(1+k)^3-311=4043k$

$(1+k)^3-13k-1=0$

$k^3+3k^2-10k=0$

Using GC, $k=-5,\ 0,\ 2$

Since $k>0$, $k=2$



Method 2

$u_1=311 \qquad u_2=(1+k)311 \qquad u_3=(1+k)^2 311$

$311+(1+k)311+(1+k)^2 311=4043$

$1+1+k+(1+k)^2=13$

$k^2+3k-10=0$

$k=2$ or -5 (NA since $k>0$)

(iii)

Since common ratio $r=3>1$, the series does not converge as $n\to\infty$. Sum to infinity does not exist and hence there is no limit to total number of daily views in the long run.

(iv)

r	v_r
1	$v_1=u_1=311$
2	$v_2=u_2=(3)311$
3	$v_3=u_3=(3^2)311$
4	$v_4=u_4=(3^3)311$
5	$v_5=v_4+80$
6	$v_6=v_5+80$
⋮	

(rows 4 onwards) AP with 1st term $(3^3)311=8397$, common difference 80

From v_4 to v_r, no. of terms $=r-4+1=r-3$

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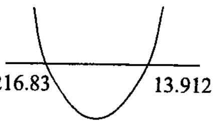
$\therefore\ S_r = S_3 + \frac{r-3}{2}[2(8397)+(r-4)(80)]$
$= 4043 + 8397r - 25191 + 40(r-3)(r-4)$
$= 40r^2 + 8117r - 20668$ where $t = 20668$ (shown)

(v)

$40r^2 + 8117r - 20668 > 100000$
$40r^2 + 8117r - 120668 > 0$
$r < -216.83$ (NA since $r \ge 1$) or $r > 13.912$
Least number of days = 14



Or using GC Table

r	$40r^2 + 8117r - 120668$
13	-8387 < 0
14	810 > 0
15	10087 > 0

Least number of days = 14

Q12

(i)

Plane contains $\overrightarrow{OA} = \begin{pmatrix}0\\0\\3\end{pmatrix}$ and is parallel to $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}4\\5\\1\end{pmatrix}$.

Normal of plane, $\mathbf{n} = \begin{pmatrix}0\\1\\0\end{pmatrix} \times \begin{pmatrix}4\\5\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-4\end{pmatrix}$

Equation of plane,

$\mathbf{r} \cdot \begin{pmatrix}1\\0\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\3\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-4\end{pmatrix}$

$\mathbf{r} \cdot \begin{pmatrix}1\\0\\-4\end{pmatrix} = -12$

$x - 4z = -12$ (Shown)

(ii)

Normal of wall, $\mathbf{n}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and normal of roof, $\mathbf{n}_2 = \begin{pmatrix}1\\0\\-4\end{pmatrix}$

Let θ be acute angle between wall and roof.

$\cos\theta = \dfrac{\left|\begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-4\end{pmatrix}\right|}{(1)\left(\sqrt{17}\right)}$

$= \dfrac{1}{\sqrt{17}}$

$\theta = 75.9638°$

Hence, obtuse angle between wall and roof $= 180° - 75.9638°$
$= 104.0362$
$\approx 104°$ (nearest degree)

(iii)

Method 1

$\overrightarrow{AP} = \begin{pmatrix}4\\3\\0\end{pmatrix} - \begin{pmatrix}0\\0\\3\end{pmatrix} = \begin{pmatrix}4\\3\\-3\end{pmatrix}$ or $\overrightarrow{BP} = \begin{pmatrix}4\\3\\0\end{pmatrix} - \begin{pmatrix}0\\5\\3\end{pmatrix} = \begin{pmatrix}4\\-2\\-3\end{pmatrix}$

Minimum length required = $\left|\overrightarrow{AP}\cdot \mathbf{n}_2\right|$ or $\left|\overrightarrow{BP}\cdot \mathbf{n}_2\right|$

$$= \frac{\left|\begin{pmatrix}4\\3\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}\right|}{\sqrt{17}} = \frac{16}{\sqrt{17}} \quad \text{or} \quad = \frac{\left|\begin{pmatrix}4\\-2\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix}\right|}{\sqrt{17}} = \frac{16}{\sqrt{17}}$$

Method 2: (Finding foot of perpendicular from P to roof)

Eq of roof: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix} = -12$ and Eq of wooden strut: $\mathbf{r} = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-4\end{pmatrix}$, $\lambda\in\mathbb{R}$

Then, $\begin{pmatrix}4+\lambda\\3\\-4\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-4\end{pmatrix} = -12$

$$4+\lambda+16\lambda = -12$$
$$17\lambda = -16$$
$$\lambda = -\frac{16}{17}$$

Length of wooden strut = $\left|\overrightarrow{PF}\right| = \left|\overrightarrow{OF} - \overrightarrow{OP}\right|$

$$= \left|\begin{pmatrix}4\\3\\0\end{pmatrix} - \frac{16}{17}\begin{pmatrix}1\\0\\-4\end{pmatrix} - \begin{pmatrix}4\\3\\0\end{pmatrix}\right|$$
$$= \left|\begin{pmatrix}-\frac{16}{17}\\0\\\frac{64}{17}\end{pmatrix}\right|$$
$$= \sqrt{\left(-\tfrac{16}{17}\right)^2 + \left(\tfrac{64}{17}\right)^2}$$
$$= \frac{16}{\sqrt{17}}$$

(iv)

Equation of wooden strut: $\mathbf{r} = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-4\end{pmatrix}$, $\lambda\in\mathbb{R}$

Equation of light ray: $\mathbf{r} = \begin{pmatrix}0\\5\\0\end{pmatrix} + \mu\begin{pmatrix}7\\-4\\4\end{pmatrix}$, $\mu\in\mathbb{R}$

Assuming they intersect at X,

$\overrightarrow{OX} = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-4\end{pmatrix} = \begin{pmatrix}0\\5\\0\end{pmatrix} + \mu\begin{pmatrix}7\\-4\\4\end{pmatrix}$, for some $\lambda,\mu\in\mathbb{R}$

$4+\lambda = 0+7\mu \quad -(1) \Rightarrow \quad \lambda - 7\mu = -4 \quad -(4)$

$3 = 5-4\mu \quad -(2) \Rightarrow \quad \mu = \frac{1}{2}$

$-4\lambda = 4\mu \quad -(3) \Rightarrow \quad \lambda = -\mu = -\frac{1}{2}$

Substitute $\lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}$ into (4): LHS = $\lambda - 7\mu = -\frac{1}{2} - 7\left(\frac{1}{2}\right) = -4$ = RHS

Hence, $\lambda = -\frac{1}{2}$ and $\mu = \frac{1}{2}$ satisfy all 3 equations.

The light ray meets the strut.

And $\overrightarrow{OX} = \begin{pmatrix}4\\3\\0\end{pmatrix} + \left(-\frac{1}{2}\right)\begin{pmatrix}1\\0\\-4\end{pmatrix} = \begin{pmatrix}3.5\\3\\2\end{pmatrix}$ or $\overrightarrow{OX} = \begin{pmatrix}0\\5\\0\end{pmatrix} + \left(\frac{1}{2}\right)\begin{pmatrix}7\\-4\\4\end{pmatrix} = \begin{pmatrix}3.5\\3\\2\end{pmatrix}$

The light ray <u>meets</u> the wooden strut at $(3.5, 3, 2)$.

BP~364

Q1

(i)

$\frac{dx}{dt} = k(P-x), k>0$

$\int \frac{1}{P-x}\,dx = \int k\,dt$

$-\ln(P-x) = kt + C$ since $P > x$

$\ln(P-x) = -kt - C$

$P-x = e^{-kt-C}$

$P-x = e^{-C} \cdot e^{-kt}$

$P-x = Ae^{-kt}$, $A = e^{-C}$

$x = P - Ae^{-kt}$

When $t=0$, $x = 0 \Rightarrow A = P$

$x = P - Pe^{-kt}$

$x = P\left(1-e^{-kt}\right)$ (Shown)

(ii)

When $t=12$, $x = \frac{1}{2}P$

$\frac{1}{2}P = P\left(1-e^{-12k}\right)$

$e^{-12k} = \frac{1}{2}$

$-12k = \ln\frac{1}{2} = -\ln 2$

$k = \frac{1}{12}\ln 2$

$x = P\left(1-e^{-\frac{t}{12}\ln 2}\right)$

When $x = 0.8P$,

$0.8P = P\left(1-e^{-\frac{t}{12}\ln 2}\right)$

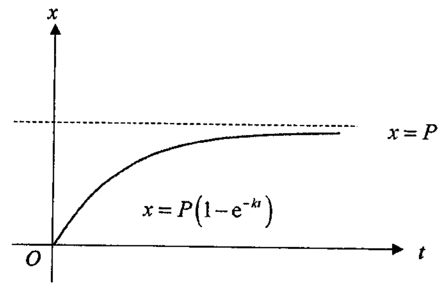
$e^{-\frac{t}{12}\ln 2} = 0.2$

$-\frac{t}{12}\ln 2 = \ln 0.2$

$t = \frac{-12\ln 0.2}{\ln 2}$

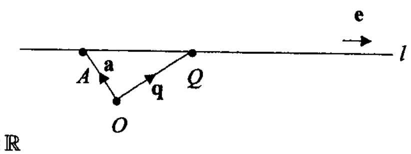
$t = 27.9 \approx 28$ hours (nearest hour)

(iii)



Q2

(i)



Method 1:

Equation of l: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{e}$, $\lambda \in \mathbb{R}$

Since Q lies on l, $\mathbf{q} = \mathbf{a} + \lambda\mathbf{e}$, for some value of $\lambda \in \mathbb{R}$

$(\mathbf{q}-\mathbf{a})\times\mathbf{e} = (\mathbf{a}+\lambda\mathbf{e}-\mathbf{a})\times\mathbf{e}$

$= \lambda\mathbf{e}\times\mathbf{e}$

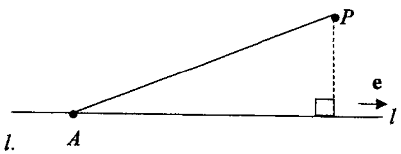
$= \mathbf{0}$ (Shown)

Method 2:

$\overrightarrow{AQ} = \mathbf{q} - \mathbf{a}$

Since, $\overrightarrow{AQ} // \mathbf{e}$, $(\mathbf{q}-\mathbf{a})\times\mathbf{e} = \mathbf{0}$. (Shown)

(ii)



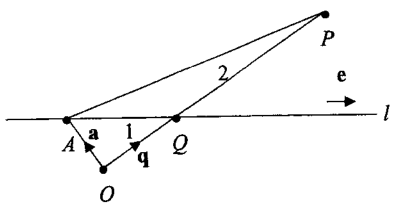
$|(\mathbf{p}-\mathbf{a})\times\mathbf{e}| = |\overrightarrow{AP}\times\mathbf{e}|$

$|(\mathbf{p}-\mathbf{a})\times\mathbf{e}|$ represent the perpendicular distance from P to l.

OR area of parallelogram with adjacent sides parallel and equal in magnitude to $\overrightarrow{AP}$ and **e**

(iii)

Method 1:



Area of triangle $APQ = \frac{1}{2}\left|(\mathbf{p}-\mathbf{a})\times\mathbf{e}\right|\left|\overline{AQ}\right|$

$= \frac{1}{2}\left|(3\mathbf{q}-\mathbf{a})\times\mathbf{e}\right|(2)$

$= \left|(2\mathbf{q}+\mathbf{q}-\mathbf{a})\times\mathbf{e}\right|$

$= \left|(2\mathbf{q})\times\mathbf{e}+(\mathbf{q}-\mathbf{a})\times\mathbf{e}\right|$

$= \left|2(\mathbf{q}\times\mathbf{e})+\mathbf{0}\right|$ (Using (i))

$= 2\left|\mathbf{q}\times\mathbf{e}\right|$

Method 2:

Area of triangle $APQ = \frac{1}{2}\left|(\mathbf{p}-\mathbf{a})\times\mathbf{e}\right|\left|\overline{AQ}\right|$

$= \frac{1}{2}\left|(3\mathbf{q}-\mathbf{a})\times\mathbf{e}\right|(2)$

$= \left|(3\mathbf{q}-\mathbf{q}+\lambda\mathbf{e})\times\mathbf{e}\right|$ (Since $\mathbf{q}=\mathbf{a}+\lambda\mathbf{e} \Rightarrow \mathbf{a}=\mathbf{q}-\lambda\mathbf{e}$)

$= \left|(2\mathbf{q})\times\mathbf{e}+\lambda\mathbf{e}\times\mathbf{e}\right|$

$= \left|2(\mathbf{q}\times\mathbf{e})+\mathbf{0}\right|$

$= 2\left|\mathbf{q}\times\mathbf{e}\right|$

Method 3:

Area of triangle $APQ = \frac{1}{2}\left|\overline{AQ}\times\overline{QP}\right|$ or $\frac{1}{2}\left|\overline{AQ}\times\overline{AP}\right|$ or $\frac{1}{2}\left|\overline{AP}\times\overline{PQ}\right|$

$= \frac{1}{2}\left|(\mathbf{q}-\mathbf{a})\times(\mathbf{p}-\mathbf{q})\right|$

$= \frac{1}{2}\left|(\lambda\mathbf{e})\times(3\mathbf{q}-\mathbf{q})\right|$ (Since $\mathbf{q}=\mathbf{a}+\lambda\mathbf{e} \Rightarrow \mathbf{q}-\mathbf{a}=\lambda\mathbf{e}$)

$= \frac{1}{2}\left|\lambda\right|\left|\mathbf{e}\times(2\mathbf{q})\right|$

$= 2\left|\mathbf{q}\times\mathbf{e}\right|$ (Since $\left|\overline{AQ}\right| = 2$, $\left|\lambda\mathbf{e}\right|=2 \Rightarrow \left|\lambda\right|=2$)

> Note: $\overline{AQ}=\mathbf{q}-\mathbf{a}=\lambda\mathbf{e}$
> Since $\left|\overline{AQ}\right| = 2$,
> $\overline{AQ} = 2\mathbf{e}$ or $-2\mathbf{e}$
> Hence, $\left|\lambda\right|=2$

(iv)

Method 1:

Area $= 2\left|\mathbf{q}\times\mathbf{e}\right| = 3$

$2\left|\mathbf{q}\right|\left|\mathbf{e}\right|\sin\theta = 3$

$2(2)(1)\sin\theta = 3$

$\sin\theta = \frac{3}{4}$

$\theta = 48.590° \approx 48.6°$ (1dp)

Acute angle between l and PQ = Acute angle between **q** and **e** = 48.6°

Method 2:

Area $= \frac{1}{2}\left|\overline{AQ}\right|\left|\overline{QP}\right|\sin\theta = 3$

$\frac{1}{2}(2)\left|\mathbf{p}-\mathbf{q}\right|\sin\theta = 3$

$\left|2\mathbf{q}\right|\sin\theta = 3$ Since $\mathbf{p} = 3\mathbf{q}$

$2\left|\mathbf{q}\right|\sin\theta = 3$

$\sin\theta = \frac{3}{2(2)}$

$\theta = 48.590° \approx 48.6°$

Acute angle between l and PQ = Acute angle between **q** and **e** = 48.6°

Q3

(i)

$x = \sin\theta + 1$ and $y = \sqrt{3}\cos\theta - 1$

$\frac{dx}{d\theta} = \cos\theta$ and $\frac{dy}{d\theta} = -\sqrt{3}\sin\theta$

$\frac{dy}{dx} = \frac{-\sqrt{3}\sin\theta}{\cos\theta} = -\sqrt{3}\tan\theta$ (Shown)

(ii)

At $T\left(\sin t + 1, \sqrt{3}\cos t - 1\right)$, $\theta = t$

Tangent at T makes an angle of $\frac{3\pi}{4}$ with the positive x-axis: $\frac{dy}{dx} = -\sqrt{3}\tan t = \tan\frac{3\pi}{4} = -1$

BP~366

$\tan t = \frac{1}{\sqrt{3}}$

$t = \frac{\pi}{6}$

Hence, $T\left(\sin\frac{\pi}{6}+1, \sqrt{3}\cos\frac{\pi}{6}-1\right) = \left(\frac{3}{2}, \frac{1}{2}\right)$

Gradient of normal at $T = 1$

Equation of normal at T: $y - \frac{1}{2} = (1)\left(x - \frac{3}{2}\right)$

$y = x - 1$

(iii)

For tangent to C to be parallel to the x-axis, $\frac{dy}{dx} = -\sqrt{3}\tan\theta = 0 \Rightarrow \theta = 0$

Hence, $y = \sqrt{3}\cos 0 - 1 = \sqrt{3} - 1$

(iv)

At Q,

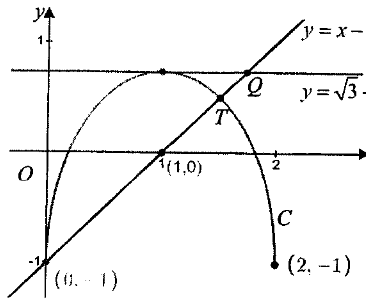
$x - 1 = \sqrt{3} - 1$

$x = \sqrt{3}$

Area of quadrilateral

$= \frac{1}{2}\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)$

$= 1$



Q4

(a)

Method 1

Since polynomial has only real coefficients, if $3 - \mathrm{i}$ is a root, $3 + \mathrm{i}$ is another root.

$[w-(3-\mathrm{i})][w-(3+\mathrm{i})]$

$= [(w-3)+\mathrm{i}][(w-3)-\mathrm{i}]$

$= (w-3)^2 - \mathrm{i}^2$

$= w^2 - 6w + 10$

$5w^3 + pw^2 + 68w + q = \left(w^2 - 6w + 10\right)\left(5w + a\right)$

$= 5w^3 + aw^2 - 30w^2 - 6aw + 50w + 10a$

Comparing coefficients of

w^2: $a - 30 = p$ - - - (1)

w: $-6a + 50 = 68$ - - - (2)

Constant: $10a = q$ - - - (3)

From (2): $-6a + 50 = 68 \Rightarrow a = -3$

Subt into (1): $-3 - 30 = p \Rightarrow p = -33$

Subt into (3): $q = -30$

$5w^3 - 33w^2 + 68w - 30 = \left(w^2 - 6w + 10\right)\left(5w - 3\right) = 0$

The other roots are $3 + \mathrm{i}$ and $\frac{3}{5}$.

Method 2

$w = 3 - \mathrm{i}$

$w^2 = (3-\mathrm{i})^2 = 9 - 6\mathrm{i} + \mathrm{i}^2 = 8 - 6\mathrm{i}$

$w^3 = (3-\mathrm{i})(8-6\mathrm{i}) = 24 - 18\mathrm{i} - 8\mathrm{i} + 6\mathrm{i}^2 = 18 - 26\mathrm{i}$

$5w^3 + pw^2 + 68w + q = 0$

$5(18-26\mathrm{i}) + p(8-6\mathrm{i}) + 68(3-\mathrm{i}) + q = 0$

$90 - 130\mathrm{i} + 8p - 6p\mathrm{i} + 204 - 68\mathrm{i} + q = 0$

Comparing real and imaginary parts:

Real: $90 + 8p + 204 + q = 0$ - - - - - - - (1)

Im: $-130 - 6p - 68 = 0 \Rightarrow p = -33$

Sub into (1)

$90 + 8(-33) + 204 + q = 0 \Rightarrow q = -30$

Since polynomial has only real coefficients, if $3 - \mathrm{i}$ is a root, $3 + \mathrm{i}$ is another root.

$[w-(3-\mathrm{i})][w-(3+\mathrm{i})]$

$= [(w-3)+\mathrm{i}][(w-3)-\mathrm{i}]$

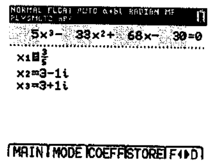
$= (w-3)^2 - \mathrm{i}^2$

$= w^2 - 6w + 10$

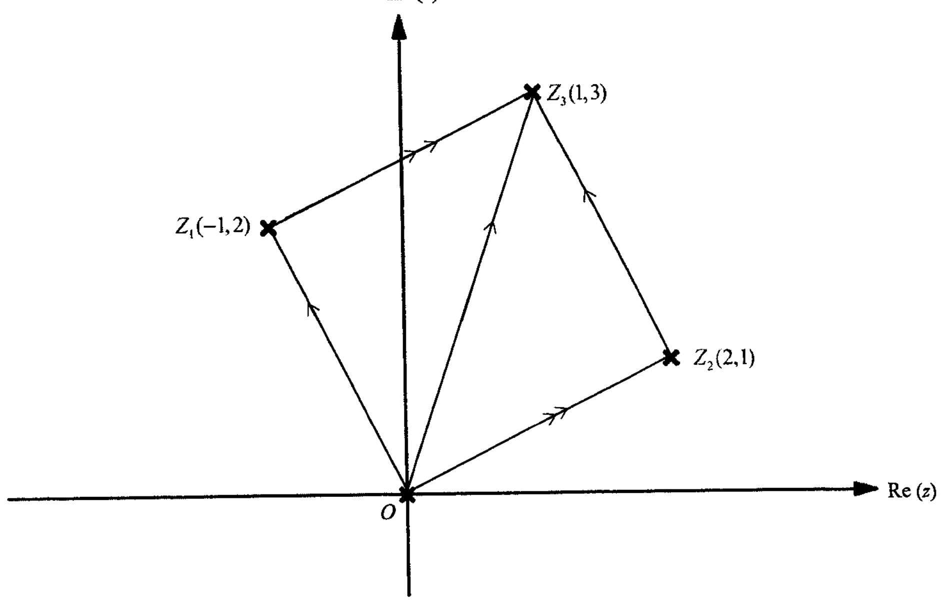
$5w^3 - 33w^2 + 68w - 30 = \left(w^2 - 6w + 10\right)\left(5w - 3\right) = 0$

The other roots are $3 + \mathrm{i}$ and $\frac{3}{5}$.

Use GC polyroot finder to check the answer.



(b)



(i)

$$\frac{z_1}{z_2}=\frac{-1+2i}{2+i}\times\frac{2-i}{2-i}=\frac{-2+i+4i+2}{5}=i$$

(ii)

$$\frac{z_1}{z_2}=i\Rightarrow z_1=iz_2$$

Rotate Z_2 by $\frac{\pi}{2}$ radians anti-clockwise about O to get Z_1

(1) $\therefore Z_1OZ_2$ is a right angle.

(2) $OZ_1=OZ_2$ (since $z_1=iz_2$ or since $|z_1|=|-1+2i|=\sqrt{5}$ and $|z_2|=|2+i|=\sqrt{5}$)

(3) Also, since $\overline{OZ_3}=\overline{OZ_1}+\overline{OZ_2}$, $OZ_1Z_3Z_2$ is a parallelogram.

From (1) + (2) + (3), we can deduce that $OZ_2Z_3Z_1$ is a square.

Q5

(i)

Number of ways to draw 4 orbs = ${}^{12}C_4=495$

(ii)

Number of ways to draw at least 2 colours = $495-{}^5C_4-{}^4C_4=489$

(iii)

Case 1: 2R, 1B, 1G

Number of ways = ${}^5C_2\times{}^4C_1\times{}^3C_1=120$

Case 2: 1R, 2B, 1G

Number of ways = ${}^5C_1\times{}^4C_2\times{}^3C_1=90$

Case 3: 1R, 1B, 2G

Number of ways = ${}^5C_1\times{}^4C_1\times{}^3C_2=60$

Total number of ways = 120 + 90 + 60 = 270

Q6

(i)

The team should conduct a 1-tail test as they are verifying the claim that users are spending more than 15 minutes.

(ii)

Central Limit Theorem states that sample means will follow a normal distribution approximately when the sample size is more than 30. Since the sample size is 80, the team is able to carry out a hypothesis test without knowing anything about the distribution of the times spend by the users.

(iii)

X: time spent per visit
μ : population mean time spent per visit

$H_0:\mu=15$
$H_1:\mu>15$

Under H₀, since n = 80 is large, by Central Limit Theorem,

$\overline{X}\sim N\left(15,\frac{\sigma^2}{80}\right)$ approximately

Test statistic, $z=\frac{\bar{x}-\mu}{\sqrt{\frac{\sigma^2}{n}}}=\frac{16-15}{\sqrt{\frac{\sigma^2}{80}}}=\frac{1}{\sqrt{\frac{\sigma^2}{80}}}$

BP-368

For 5% level of significance, reject H_0 if $z > 1.6449$

In order for platform's claim to be valid, H_0 must be rejected, hence,

$$\frac{1}{\sqrt{\frac{\sigma^2}{80}}} > 1.6449$$

$$\sqrt{\frac{\sigma^2}{80}} < 0.6079396$$

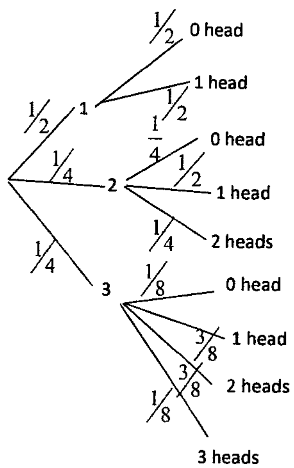
$$0 < \sigma < 5.44$$

(iv)

There is a probability of 0.05 of concluding that the population mean time spent per visit is more than 15 minutes when the population mean time spent is 15 minutes.

Q7

(i)



$$a = P(X=0) = \left(\frac{1}{2}\times\frac{1}{2}\right)+\left(\frac{1}{4}\times\left(\frac{1}{2}\right)^2\right)+\left(\frac{1}{4}\times\left(\frac{1}{2}\right)^3\right)=\frac{11}{32}$$

$$b = P(X=2) = \left(\frac{1}{4}\times\left(\frac{1}{2}\right)^2\right)+\left(\frac{1}{4}\times {}^3C_2\times\left(\frac{1}{2}\right)^3\right)=\frac{5}{32}$$

Or

$$b = P(X=2) = 1-\frac{11}{32}-\frac{15}{32}-\frac{1}{32}=\frac{5}{32}$$

$$a=\frac{11}{32},\ b=\frac{5}{32}$$

(ii)

Probability distribution of X:

x	0	1	2	3
$P(X=x)$	$\frac{11}{32}$	$\frac{15}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

$$E(X)=\sum xP(X=x)=(0)\left(\frac{11}{32}\right)+(1)\left(\frac{15}{32}\right)+(2)\left(\frac{5}{32}\right)+(3)\left(\frac{1}{32}\right)=\frac{7}{8}$$

$$E(X^2)=\sum x^2P(X=x)=(0)^2\left(\frac{11}{32}\right)+(1)^2\left(\frac{15}{32}\right)+(2)^2\left(\frac{5}{32}\right)+(3)^2\left(\frac{1}{32}\right)=\frac{11}{8}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2=\frac{11}{8}-\left(\frac{7}{8}\right)^2=\frac{39}{64}\ \text{(Shown)}$$

(iii)

Let X = number of heads obtained in one game

$\bar{X}$ = mean number of heads obtained per game

Since n =50 is large, by the Central Limit Theorem, $\bar{X} \sim N\left(\frac{7}{8},\frac{39/64}{50}\right)=N\left(\frac{7}{8},\frac{39}{3200}\right)$ approximately

$P(\bar{X}>1)=0.12876\approx 0.129$

Q8

(i)

(1)The condition of a portable speaker is independent of the condition of any other portable speakers.

Or

Whether a portable speaker is faulty is independent of the condition of any other portable speakers.

(2) The probability that any portable speaker is faulty is constant.

(ii)

Let X = number of faulty portable speakers out of 24

$X \sim B(24, 0.02)$

$P(X \le 1) = 0.917387 \approx 0.917$

(iii)

Method 1:

Let Y = number of substandard cartons out of 50

$P(X>1)=1-0.917387=0.082613$

$Y \sim B(50, 0.082613)$

$P(Y=0)=0.013416\approx 0.0134$

Method 2:

Required Probability $=[P(X\le 1)]^{50}=[0.917387]^{50}=0.0134$

(iv)

Let F = the number of faulty flash drives out of 3

$F \sim B(3, p)$

P(batch is accepted)

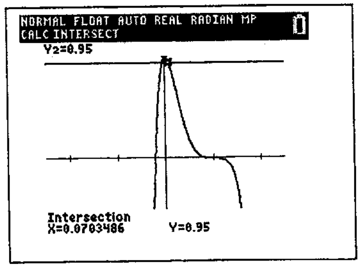
$=P(F=0)+P(F=1)\times P(F=0)$

$=(1-p)^3+{}^3C_1p^1(1-p)^2\times(1-p)^3$

$=(1-p)^3\left[1+3p(1-p)^2\right]$

At least 95% of the batches are accepted

$(1-p)^3\left[1+3p(1-p)^2\right]\geq 0.95$



From the GC, $0<p\leq 0.0703$

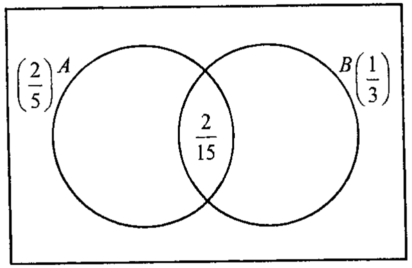
Q9

(i)

Given $P(A|B)=\frac{2}{5}$,

$$\frac{P(A\cap B)}{P(B)}=\frac{2}{5}$$

$$P(A\cap B)=\frac{2}{5}\times\frac{1}{3}=\frac{2}{15}$$



Also, $P(A\cup B)=\frac{3}{5}$

$$P(A)+P(B)-P(A\cap B)=\frac{3}{5}$$

$$P(A)+\frac{1}{3}-\frac{2}{15}=\frac{3}{5}$$

$$P(A)=\frac{2}{5}$$

Since $P(A)=P(A|B)=\frac{2}{5}$, events A and B are independent.

Or

Since $P(A)\cdot P(B)=\frac{2}{5}\times\frac{1}{3}=\frac{2}{15}=P(A\cap B)$, events A and B are independent.

(ii)

$P(A'\cap B)=P(A\cup B)-P(A)$ $=\frac{3}{5}-\frac{2}{5}=\frac{1}{5}$ **or** $P(A'\cap B)=P(B)-P(A\cap B)$ $=\frac{1}{3}-\frac{2}{15}=\frac{1}{5}$

Or since A and B are independent, A' and B are also independent.

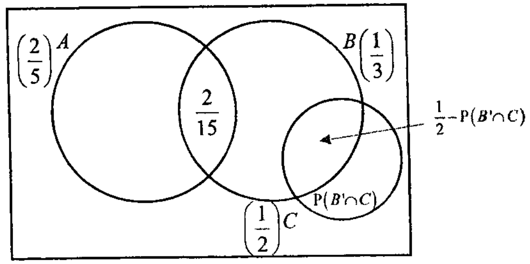
$P(A'\cap B)=P(A')\times P(B)$

$=\frac{3}{5}\times\frac{1}{3}=\frac{1}{5}$

(iii)

Method 1:

Since A and C are mutually exclusive, we have



From the Venn diagram,

$P(B\cap C)\leq P(A'\cap B)$

$\frac{1}{2}-P(B'\cap C)\leq\frac{1}{5}$

$P(B'\cap C)\geq\frac{1}{2}-\frac{1}{5}=\frac{3}{10}$

Also, $P(A\cup B\cup C)\leq 1$

$P(A\cup B)+P(B'\cap C)\leq 1$

$\frac{3}{5}+P(B'\cap C)\leq 1$

$P(B'\cap C)\leq\frac{2}{5}$

Hence, the least possible value of $P(B'\cap C)=\frac{3}{10}$

And the greatest possible value of $P(B'\cap C)=\frac{2}{5}$.

Method 2:

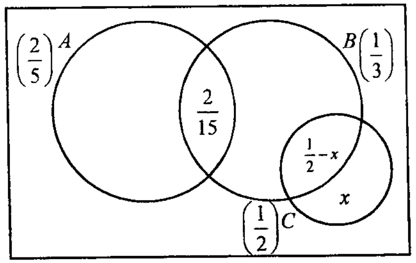
Let $P(B'\cap C)=x$

$P(B\cap C)=\frac{1}{2}-x$

$P(A'\cap B\cap C')=P(A'\cap B)-P(B\cap C)$

$=\frac{1}{5}-\left(\frac{1}{2}-x\right)$

$=x-\frac{3}{10}$



$P(A'\cap B'\cap C')=1-P(A\cup B)-P(B'\cap C)$

$=1-\frac{3}{5}-x$

$=\frac{2}{5}-x$

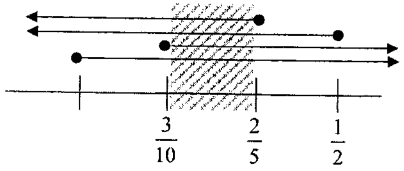
Since they are all probability, we have

$x \geq 0$

$\frac{1}{2} - x \geq 0 \Rightarrow x \leq \frac{1}{2}$

$x - \frac{3}{10} \geq 0 \Rightarrow x \geq \frac{3}{10}$

$\frac{2}{5} - x \geq 0 \Rightarrow x \leq \frac{2}{5}$



Solving the inequalities, $\frac{3}{10} \leq x \leq \frac{2}{5}$.

Hence, the least possible value of $P(B' \cap C) = \frac{3}{10}$

And the greatest possible value of $P(B' \cap C) = \frac{2}{5}$.

Q10

(i)

Let A = length of a ball of 2-ply yarn from Machine A

$A \sim N(\mu, \sigma^2)$

$P(A < 340) = 0.8$		$P(A < 280) = 0.1$
$P\left(Z < \frac{340-\mu}{\sigma}\right) = 0.8$		$P\left(Z < \frac{280-\mu}{\sigma}\right) = 0.1$
$\frac{340-\mu}{\sigma} = 0.84162$	&	$\frac{280-\mu}{\sigma} = -1.28155$
$340 - \mu = 0.84162\sigma$		$280 - \mu = -1.28155\sigma$
$\mu + 0.84162\sigma = 340 ----(1)$		$\mu - 1.28155\sigma = 280 ----(2)$

Solve (1) and (2)

$\mu = 316.216 \approx 316$

$\sigma = 28.259 \approx 28.3$ (AG)

(ii)

Let B = length of a ball of 3 ply yarn from Machine B

$B \sim N(220, 6^2)$

$$[P(B > 230)]^2 \times [P(B < 215)]^3 \times \frac{5!}{3!2!} = 0.000189169$$
$$\approx 0.000189$$

(iii)

$A \sim N(300, 20^2),\ B \sim N(220, 6^2)$

$T = A_1 + A_2 + B_1 + B_2 + B_3 \sim N(2 \times 300 + 3 \times 220, 2 \times 20^2 + 3 \times 6^2) = N(1260, 908)$

$P(T > k) = 0.385$

$k = 1268.81$ (2dp)

(iv)

$A \sim N(300, 20^2),\ B \sim N(220, 6^2)$

$X = 0.91(A_1 + A_2 + A_3) \sim N(0.91 \times 3 \times 300,\ 0.91^2 \times 3 \times 20^2) = N(819, 993.72)$

$Y = 0.95(B_1 + \cdots + B_4) \sim N(0.95 \times 4 \times 220, 0.95^2 \times 4 \times 6^2) = N(836, 129.96)$

$X - Y \sim N(819 - 836,\ 993.72 + 129.96) = N(-17, 1123.68)$

$$P(|X - Y| < 50) = P(-50 < X - Y < 50)$$
$$= 0.814733$$
$$\approx 0.815$$

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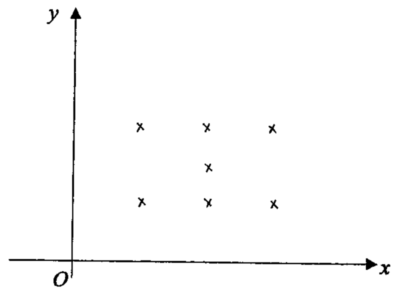
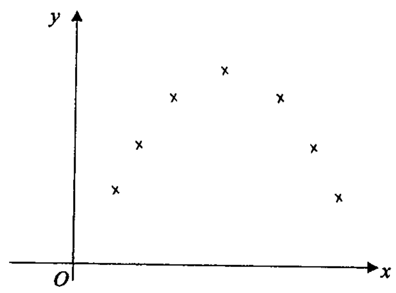
(a)(i)

Possible scatter diagrams:





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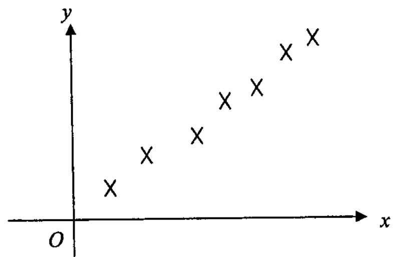
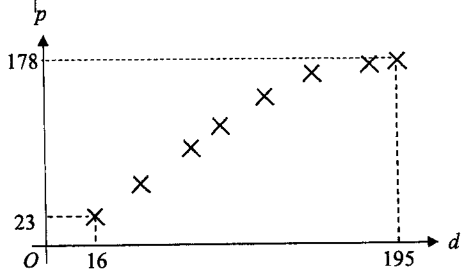
(a)(ii)

One possible scatter diagram:



(b)(i)



(b)(ii)

The values of the residuals may be positive or negative. To avoid the cancellation effect in adding them, the squares of the residuals, which are non-negative, should be used.

(b)(iii)

The linear relationship is not appropriate as the scatter diagram indicates that as d increases, p increases at a decreasing rate.

(b)(iv)

$p = -184.5953136 + 68.95820682 \ln d$

$\approx -185 + 69.0 \ln d$ (to 3 sf)

$p = -184.5953136 + 68.95820682 \ln(210) \approx 184.13$ (to 2 dp)

(b)(v)

Since the range of values of d is from 16 to 195, $d = 210$ is out of the given data range. To estimate the value of p when $d = 210$ is an extrapolation, which may not necessary be reliable as the linear relationship between $\ln d$ and p may not hold at the extrapolated values.

(b)(vi)

The value of t will remain the same while the value of s will increase by 10.