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RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

H2 Math Year 6 Preliminary Examination Paper 1: Solutions with comments

1 A curve has equation $y = ax + b + \frac{c}{x^2 - 1}$, where a, b and c are real constants. It is given that the curve crosses the x-axis at $x = 2$. The normal to the curve at the point $(0, 3)$ meets the x-axis at $x = 7.5$. Find the values of a, b and c. [4]

	Solution	Comments
[4]	$y = ax + b + \frac{c}{x^2 - 1}$ Since $(2, 0)$ and $(0, 3)$ lies on the curve, $2a + b + \frac{1}{3}c = 0$ --- (1) $b - c = 3$ --- (2) $\frac{dy}{dx} = a - \frac{2cx}{(x^2-1)^2}$ Gradient of normal at $(0, 3) = \frac{3-0}{0-7.5} = -\frac{2}{5}$. Gradient of the tangent to the curve at $(0, 3)$ is $\frac{5}{2}$ and thus $a = \frac{5}{2}$ --- (3) Using GC, $a = \frac{5}{2}$, $b = -3$ and $c = -6$.	There are quite many sign/conceptual errors in the attempt to find $\frac{dy}{dx}$. Do note that the gradient of the normal to the curve at the point (0, 3) means that we need to substitute $x = 0$ in $\frac{dy}{dx}$, and get $-\frac{1}{a}$ as the gradient of the normal. The two points (0, 3) and (7.5, 0) gives the gradient of the normal as $-\frac{2}{5}$.

2 The point P travels along the curve C with equation $y = x\sin^{-1}x$, $-1 < x < 1$. Let the gradient of the curve C at the point P be m.

If the x-coordinate of P is increasing at the rate of 9 units per second when $x = \frac{1}{2}$, find the exact value of the rate at which m is changing at this instant. [4]

	Solution	Comments
[4]	$y = x\sin^{-1}x, -1 < x < 1$ Differentiate with respect to x: $m = \frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ Differentiate with respect to x again: $\frac{dm}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2} - x\left(\frac{1}{2\sqrt{1-x^2}}\right)(-2x)}{(1-x^2)}$ $= \frac{1}{\sqrt{1-x^2}} + \frac{(1-x^2)+x^2}{\sqrt{1-x^2}(1-x^2)}$ $= \frac{2-x^2}{\sqrt{(1-x^2)^3}}$ When $x = \frac{1}{2}$, $\frac{dm}{dt} = \frac{dm}{dx} \cdot \frac{dx}{dt} = \frac{2 - \frac{1}{4}}{\sqrt{\left(1-\frac{1}{4}\right)^3}} \times 9 = 14\sqrt{3}$ Therefore, required rate is $14\sqrt{3}$.	Most students did well but many did not simplify the answer to the lowest terms. Answer mark was not awarded to non-exact answers and unsimplified answers. Some students substituted $x=p$, assuming the parameter x takes the value of p at point P, note that p will be taken as a constant in this case and we shall not differentiate with respect to p.

3 Relative to the origin O, the points A and C have position vectors $\mathbf{a}$ and $2\mathbf{b}$ such that

$$\mathbf{a} = 5p\mathbf{i} - 2p\mathbf{j} + 4p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where p is a positive constant. It is given that $|\mathbf{a}| = 2|\mathbf{b}|$.

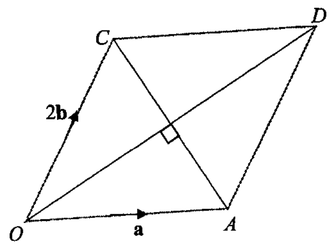
(a) Find the exact value of p. [2]

(b) Evaluate $(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})$. [1]

(c) Evaluate $\mathbf{a}\times\mathbf{b}$ and hence find the area of triangle OAC. [3]

(d) Use a geometrical reason to explain why $|\mathbf{a}\times\mathbf{b}| = \frac{1}{4}(|\mathbf{a}+2\mathbf{b}|)(|\mathbf{a}-2\mathbf{b}|)$. [2]

	Solution	Comments
(a) [2]	Since $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$, $\sqrt{(5p)^2+(-2p)^2+(4p)^2} = 2\sqrt{1^2+(-2)^2+2^2}$ $3\sqrt{5}p = 6$ (given $p>0$) $p = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$	Most students did well but many did not simplify the answer or made careless mistakes, which affected the rest of the parts, too.
(b) [1]	$(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-2\mathbf{b})$ $= \mathbf{a}.\mathbf{a} - 2\mathbf{a}.\mathbf{b} + 2\mathbf{b}.\mathbf{a} - 4\mathbf{b}.\mathbf{b}$ $= \lvert\mathbf{a}\rvert^2 - 4\lvert\mathbf{b}\rvert^2$ since $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$ $= 0$	Most students did well.
(c) [3]	$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}5p\\-2p\\4p\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix} = p\begin{pmatrix}5\\-2\\4\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix} = p\begin{pmatrix}4\\-6\\-8\end{pmatrix} = \frac{4\sqrt{5}}{5}\begin{pmatrix}2\\-3\\-4\end{pmatrix}$ Area of triangle OAC $= \frac{1}{2}\lvert\mathbf{a}\times 2\mathbf{b}\rvert$ $= \lvert\mathbf{a}\times\mathbf{b}\rvert$ $= \frac{4\sqrt{5}}{5}\sqrt{2^2+(-3)^2+(-4)^2} = \frac{4\sqrt{145}}{5}$	Some students did not evaluate $\mathbf{a}\times\mathbf{b}$ to answer the first part of the question. Many students did not rationalise the denominator or simplify the answer.

(d) [2]	Observe that $\overrightarrow{OD} = \mathbf{a}+2\mathbf{b}$ and $\overrightarrow{CA} = \mathbf{a}-2\mathbf{b}$. From **(b)**, OD is perpendicular to CA. In addition, CA and OD, which are the diagonals of the parallelogram $OADC$, bisect each other.  From **(c)**, area of triangle $OAC = \lvert\mathbf{a}\times\mathbf{b}\rvert$. But area of triangle OAC is also $= \frac{1}{2}(CA)\left(\frac{1}{2}OD\right)$ $= \frac{1}{4}(\lvert\mathbf{a}-2\mathbf{b}\rvert)(\lvert\mathbf{a}+2\mathbf{b}\rvert)$ Hence $\lvert\mathbf{a}\times\mathbf{b}\rvert = \frac{1}{4}(\lvert\mathbf{a}+2\mathbf{b}\rvert)(\lvert\mathbf{a}-2\mathbf{b}\rvert)$.	Marks are awarded only if there are: 1. correct discussion on the perpendicular property of $\mathbf{a}+2\mathbf{b}$ and $\mathbf{a}-2\mathbf{b}$. 2. correct discussion on the relationship between the areas, building upon Point 1.

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4 (a) Find $\int \frac{9x}{(2x-1)(x+1)^2}\,dx$. [4]

(b) (i) Differentiate $\frac{1}{x^2+1}$ with respect to x. [1]

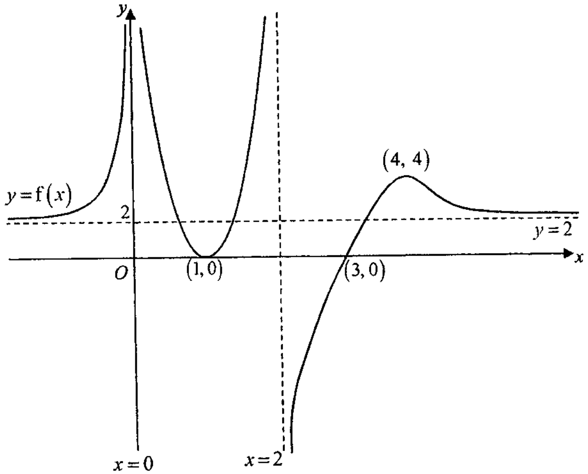
(ii) Differentiate $\ln\sqrt{x^2+1}$ with respect to x. [1]

(iii) Hence find $\int \frac{x\ln\sqrt{x^2+1}}{\left(x^2+1\right)^2}\,dx$. [4]

	Solution	Comments
(a) [4]	Let $\frac{9x}{(2x-1)(x+1)^2}=\frac{A}{2x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$. $9x\equiv A(x+1)^2+B(2x-1)(x+1)+C(2x-1)$ Sub. $x=\frac{1}{2}$, $A=2$. Sub. $x=-1$, $C=3$. Compare the coefficient of x^2, $A+2B=0\Rightarrow B=-1$. $\int\frac{9x}{(2x-1)(x+1)^2}\,dx=\int\frac{2}{2x-1}-\frac{1}{x+1}+\frac{3}{(x+1)^2}\,dx$ $=\ln\lvert 2x-1\rvert-\ln\lvert x+1\rvert-\frac{3}{x+1}+c,\quad c\in\mathbb{R}$ $=\ln\left\lvert\frac{2x-1}{x+1}\right\rvert-\frac{3}{x+1}+c$	Most students did well but many made careless mistakes, which affected the rest of the parts. A handful is still not familiar with splitting into appropriate partial fractions in order to integrate.
(b) (i) [1]	$\frac{d}{dx}\left(\frac{1}{x^2+1}\right)=\frac{-2x}{\left(x^2+1\right)^2}$	Most students did well but some made careless mistake.s
(b) (ii) [1]	$\frac{d}{dx}\ln\sqrt{x^2+1}=\frac{d}{dx}\left[\frac{1}{2}\ln\left(x^2+1\right)\right]=\frac{x}{x^2+1}$	Most students did well but again some made careless mistakes particularly for

		those who did not simplify first using the property of logarithm.
(b) (iii) [4]	$u=\ln\sqrt{x^2+1}=\frac{1}{2}\ln\left(x^2+1\right)$, $\frac{dv}{dx}=\frac{-2x}{(x^2+1)^2}$ $\frac{du}{dx}=\frac{x}{x^2+1}$, $v=\frac{1}{x^2+1}$ $\int\frac{x\ln\sqrt{x^2+1}}{\left(x^2+1\right)^2}\,dx$ $=-\frac{1}{2}\int\frac{-2x}{(x^2+1)^2}\left(\ln\sqrt{x^2+1}\right)dx$ $=-\frac{1}{2}\left[\left(\frac{1}{x^2+1}\right)\ln\sqrt{x^2+1}-\int\left(\frac{1}{x^2+1}\right)\frac{x}{x^2+1}\,dx\right]$ ---integrate by parts $=-\frac{\ln\sqrt{x^2+1}}{2\left(x^2+1\right)}+\frac{1}{2}\int\frac{x}{\left(x^2+1\right)^2}\,dx$ $=-\frac{\ln\sqrt{x^2+1}}{2\left(x^2+1\right)}-\frac{1}{4\left(x^2+1\right)}+c$ $=-\frac{\ln\left(x^2+1\right)+1}{4\left(x^2+1\right)}+c$	Those students who could not do this part fail to observe the role of the previous two parts and the word "hence". As a result, they could not select the correct terms for integration by parts. Still there are many who made careless mistakes in the integrations.

5 **(a)** The diagram below shows the sketch of the graph of $y = f(x)$. The curve passes through the points with coordinates $(1, 0)$ and $(3, 0)$, and has turning points at $(1, 0)$ and $(4, 4)$. The asymptotes are $x = 0$, $x = 2$ and $y = 2$.



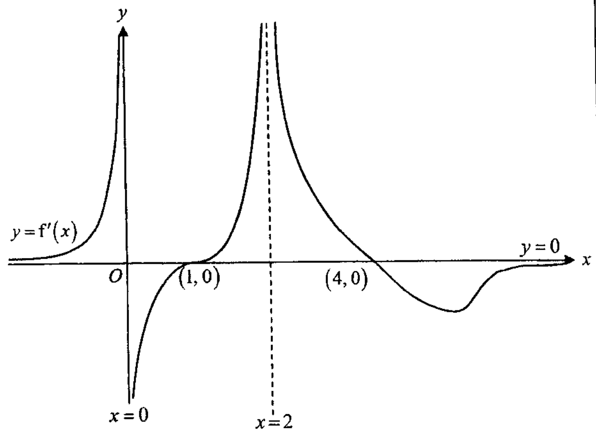
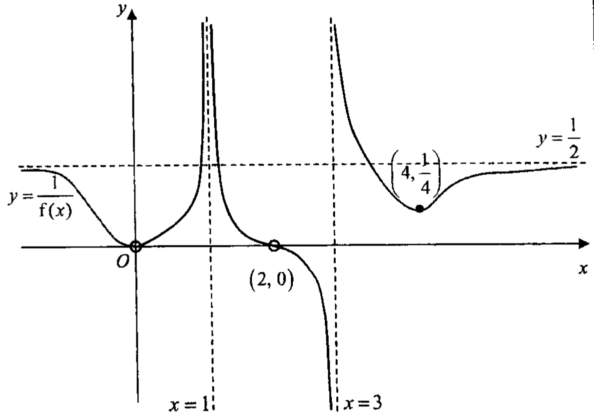
Sketch on separate diagrams, the graphs of

(i) $y = f'(x)$, [3]

(ii) $y = \dfrac{1}{f(x)}$, [3]

showing clearly the main features of the graphs.

(b) Describe a sequence of transformations which transforms the graph of $y = \dfrac{x^2+1}{x}$ to the graph of $y = \dfrac{2x^2-5x+4}{x-2}$. [3]

	Solution	Comments
(a)(i) [3]		Students need to ensure the graphs are drawn proportionally along the axis according to their defined scale. Furthermore, the graphs should visibly demonstrate asymptotic behavior (sketches should tend to the asymptotes when applicable).
(a)(ii) [3]		

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(b) [3]	$y=\frac{x^2+1}{x}$ Replace x by $(x-2)$, $y=\frac{(x-2)^2+1}{(x-2)}=\frac{x^2-4x+5}{x-2}$ **(A)** Translation in the positive x-direction by 2 units Replace y by $\frac{y}{2}$, $\frac{y}{2}=\frac{x^2-4x+5}{x-2}$ $\Rightarrow y=\frac{2x^2-8x+10}{x-2}=\frac{2x^2-5x+4-3(x-2)}{x-2}$ $\Rightarrow y=\frac{2x^2-5x+4}{x-2}-3$ **(B)** Scaling parallel to the y-axis by a factor of 2 Replace y by $(y-3)$, $y-3=\frac{2x^2-5x+4}{x-2}-3\Rightarrow y=\frac{2x^2-5x+4}{x-2}$ **(C)** Translation in the positive y-direction by 3 units **OR** (B) (C) (A) **OR** **(A')** Translation in the positive x-direction by 2 units **(B')** Translation in the positive y-direction by $\frac{3}{2}$ units **(C')** Scaling parallel to the y-axis by a factor of 2 **OR** (B')(A')(C') **Alternative** $y=\frac{x^2+1}{x}=x+\frac{1}{x}$ and $y=\frac{2x^2-5x+4}{x-2}=2x-1+\frac{2}{x-2}$ $y=\frac{x^2+1}{x}=x+\frac{1}{x}$	Students should use proper and accurate phrasing as taught in lectures, such as "translate" instead of "shift/move".

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	Replace x by $(x-2)$, $y=x-2+\frac{1}{x-2}$ Replace y by $\frac{y}{2}$, $y=2(x-2)+\frac{2}{x-2}$ Replace y by $(y-3)$, $y=2(x-2)+\frac{2}{x-2}+3=2x-1+\frac{2}{x-2}$. **(A)** Translation in the positive x-direction by 2 units **(B)** Scaling parallel to the y-axis by a factor of 2 **(C)** Translation in the positive y-direction by 3 units	

6 (a) A geometric series has first term 2 and common ratio 0.8. Find algebraically, the least value of m for the sum of the first $4m$ terms of the series to be greater than 99% of the sum to infinity. [3]

(b) A finite arithmetic progression A has n terms, first term a and common difference d.
In another progression B, the kth term is obtained by adding the kth positive odd integer to the corresponding term of A. That is, the first term of B is obtained by adding 1 to the first term of A, the second term of B is obtained by adding 3 to the second term of A and so on.

It is given that the twelfth term of A is 25, the sum of all the terms of A is 676 and the sum of all the terms of B is twice the sum of all the terms of A.

(i) Find the values of n, a and d. [4]

(ii) Obtain the sum of the first ten terms of B. [2]

	Solution	Comments
(a) **[3]**	$S_{4m}=\dfrac{2\left(1-0.8^{4m}\right)}{0.2}=10\left(1-0.8^{4m}\right)$ Sum to infinity, $S=\dfrac{2}{1-0.8}=10$ $S_{4m}>0.99S$ $10\left(1-0.8^{4m}\right)>0.99(10)$ $0.8^{4m}<0.01$ $4m>\dfrac{\ln 0.01}{\ln 0.8}=20.637$ Since $4m$ is a positive integer, $4m\ge 21$ $\therefore m\ge\dfrac{21}{4}$ or 5.25 Least $m=\dfrac{21}{4}$.	Candidates often overlooked that the question required you to solve **algebraically**. Using a table to solve for values of m is not sufficient. In addition, the question does not state that m itself must be an integer—only that $4m$ must be an integer, since it refers to the "first $4m$ terms". Hence, answer like $m=6$ is not accepted. Unless otherwise specified, we should work with the largest possible set of values of m that satisfy the given conditions.

(b)(i) **[4]**	Sum of all the terms of the arithmetic progression A, $\dfrac{n}{2}\left[2a+(n-1)d\right]=676$ --- (1) Sum of all the terms of the arithmetic progression B, $\dfrac{n}{2}\left[2a+(n-1)d\right]+1+3+\cdots+2n-1=2(676)$ Applying (1), $\dfrac{n}{2}[1+2n-1]=676$ $\Rightarrow n=26$ Alternatively, since the kth term of B is obtained by adding the kth positive odd integer to the corresponding term of A and this results in the sum of the first n terms of B to be twice the sum of the first n terms of A, the sum of the increases is 676. Hence $\dfrac{n}{2}\left[2(1)+(n-1)2\right]=676\Rightarrow n=26$ Substitute into (1), $\dfrac{26}{2}(2a+25d)=676$ $\Rightarrow 2a+25d=52$ --- (2) $u_{12}=a+11d=25$ --- (3) From GC, $a=\dfrac{53}{3}$ and $d=\dfrac{2}{3}$.	Most candidates started well by correctly formulating the equations for the sums of the two progressions. However, many who tried to equate the two directly ended up with messy algebra and were not successful. Candidates who used **subtraction or substitution** tended to be more successful, though a significant number still made algebraic slips and did not reach the correct answers.
(b)(ii) **[2]**	B is an arithmetic progression with first term $(a+1)$ and common difference $(d+2)$. Sum of the first ten terms of B $=\dfrac{10}{2}\left[2\left(\dfrac{56}{3}\right)+9\left(\dfrac{8}{3}\right)\right]=\dfrac{920}{3}$ Or Sum of the first ten terms of B = Sum of the first 10 terms of A + sum of the first 10 positive odd integers	A significant number of candidates who obtained the correct answers in Part (b)(i) went on to make mistakes in computing the sum here. Common errors included: 1. Mixing up the values of a and d, 2. Reusing 676 as the sum even

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	$=\frac{10}{2}\left[2\left(\frac{53}{3}\right)+9\left(\frac{2}{3}\right)\right]+\frac{10}{2}[2(1)+9(2)]=\frac{920}{3}$	though this part asked for the sum of the **first 10 terms**, not all the terms, 3. Substituting $n=26$ instead of $n=10$.

7 The function f is defined by $\mathrm{f}: x \mapsto 1+\frac{3}{\mathrm{e}^x-2}$ for $x \in \mathbb{R}, x>\ln 2$.

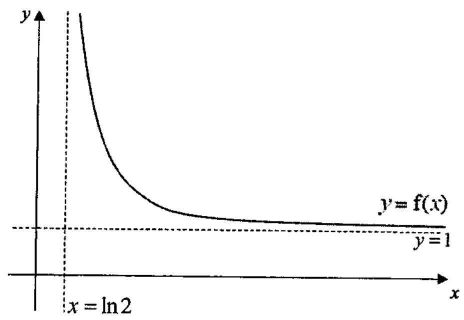
(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

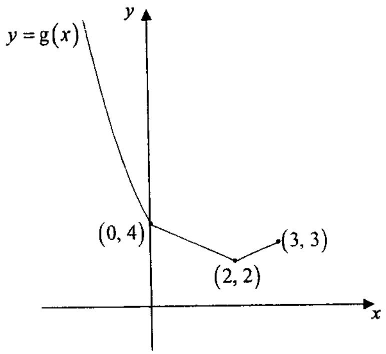
Another function g is defined by

$$\mathrm{g}: x \mapsto \begin{cases} 2(x-1)^2+2 & \text{for } x \le 0, \\ 2+|2-x| & \text{for } 0<x\le 3. \end{cases}$$

(b) Sketch the graph of $y=\mathrm{g}(x)$ and explain why the composite function $\mathrm{f}^{-1}\mathrm{g}$ exists. [4]

(c) Hence find the exact value of k such that $\mathrm{f}^{-1}\mathrm{g}(k)=\ln\frac{5}{2}$. [3]

	Solution	Comments
(a) **[3]**	 $\mathrm{R_f}=(1,\infty)$	Graph is not required but presented here to show how the domain of f^{-1} (which is the range of f) is obtained.
	Let $y=1+\frac{3}{\mathrm{e}^x-2}$ $\Rightarrow \mathrm{e}^x-2=\frac{3}{y-1}$ $\Rightarrow x=\ln\left(2+\frac{3}{y-1}\right)$ $\mathrm{f}^{-1}(x)=\ln\left(2+\frac{3}{x-1}\right),\ x\in\mathbb{R},\ x>1$	Do not write $x=\ln\left\lvert 2+\frac{3}{y-1}\right\rvert$ as the modulus sign should be removed, since $\mathrm{R_f}=(1,\infty)$

<table>
<tr><td></td><td>$D_{f^{-1}} = (1, \infty)$</td><td>$\Rightarrow y > 1 \Rightarrow y-1>0$
$\Rightarrow 2+\dfrac{3}{y-1} > 2 > 0$
Students who left their answer with the modulus sign do not obtain full credit.</td></tr>
<tr><td>(b)
[4]</td><td>

Since $R_g = [2, \infty) \subseteq D_{f^{-1}} = (1, \infty)$, $f^{-1}g$ exists.</td><td>At $(0, 4)$, the gradient is undefined as $\lim_{x\to 0^-} g'(x) = -4$ while $\lim_{x\to 0^+} g'(x) = -1$.
Hence the curve has a sharp point at $(0, 4)$.

Use mathematical terms like 'subset' and not phrases like 'lies within'.</td></tr>
<tr><td>(c)
[3]</td><td>$f^{-1}g(k) = \ln\frac{5}{2}$
$\Rightarrow g(k) = f\left(\ln\frac{5}{2}\right) = 1 + \dfrac{3}{\frac{5}{2}-2} = 7 > 4$
Therefore,
$g(k) = 2(k-1)^2 + 2 = 7$
$\Rightarrow (k-1)^2 = \frac{5}{2}$
$\Rightarrow k = 1-\sqrt{\frac{5}{2}}$ since $k \le 0$
Hence $k = 1-\sqrt{\frac{5}{2}} = 1-\frac{\sqrt{10}}{2}$.</td><td>Need to state explicitly, the value of k.</td></tr>
</table>

<table>
<tr><td></td><td>*Alternative method:*
$f^{-1}g : x \mapsto \begin{cases} \ln\left(2+\dfrac{3}{2(x-1)^2+1}\right) & \text{for } x \le 0, \\ \ln\left(2+\dfrac{3}{1+\lvert 2-x\rvert}\right) & \text{for } 0 < x \le 3. \end{cases}$
For $k \le 0$,
$f^{-1}g(k) = \ln\frac{5}{2}$
$\ln\left(2+\dfrac{3}{2(k-1)^2+1}\right) = \ln\frac{5}{2}$
$\dfrac{3}{2(k-1)^2+1} = \frac{1}{2}$
$2(k-1)^2+1 = 6$
$(k-1)^2 = \frac{5}{2}$
$k = 1-\sqrt{\frac{5}{2}}$ (reject $k = 1+\sqrt{\frac{5}{2}}$ since $k \le 0$)

For $0 < k \le 3$,
$f^{-1}g(k) = \ln\frac{5}{2}$
$\ln\left(2+\dfrac{3}{1+\lvert 2-k\rvert}\right) = \ln\frac{5}{2}$
$\dfrac{3}{1+\lvert 2-k\rvert} = \frac{1}{2}$
$1+\lvert 2-k\rvert = 6$
$\lvert 2-k\rvert = 5$
$2-k = \pm 5$
$k = -3$ or 7 (reject both as $0 < k \le 3$)
Hence $k = 1-\sqrt{\frac{5}{2}} = 1-\frac{\sqrt{10}}{2}$.</td><td>The alternative method (not recommended) is presented here for students who attempted to solve this way.</td></tr>
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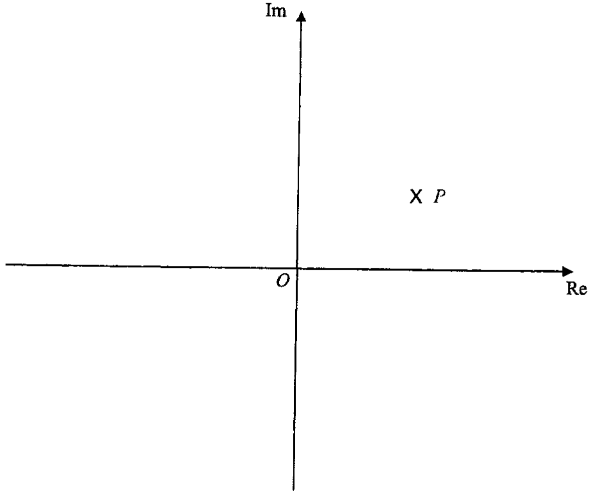
8 **Do not use a calculator in answering this question.**

(a)



The point P on the Argand diagram represents the complex number w given by $w = u + \mathrm{i}v$, where u and v are real and positive.

(i) Explain algebraically why $\arg(kw) = \arg(w)$ for any real constant $k > 1$. [1]

Points Q and R represent the complex numbers kw and $\mathrm{i}kw$, where k is a real constant, and $k > 1$.

(ii) On the same Argand diagram in the Printed Answer Booklet, plot the points Q and R. Show clearly the geometrical relationship between the points P, Q and R. [2]

(b) In another Argand diagram, the points A, B and C represent the complex numbers z, $\mathrm{f}(z)$ and $\mathrm{f}(\mathrm{f}(z))$ respectively, where $\mathrm{f}(z) = z^2 - 2z$.

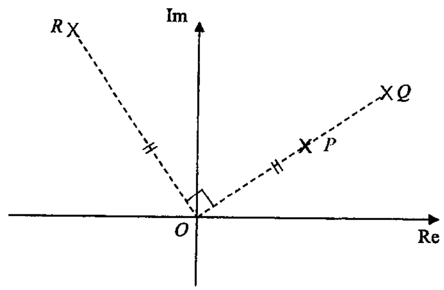
(i) Show that $\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z) = z(z-2)(z-3)(z+1)$. [2]

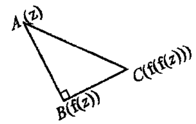
It is given that ABC is a right-angled triangle, described in an anticlockwise sense, with a right angle at B, and $BC = mBA$, where m is a positive real constant.

(ii) By considering $\dfrac{\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z)}{\mathrm{f}(z) - z}$, or otherwise, show that

$$(z-2)(z+1) = m\mathrm{i}.$$ [2]

(iii) In the case where $z = x + 2\mathrm{i}$, where x is a positive real number, find x and m. Hence obtain the complex number represented by the point B. [5]

	Solution	Comments
(a)(i) **[1]**	Since $w = u + \mathrm{i}v$, $u > 0$ and $v > 0$, we have, $kw = ku + \mathrm{i}kv$, $ku > 0$ and $kv > 0$. Therefore, $\arg(w) = \tan^{-1}\frac{v}{u}$ $\arg(kw) = \tan^{-1}\frac{kv}{ku} = \tan^{-1}\frac{v}{u} = \arg(w)$	Note that algebraically means that the definition of $\arg(w)$ needs to be given. Those who use the property $\arg(kw)$ $= \arg(k) + \arg(w)$ need to state that $\arg(k) = 0$ since $k > 0$.
(a)(ii) **[2]**		The geometrical relationship is that OR is perpendicular to OQ. This needs to be shown on the diagram. Also, since $k > 1$, $OQ > OP$. Since OR is the rotation of OQ 90° anticlockwise about the origin, then $OR = OQ$.
(b)(i) **[2]**	$\mathrm{f}(\mathrm{f}(z)) - \mathrm{f}(z) = (z^2-2z)^2 - 2(z^2-2z) - (z^2-2z)$ $= (z^2-2z)(z^2-2z-3)$ $= z(z-2)(z-3)(z+1)$	Note that this question does not allow the use of a calculator, so

		factorization needs to be shown clearly.
(b)(ii) **[2]**	 Rotate $\overrightarrow{BC}$ anticlockwise by $\frac{\pi}{2}$ about B to obtain $m\overrightarrow{BA}$: $\mathrm{i}\left[\mathrm{f}(\mathrm{f}(z))-\mathrm{f}(z)\right]=m(z-\mathrm{f}(z))$ $\frac{\mathrm{f}(\mathrm{f}(z))-\mathrm{f}(z)}{\mathrm{f}(z)-z}=-\frac{m}{\mathrm{i}}$ $\frac{z(z-2)(z-3)(z+1)}{z(z-3)}=m\mathrm{i}$ $(z-2)(z+1)=m\mathrm{i}$	The explanation of rotating $\overrightarrow{BC}$ 90° anticlockwise about B to get $m\overrightarrow{BA}$ needs to be given. The directions of the "vectors" need to be noted – just stating that BC is perpendicular to AB is not sufficient to explain for the "i" factor. No marks are awarded for the simplification of $\frac{z(z-2)(z-3)(z+1)}{z(z-3)}$ $=(z-2)(z+1)$ Note that $\frac{BC}{AB}=m\mathrm{i}$ is incorrect: LHS is a ratio of lengths, RHS is a complex number.
(b) **(iii)** **[5]**	Since $z=x+2\mathrm{i}$, $(z-2)(z+1)=m\mathrm{i}$ $z^2-z-2=m\mathrm{i}$ $(x+2\mathrm{i})^2-(x+2\mathrm{i})-2=m\mathrm{i}$ $x^2-x-6+\mathrm{i}(4x-2)=m\mathrm{i}$ Compare real and imaginary parts, $x^2-x-6=0$ ---(1) $4x-2=m$ ---(2) From (1), $(x-3)(x+2)=0\Rightarrow x=3$ or -2 (rej. since x is positive).	There are quite many sign errors in the expansion of the 4th line. Students are reminded to check their working and write clearly so that careless mistakes would not lead to subsequent deductions being incorrect.

	Sub. (2), $m=10$. $\mathrm{f}(3+2\mathrm{i})=(3+2\mathrm{i})^2-2(3+2\mathrm{i})$ $=9+12\mathrm{i}-4-6-4\mathrm{i}$ $=-1+8\mathrm{i}$ B represents the complex number $-1+8\mathrm{i}$.	

BP-504

9 Decontamination is a water treatment method to remove a certain undesirable chemical from contaminated water. As the liquid agent is continuously added to the contaminated water in a large tank, the undesirable chemical in the water is removed gradually.

The rate of decrease of the concentration of the undesirable chemical is proportional to the product of its current concentration and the rate at which the liquid agent flows into the tank. At time t minutes after the start of the treatment process, the concentration of the undesirable chemical is C mg/L and the liquid agent flows into the tank at a rate of $\frac{1}{1+t^2}$ L/min. It is known that the initial concentration of the undesirable chemical in the tank is 50 mg/L and after 10 minutes, its concentration is measured to be 20 mg/L.

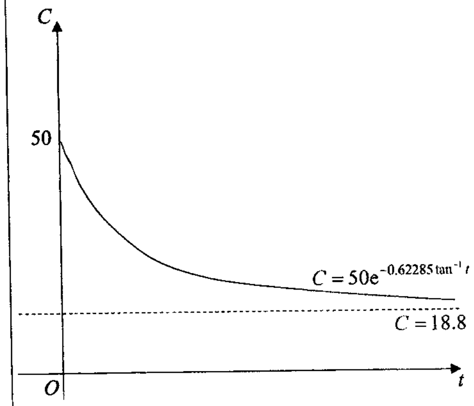
(a) By setting up and solving a differential equation relating C and t, show that $C = 50e^{q\tan^{-1}t}$, giving the value of q correct to 5 decimal places. [7]

(b) Determine the value of C when $t = 50$. [1]

(c) Sketch the graph of C against t and state what happens to C in the tank for large values of t. [3]

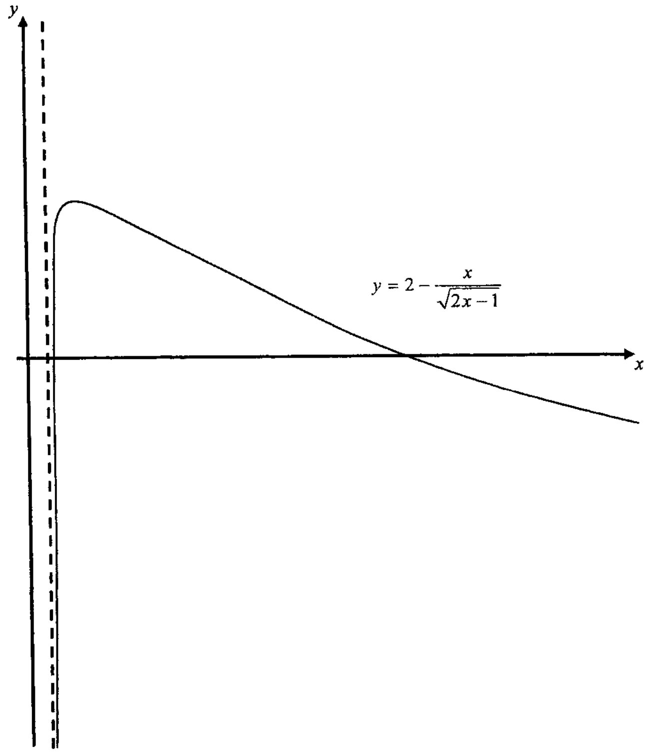
	Solution	Comments
(a) **[7]**	$\frac{dC}{dt} = -kC\left(\frac{1}{1+t^2}\right)$ where $k>0$. (Note: Since $C=50$ when $t=0$, $C\neq 0$.) $\int \frac{1}{C}\,dC = -k\int \frac{1}{1+t^2}\,dt$ $\ln\lvert C\rvert = -k\tan^{-1}t + c_1$, where $c_1 \in \mathbb{R}$ $\lvert C\rvert = e^{-k\tan^{-1}t+c_1}$ As $C>0$, $C = c_2e^{-k\tan^{-1}t}$, where $c_2 = e^{c_1}$ (Alternatively, $C=50$ when $t=0 \Rightarrow \lvert C\rvert = C$) When $t=0$ and $C=50$, $50 = c_2e^0$ $c_2 = 50$ $C = 50e^{-k\tan^{-1}t}$	Choice of ***constant of proportionality***: It is more convenient to write this as "$-k$" and work with a positive k, noting that the required value of $q=-k$. $\int \frac{1}{C}dC = \ln\lvert C\rvert + d$. A reminder that "$\lvert C\rvert$" is expected in general, unless the student explicitly discussed that $C>0$ based on contextual clue. Algebraic manipulation leading to $C = c_2e^{-k\tan^{-1}t}$ ought to be clearly shown. Choice of ***constant of integration***:

	When $t=10$ and $C=20$, $20 = 50e^{-k\tan^{-1}10}$ $e^{-k\tan^{-1}10} = \frac{2}{5}$ $-k\tan^{-1}10 = \ln\frac{2}{5}$ $k = -\frac{1}{\tan^{-1}10}\ln\frac{2}{5} \approx 0.62285$ $C = 50e^{-0.62285\tan^{-1}t}$ where $q = -0.62285$ (5 d.p.) (shown).	Since the letter C is already used in the question as a variable, it is preferable to choose another letter. The value of q should be stated to 5 dp. Many errors arose from using insufficient accuracy in intermediate steps. *** In Calculus, all angles must be in **radians**, not degrees. Hence, $\tan^{-1}(10) = 1.471127674$ (and <u>**not**</u> 84.28940686). Answer like $C = 50e^{-0.01087\tan^{-1}t}$ is a clear sign that the wrong unit was being used for angle measurement.
(b) **[1]**	When $t=50$, $C = 50e^{-0.62285\tan^{-1}50}$ $= 19.0317 = 19.0$ (correct to 3sf)	This is a good example of how incorrect working can still appear to give the "correct" final answer. Working such as $C = 50e^{-0.01087\tan^{-1}(50)}$ $= 19.0$ (3 sf) <u>**cannot**</u> be accepted. The value "-0.01087" arose because the wrong unit was used in earlier steps. The student must have repeated the calculation using the same incorrect unit (for $\tan^{-1}(50)$), which happened to produce the expected numerical value.

(c) [3]	 Therefore C decreases and **approaches** 18.8 mg/L for large values of t. **Note** (for reference only): When $t\to\infty$, $\tan^{-1}t\to\frac{\pi}{2}$ and $C\to 50e^{-0.62285(\pi/2)}=18.8$.	The sketch should be drawn only for the range $t\geq 0$, as required by the context of the question. Axial intercept and the horizontal asymptote ($y=m$, with m explicitly evaluated.) should be labelled. For questions that ask *"what happens ..."* or *"explain ..."*, the response should be given in **words**, taking the context into account, rather than only using mathematical expressions. In describing trend, it is important to mention whether the function is increasing or decreasing, and, where a limiting value exists, to state that value clearly.

10 **(a)** Using the substitution $u=\sqrt{2x-1}$, show that $\int \frac{x}{\sqrt{2x-1}}\,dx=\int \frac{u^2+1}{2}\,du$. [2]

(b) **(i)** The diagram below shows the graph of $y=2-\frac{x}{\sqrt{2x-1}}$. Indicate on the same diagram in the Printed Answer Booklet, the equation of the asymptote of the curve and the coordinates of its turning point and points of intersection with the x-axis. [2]



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(ii) The region R is bounded by the curve $y=2-\frac{x}{\sqrt{2x-1}}$ and the lines $x=1$, $x=3$ and $y=\frac{1}{2}$. Find the exact area of R. [5]

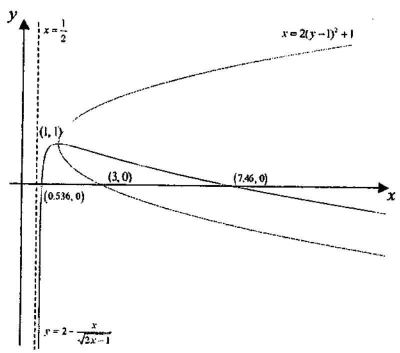
(c) The region S is bounded by the curves $y=2-\frac{x}{\sqrt{2x-1}}$, $x=2(y-1)^2+1$, the lines $x=1$, $x=4$ and the x-axis.

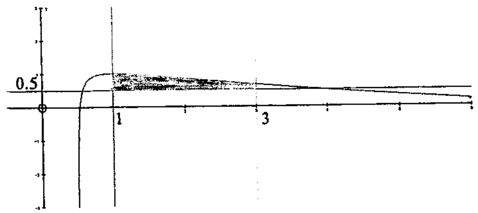
(i) Express $x=2(y-1)^2+1$ in the form $y=f(x)$. [1]

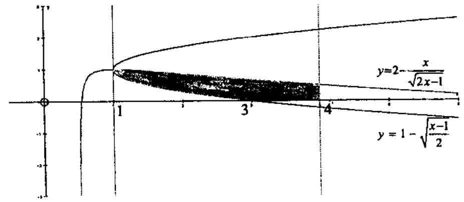
(ii) On the same diagram as in part **(b)(i)**, sketch the graph of $x=2(y-1)^2+1$. [2]

(iii) Find the volume of the solid generated when S is rotated through 2π radians about the x-axis. Give your answer correct to 3 decimal places. [3]

	Solution	Comments
(a) **[2]**	$u=\sqrt{2x-1}\Rightarrow u^2=2x-1$ Hence $u\frac{du}{dx}=1\Rightarrow\frac{dx}{du}=u$. $\int\frac{x}{\sqrt{2x-1}}dx=\int\frac{\frac{u^2+1}{2}}{u}\left(\frac{dx}{du}\right)du=\int\frac{u^2+1}{2u}(u)du=\int\frac{u^2+1}{2}du$ (shown) Alternatively, $u=\sqrt{2x-1}\Rightarrow\frac{du}{dx}=\frac{1}{\sqrt{2x-1}}$ and therefore $\int\frac{x}{\sqrt{2x-1}}dx=\int x\cdot\frac{1}{\sqrt{2x-1}}dx=\int\frac{u^2+1}{2}\left(\frac{du}{dx}\right)dx=\int\frac{u^2+1}{2}du$	As this is a show question, simply differentiating the substitution given and quoting what you are asked to show $\int\frac{x}{\sqrt{2x-1}}dx$ $=\int\frac{u^2+1}{2}du$ is insufficient.

(b)(i) **[2]** **(c)(ii)** **[2]**	 **Note:** Detailed workings for intercepts and turning points (Optional, since more efficient to use GC) When $y=0$, $\frac{x}{\sqrt{2x-1}}=2$ $\Rightarrow x^2=4(2x-1)$ $\Rightarrow x=4\pm2\sqrt{3}$ $\frac{dy}{dx}=-\frac{\sqrt{2x-1}-\frac{x}{\sqrt{2x-1}}}{2x-1}=\frac{1-x}{\sqrt{(2x-1)^3}}=0\Rightarrow x=1\Rightarrow y=1$	**(b)(i)** Do take note that there are 2 intercepts, it is quite common for students to miss out 0.536 as it might not be shown clearly on the GC (but it is clearly there in the sketch in the answer booklet!) **(c)(ii)** For the sketch of $x=2(y-1)^2+1$, to obtain any credit, you should recognize that there are 2 branches, with the vertex at (1, 1). Students who do so should also pay attention that the position of the x-intercept 3 should be considered carefully in relation to the origin and the x-intercept of the first curve which is 7.46.

(b)(ii) **[5]**	 Required area of R $=\int_1^3\left(2-\frac{x}{\sqrt{2x-1}}-\frac{1}{2}\right)\mathrm{d}x$ $=\int_1^3\frac{3}{2}\mathrm{d}x-\int_1^3\frac{x}{\sqrt{2x-1}}\mathrm{d}x$ $=\frac{3}{2}[x]_1^3-\int_1^{\sqrt{5}}\frac{u^2+1}{2}\mathrm{d}u$ $=3-\frac{1}{2}\left[\frac{u^3}{3}+u\right]_1^{\sqrt{5}}$ $=3-\frac{1}{2}\left[\frac{5\sqrt{5}}{3}+\sqrt{5}-\frac{1}{3}-1\right]$ $=3-\left[\frac{4\sqrt{5}}{3}-\frac{2}{3}\right]$ $=\frac{11-4\sqrt{5}}{3}$ $u=\sqrt{2x-1}$ When $x=3$, $u=\sqrt{5}$ When $x=1$, $u=\sqrt{1}=1$	It is easiest to use the GC to sketch the graph and the line $y=\frac{1}{2}$ to ascertain the desired region correctly. Be careful that the result in (a) only applies to $\int\frac{x}{\sqrt{2x-1}}\mathrm{d}x$ $=\int\frac{u^2+1}{2}\mathrm{d}u$ Hence the constants should not be affected by the change in the limits. The final answer should be simplified whenever possible if an exact answer is required, in particular the surds in this case are all of the form $\sqrt{5}$.
(c) **(i)** **[1]**	$x=2(y-1)^2+1\Rightarrow y=1\pm\sqrt{\frac{x-1}{2}}$	This is a quadratic in y. Hence when expressing y in terms of x, there are 2 possible values.

(c) **(ii)** **[2]**	Please refer to part (b)(i)	
(c) **(iii)** **[3]**	 Required volume $=\pi\int_1^4\left(2-\frac{x}{\sqrt{2x-1}}\right)^2\mathrm{d}x-\pi\int_1^3\left(1-\sqrt{\frac{x-1}{2}}\right)^2\mathrm{d}x$ $=4.518334$ $=4.518$ units^3 (3 d.p.)	A very common mistake is that the difference between two volumes is evaluated as $\pi\int_a^b(y_1-y_2)^2\mathrm{d}x$ This is a bad conceptual error. The correct way to understand is it is the difference of 2 volumes and thus $\pi\int_a^b(y_1)^2\mathrm{d}x$ $-\pi\int_c^d(y_2)^2\mathrm{d}x$ As the question does not require an exact answer, you should immediately use the GC to evaluate the definite integral and not waste time evaluating it algebraically.

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11 A curve C has parametric equations

$$x = 2\cos\theta + \cos 2\theta,\ y = 2\sin\theta - \sin 2\theta,$$

where $0 < \theta < 2\pi$ and $\theta \neq \frac{2\pi}{3}, \frac{4\pi}{3}$.

The point P with parameter p lies on C, and the lines M_p and N_p are the tangent and normal to C at P respectively.

(a) It is given that

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \text{ and}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Using these two results, show that the gradient of M_p is $-\tan\left(\frac{p}{2}\right)$. [2]

(b) For $p = \frac{\pi}{3}$, M_p and N_p cut the x-axis at points Q and R respectively. Find the area of ΔPQR. [4]

(c) The point S with parameter s lies on C, and the line N_s is the normal to C at S. It is given that M_p and N_s are parallel.

Given $p \neq \frac{\pi}{3}, \pi$ and $p < s$, by using the result $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$, find an equation relating p and s. [2]

	Solution	Comments
(a) [2]	$x = 2\cos\theta + \cos 2\theta \Rightarrow \frac{dx}{d\theta} = -2\sin\theta - 2\sin 2\theta$ $y = 2\sin\theta - \sin 2\theta \Rightarrow \frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$	**Note 1** Since this is a 'SHOW' question, working presented needs to be complete and with clear explanation. No steps should be skipped or assumed, and in doing so it may result in losing marks. The most glaring 'error' was writing

	Solution	Comments
	$\therefore \frac{dy}{dx} = \frac{-2(\cos 2\theta - \cos\theta)}{-2(\sin 2\theta + \sin\theta)}$ $= \frac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$ (factor formula for $\cos 2\theta - \cos\theta$ and $\sin 2\theta + \sin\theta$) $= -\tan\left(\frac{\theta}{2}\right)$ Alternatively, $\therefore \frac{dy}{dx} = \frac{2(\cos\theta - \cos 2\theta)}{-2(\sin 2\theta + \sin\theta)}$ $= \frac{-2\sin\left(\frac{3\theta}{2}\right)\sin\left(-\frac{\theta}{2}\right)}{-2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$ (factor formula for $\cos\theta - \cos 2\theta$ and $\sin 2\theta + \sin\theta$) $= \frac{\sin\left(-\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} = -\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$ ($\sin\left(-\frac{\theta}{2}\right) = -\sin\left(\frac{\theta}{2}\right)$ as the sine function is an odd function) $= -\tan\left(\frac{\theta}{2}\right)$ When $\theta = p$, $\frac{dy}{dx} = -\tan\left(\frac{p}{2}\right)$	$\cos\theta - \cos 2\theta = 2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$ without any explanation **Note 2** In this question θ is the variable and p is a constant that θ takes on at different points on C. As such, any differentiation should be done in terms of θ $\left(\frac{dx}{d\theta} \text{ and } \frac{dy}{d\theta}\right)$, and not p $\left(\frac{dx}{dp} \text{ and } \frac{dy}{dp}\right)$. **Note 3** Remember to answer the question and find an expression for the gradient of the tangent line at $\theta = p$.
(b) [4]	At $p = \frac{\pi}{3}$: $x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}, \frac{dy}{dx} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ Equation of M_P: $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right) \Rightarrow y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}} \Rightarrow Q(2,0)$ Equation of N_P: $y - \frac{\sqrt{3}}{2} = \sqrt{3}\left(x - \frac{1}{2}\right) \Rightarrow y = \sqrt{3}x \Rightarrow R(0,0)$ Area of $\Delta PQR = \frac{1}{2}(2)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$ units^2	**Note 1** In this part, $p = \frac{\pi}{3}$, and so the gradient and point P should be evaluated instead of being expressed as the trigo expressions, making the working very cumbersome.

		Note 2 Always read the question carefully. In this case, Q and R lie on the x-axis and hence should be found by letting $y=0$, instead of $x=0$. **Note 3** It may be useful to draw a diagram relating the 3 points P, Q and R to identify how to find the area of the triangle.
(c) **[2]**	Gradient of $M_p = -\tan\left(\frac{p}{2}\right)$ Gradient of $N_s = \frac{1}{\tan\left(\frac{s}{2}\right)}$ Since M_p and N_s are parallel, $\frac{1}{\tan\left(\frac{s}{2}\right)} = -\tan\left(\frac{p}{2}\right) \Rightarrow 1+\tan\left(\frac{s}{2}\right)\tan\left(\frac{p}{2}\right)=0$ $\Rightarrow \tan\left(\frac{s}{2}-\frac{p}{2}\right)$ is undefined $\Rightarrow \frac{s}{2}-\frac{p}{2}=\frac{\pi}{2}$ $\Rightarrow s-p=\pi$ **Note 2** By considering $1+\tan\left(\frac{s}{2}\right)\tan\left(\frac{p}{2}\right)=\frac{\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)}{\tan\left(\frac{s}{2}-\frac{p}{2}\right)}=0$, note that $\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)=0 \Rightarrow \frac{s}{2}=\frac{p}{2}+\pi \Rightarrow \tan\left(\frac{s}{2}-\frac{p}{2}\right)=0$, which	**Note 1** When 2 **lines** are **parallel**, their **gradients are equal**. Some mistakenly thought that the gradients would be a multiple of each other, possibly confusing it with the idea of parallel lines when the direction of one line is a multiple of the other line. **Note 2** See note on the left

	makes the term $\frac{\tan\left(\frac{s}{2}\right)-\tan\left(\frac{p}{2}\right)}{\tan\left(\frac{s}{2}-\frac{p}{2}\right)}$ undefined instead of 0. Hence we need only to consider $\tan\left(\frac{s}{2}-\frac{p}{2}\right)$ to be undefined as stated in the solution above.	

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RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 6

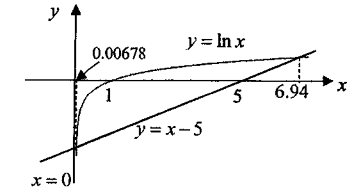
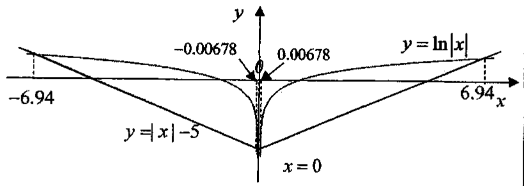
H2 Math Year 6 Preliminary Examination Paper 2: Solutions with comments

1 (a) Without using a calculator, solve the inequality $\frac{x^2-5x+6}{x^2-4}<\frac{2x-3}{x+2}$. [4]

(b) (i) Sketch on the same diagram the graphs of $y=\ln x$ and $y=x-5$, giving the equations of any asymptotes and the x-coordinates of the points of intersection between the two graphs. [2]

(ii) Hence solve the inequality $\ln|x|<|x|-5$. [2]

	Solution	Comments
(a) [4]	$\frac{x^2-5x+6}{x^2-4}<\frac{2x-3}{x+2}$, $x\neq\pm2$ $\frac{(x-2)(x-3)}{(x-2)(x+2)}<\frac{2x-3}{x+2}$ $\frac{x-3}{x+2}<\frac{2x-3}{x+2}$ $\frac{x-3-2x+3}{x+2}<0$ $\frac{-x}{x+2}<0$ $x(x+2)>0$ $x>0$ or $x<-2$ and $x\neq2$ OR $x<-2$ or $0<x<2$ or $x>2$	A number of students did not simplify the LHS to $\frac{x-3}{x+2}$ and so arrive at a more complicated inequality $x(x-2)^2(x+2)>0$. Most of them managed to arrive at the second form of the solution from here. Many students who gave the first form of the solution omitted to exclude 2 from the answer. A few students cross multiplied and did not get any mark for this part.

	Solution	Comments
(bi) [2]	 From GC, x-coordinates of intersection are $x=6.94$ (3sf) or $x=0.00678$ (3sf)	The graph of $y=\ln x$ was not well drawn with some students copying directly from the GC to have a hanging graph that does not extend to negative infinity. Some students even put in a horizontal asymptote of $y=e$! Students are reminded to round off the answers to 3 sf.
(bii) [2]	 For $\ln\|x\|<\|x\|-5$, $x<-6.94$ or $-0.00678<x<0$ or $0<x<0.00678$ or $x>6.94$ OR $x<-6.94$ or $-0.00678<x<0.00678$ or $x>6.94$ and $x\neq0$ <u>Alternative Method</u> For $\ln x<x-5$, the solution is $0<x<0.00678$ or $x>6.94$. To solve $\ln\|x\|<\|x\|-5$, we replace x by $\|x\|$. The solution is then $x<-6.94$ or $-0.00678<x<0$ or $0<x<0.00678$ or $x>6.94$	As this part carries 2 marks, one mark is for some explanation of the method used and one mark is for the completely correct answer. Hence students who put down an answer with no explanation will not get any mark unless all the intervals are correct. Students should remember that disjoint intervals should be separated by the conjunction "or" and not a comma.

2 **(a)** In a triangle ABC, $AB = 2$, angle $CAB = x$ radians and angle $CBA = \frac{\pi}{6}$ radians.

(i) Show that $AC = \dfrac{2}{\cos x + \sqrt{3}\sin x}$. [2]

(ii) Given that x is a sufficiently small angle, show that

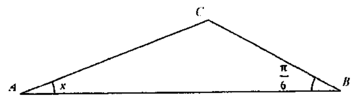
$$AC \approx a + bx + cx^2,$$

where a, b and c are constants to be determined. [3]

(b) It is given that $2xy + \ln y = \ln 3$. Show that

$$(2xy^2 + y)\frac{d^2y}{dx^2} + 4y^2\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = 0.$$

Hence find the Maclaurin series for y, up to and including the term in x^2. [5]

	Solution	Comments
(ai) [2]	 $\angle ACB = \pi - \frac{\pi}{6} - x = \frac{5\pi}{6} - x$ Using Sine Rule: $\dfrac{AC}{\sin\frac{\pi}{6}} = \dfrac{2}{\sin\left(\frac{5\pi}{6} - x\right)} = \dfrac{2}{\sin\frac{5\pi}{6}\cos x - \cos\frac{5\pi}{6}\sin x}$ $\dfrac{AC}{\frac{1}{2}} = \dfrac{2}{\frac{1}{2}\cos x - \left(-\frac{\sqrt{3}}{2}\right)\sin x}$ $AC = \dfrac{2}{\cos x + \sqrt{3}\sin x}$ (shown)	Most students were able to use the sine rule to arrive at the first step. However, there was a number of students who calculated $\angle ACB$ wrongly: • $\angle ACB = 2\pi - \frac{\pi}{6} - x = \frac{11\pi}{6} - x$ • $\angle ACB = \frac{\pi}{2} - \frac{\pi}{6} - x = \frac{\pi}{3} - x$ Students are reminded to give details in their working for a "show" question and thus they are required to give the following steps: $\dfrac{1}{\sin\frac{5\pi}{6}\cos x - \cos\frac{5\pi}{6}\sin x}$ $= \dfrac{1}{\frac{1}{2}\cos x - \left(-\frac{\sqrt{3}}{2}\right)\sin x}$ $= \dfrac{2}{\cos x + \sqrt{3}\sin x}$
(aii) [3]	When x is sufficiently small, $AC = \dfrac{2}{\left(1 - \frac{1}{2}x^2 + \ldots\right) + \sqrt{3}x + \ldots}$	This part proves to be the most challenging as a number of students did not get the hint to use "small angle

	$= 2\left[1 + \left(\sqrt{3}x - \frac{1}{2}x^2 + \ldots\right)\right]^{-1}$ $= 2\left[1 - \left(\sqrt{3}x - \frac{1}{2}x^2\right) + \left(\sqrt{3}x - \frac{1}{2}x^2\right)^2 + \ldots\right]$ $= 2\left(1 - \sqrt{3}x + \frac{1}{2}x^2 + 3x^2 + \ldots\right)$ $\approx 2 - 2\sqrt{3}x + 7x^2$ where $a = 2$, $b = -2\sqrt{3}$ and $c = 7$.	approximation" when question said "x is a sufficiently small angle". For those who knew to use small angle approximation, some got the wrong approximation for cosine. After applying small angle approximation correctly, there was a large number of students who did not know how to proceed after that. Students are strongly reminded to look through the suggested solution to remember that small angle approximation and binomial expansion are usually used in this manner for a typical question in this topic.
(b) [5]	$2xy + \ln y = \ln 3$ Differentiate implicitly with respect to x: $2x\frac{dy}{dx} + 2y + \frac{1}{y}\frac{dy}{dx} = 0$ Differentiate implicitly with respect to x again: $2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2\frac{dy}{dx} + \frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = 0$ Multiply by y^2 throughout: $2xy^2\frac{d^2y}{dx^2} + 2y^2\frac{dy}{dx} + 2y^2\frac{dy}{dx} + y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$ $(2xy^2 + y)\frac{d^2y}{dx^2} + 4y^2\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2 = 0$ (shown) When $x = 0$, $y = 3$, $2(0)\frac{dy}{dx} + 2(3) + \frac{1}{3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -18$, $3\frac{d^2y}{dx^2} + 4(3)^2(-18) - (-18)^2 = 0 \Rightarrow \frac{d^2y}{dx^2} = 324$, $y = 3 - 18x + \frac{324}{2!}x^2 + \ldots = 3 - 18x + 162x^2 + \ldots$	Students are reminded to use implicit differentiation for this type of questions as the working will usually be shorter. Students are also reminded to be careful in working out the values to be used in the Maclaurin's series as there are high accuracy marks for this type of question.

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3 The terms of the sequence U are given by

$$u_1 = k \text{ and } u_{n+1} = \frac{8u_n - 14}{u_n - 1}, \quad n \geq 1.$$

(a) For the following values of k, describe the behaviour of the sequence U.

(i) $k = 3$ [1]

(ii) $k = 10$ [1]

(b) Find the possible value(s) of k if the sequence U is a constant sequence. [2]

The nth term of the sequence V is given by $v_n = \frac{a^n}{b} + \frac{b}{a(1-a^n)} - \frac{a}{n+1}$, where a and b are non-zero real constants and $a \neq \pm 1$.

(c) For some values of a, $v_n \to L$ as $n \to \infty$. Find, with justification, the range of values of a for L to exist, and state the value of L in terms of a and b. [3]

The nth term of the sequence W is given by

$$w_n = \begin{cases} u_n & \text{when } n \text{ is even,} \\ v_n & \text{when } n \text{ is odd.} \end{cases}$$

It is given that the sequence W converges when the sequences U and V converge to the same limit. The sequence W diverges otherwise.

(d) For $k = 10$, by using part **(a)(ii)** and part **(c)**, find the range of values of b for the sequence W to converge. Hence explain whether $\sum_{r=1}^{\infty} w_r$ is a convergent series. [3]

	Solution	Comments
(a) **[2]**	From GC: (i) For $k = 3$, the terms are increasing and converging to 7. (ii) For $k = 10$, the terms are decreasing and converging to 7.	Q3 in general was difficult for most students. In (a), many students were unclear about what the "behaviour of a sequence" referred to, and what to describe. Here, the key features were the monotonicity (increasing/decreasing) and convergence to the limit 7.
(b) **[2]**	For sequence to be a constant sequence, $u_{n+1} = u_n = \cdots = u_1 = k$. $u_{n+1} = \frac{8u_n - 14}{u_n - 1} \Rightarrow k = \frac{8k-14}{k-1}$ $\Rightarrow k^2 - k = 8k - 14$ $\Rightarrow k^2 - 9k + 14 = 0$ $\Rightarrow (k-2)(k-7) = 0$ $\Rightarrow k = 2, 7$ $\therefore k = 2$ or 7.	This part was fairly successful with many students able to secure 2 marks.
(c) **[3]**	$v_n = \frac{a^n}{b} + \frac{b}{a(1-a^n)} - \frac{a}{n+1}$ Observe that as $n \to \infty$, $\frac{1}{n+1} \to 0 \Rightarrow \frac{a}{n+1} \to 0$ for any constant $a \neq 0, \pm 1$. If $\lvert a \rvert > 1$, as $n \to \infty$, $\left\lvert \frac{a^n}{b} \right\rvert$ increases without bound, which implies V is not convergent. If $-1 < a < 1$ and $a \neq 0$, as $n \to \infty$, $a^n \to 0$ Hence for V to be convergent, $-1 < a < 1$ and $a \neq 0$. The required range of values of a is $(-1,0) \cup (0,1)$ and the limiting value L is $\frac{b}{a}$.	A number of students did not attempt (c). v_n was neither an AP nor a GP, so attempts to consider $v_n - v_{n-1}$ or $\frac{v_n}{v_{n-1}}$ proved futile. Students were expected to consider $a = 0, \pm 1$, $0 < \lvert a \rvert < 1$, and also when $\lvert a \rvert > 1$.

(d) [3]	From (a)(ii), $u_n \to 7$, and from (c), $L = \frac{b}{a}$ for $-1 < a < 1$, $a \neq 0$. Hence, since W converges, $\frac{b}{a} = 7$ and $-1 < a < 1$ and $a \neq 0$ $\Rightarrow b = 7a$ and $-1 < a < 1$ and $a \neq 0$ Since $-1 < a < 1$ and $a \neq 0$, $-7 < b < 0$ or $0 < b < 7$ $\Rightarrow$ Range of values of b for W to converge is $(-7,0) \cup (0,7)$. Since the limiting value of the sequence W is a non-zero value (7), the sum of infinite number of non-zero values is arbitrary large and cannot converge to a particular value. Hence the series $\sum_{r=1}^{\infty} w_r$ is not convergent.	A number of students did not attempt (d). While challenging for most, there were many instances of students who showed good intuition to explain why the series failed to converge.

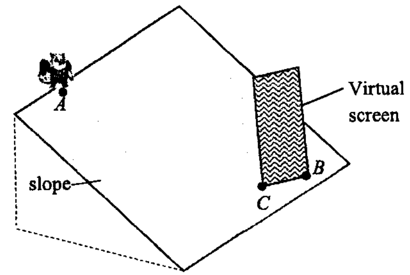
4 In a computer game, a slope can be modelled as a plane p containing three points, $A(1,0,-3)$, $B(1,4,-15)$ and $C(2,3,-14)$.

(a) Show that p has equation $\mathbf{r}.\begin{pmatrix}2\\3\\1\end{pmatrix} = -1$. [2]

In Stage 1 of the game, Griffles travels on the slope from A along a path with equation $\mathbf{r} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\m\\n\end{pmatrix}$, where m and n are constants and λ is a real parameter.

(b) Find an equation relating m and n. [1]

In the rest of the question, it is given that $m = \frac{1}{3}$, $n = -3$ and Griffles is modelled as a point travelling on the slope. An obstacle in the form of a rectangular virtual screen stands on the slope with its base modelled by the **line segment** BC (see diagram).



(c) By considering the **line segment** BC, show that Griffles is able to successfully navigate this obstacle without colliding into it. [3]

A laser gun with a sensor is mounted at the point $E(5,2,1)$ above the slope. The sensor is activated when an object is within a range of 5 units.

(d) Show that Griffles does not activate the sensor. [3]

After completing Stage 1, Griffles is teleported to another slope. This slope can be modelled as a plane q parallel to p such that E is equidistant from p and q.

(e) Find the cartesian equation of q. [3]

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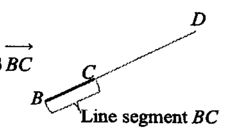
	Solution	Comments
(a) [2]	$A(1,0,-3)$, $B(1,4,-15)$ and $C(2,3,-14)$ $\Rightarrow \overrightarrow{AB} = \begin{pmatrix}0\\4\\-12\end{pmatrix} = 4\begin{pmatrix}0\\1\\-3\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}1\\3\\-11\end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ A normal to the plane ABC is $\begin{pmatrix}0\\1\\-3\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\-3\\-1\end{pmatrix} = -\begin{pmatrix}2\\3\\1\end{pmatrix}$ $\therefore$ Equation of p is $\mathbf{r}.\begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix} = 2-3 = -1$ (shown)	This part is well done. Students are reminded to provide detailed working when evaluating cross or dot product since this is a "show" question.
(b) [1]	Path is perpendicular to normal of plane $\Rightarrow \begin{pmatrix}1\\m\\n\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix} = 0 \Rightarrow 3m+n=-2$	Many misinterpreted this question and gave the cartesian equation of the linear path travelled by Griffles, which is not accepted as it contains variables x, y and z.
(c) [3]	Equation of l_1 is $\mathbf{r} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix}, \mu \in \mathbb{R}$ Line segment BC: $\mathbf{r} = \begin{pmatrix}1\\4\\-15\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\1\end{pmatrix}, \alpha \in \mathbb{R}$ and $0 \le \alpha \le 1$ Consider $\begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix} = \begin{pmatrix}1\\4\\-15\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $3\mu-\alpha=0$ $\mu+\alpha=4$ $-9\mu-\alpha=-12$	<u>Note 1-Interpretation of qn:</u> Griffles is travelling along a linear path, l_1, and the base of the obstacle is modelled by the line segment BC. To show that Griffles do not collide with the line segment BC, it is equivalent to show that l_1 and line passing through BC will intersect at a point <u>that is not on the line segment BC</u> (l_1 and line passing through BC will definitely intersect since they are both on the same slope and

	Solving, $\mu=1$ and $\alpha=3$. Since $\alpha=3>1$, this point of intersection is not on line segment BC. Hence, Griffles is able to navigate the virtual screen without colliding into it. Alternatively, Given that $m=\frac{1}{3}$ and $n=-3 \Rightarrow \begin{pmatrix}1\\m\\n\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3\\1\\-9\end{pmatrix}$ Let l_1 denote the linear path that Griffles travels along. Equation of l_1 is $\mathbf{r} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix}, \mu \in \mathbb{R}$ Line BC: $\mathbf{r} = \begin{pmatrix}1\\4\\-15\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\1\end{pmatrix}, \alpha \in \mathbb{R}$ Consider $\begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix} = \begin{pmatrix}1\\4\\-15\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $3\mu-\alpha=0$ $\mu+\alpha=4$ $-9\mu-\alpha=-12$ Solving, $\mu=1$ and $\alpha=3$ At point of intersection, D, $\overrightarrow{OD} = \begin{pmatrix}1\\4\\-15\end{pmatrix} + 3\begin{pmatrix}1\\-1\\1\end{pmatrix} = \overrightarrow{OB} + 3\overrightarrow{BC}$ $\overrightarrow{BD} = 3\overrightarrow{BC} \Rightarrow D$ is not on line segment BC  Hence, Griffles is able to navigate the virtual screen without colliding into it.	they are not parallel to each other). <u>Note 2:</u> Students who verified that $\begin{pmatrix}1\\4\\-15\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix}$ has no solution (ie B does not lie on l_1) and/or $\begin{pmatrix}2\\3\\-14\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix}$ has no solution (ie C does not lie on l_1) are not given any credit as it may still be possible for line segment BC to intersect l_1 at other points in between B and C. Similarly, students who solve $\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-3\end{pmatrix} + \mu\begin{pmatrix}3\\1\\-9\end{pmatrix}$ to show $\overrightarrow{BC}$ (LHS of equation) does not meet l_1 are not given any credit since a fixed vector ($\overrightarrow{BC}$) cannot be equivalent to position vector of a point on Griffles' path (RHS) <u>Note 3:</u> Many were not careful when copying values from the qn for use in their working, eg. omitting "–3" from "3" in the equation of l_1, etc. Such mistakes are costly and can be

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		avoided by doing a quick visual cross check against the question to confirm that the correct values have been used.
(d) [3]	$\overrightarrow{AE}=\begin{pmatrix}5\\2\\1\end{pmatrix}-\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}4\\2\\4\end{pmatrix}=2\begin{pmatrix}2\\1\\2\end{pmatrix}$ Shortest distance between E and l_1 is $\dfrac{\left\|2\begin{pmatrix}2\\1\\2\end{pmatrix}\times\begin{pmatrix}3\\1\\-9\end{pmatrix}\right\|}{\left\|\begin{pmatrix}3\\1\\-9\end{pmatrix}\right\|}=\dfrac{2}{\sqrt{91}}\left\|\begin{pmatrix}-11\\24\\-1\end{pmatrix}\right\|=\dfrac{2\sqrt{698}}{\sqrt{91}}\approx 5.54$ Since shortest distance between E and l_1 is is 5.54 >5, Griffles is never within a range of 5 units from the sensor and hence he does not activate the sensor. <u>Alternative 1:</u> Let the point G represent the position of Griffles on l_1. Distance between Griffles and point E is $\left\|\overrightarrow{EG}\right\|$ $=\left\|\begin{pmatrix}-4+3\mu\\-2+\mu\\-4-9\mu\end{pmatrix}\right\|$	<u>Note 1 -interpretation of qn:</u> This part involves distance between Griffles and the sensor at E and we are concerned whether this distance will be smaller than 5 units (which will activate the sensor). One way to determine this is to work out the shortest distance between E and l_1 and compare it to 5 units. However, many misinterpreted shortest distance between point E (sensor) and Griffles' path (l_1) to be equivalent to shortest distance between point E and the plane p (slope). Students with this misinterpretation did not receive any credit for their working.

	$=\sqrt{(-4+3\mu)^2+(-2+\mu)^2+(-4-9\mu)^2}$ $=\sqrt{36+44\mu+91\mu^2}$ $=\sqrt{91\left(\mu+\frac{22}{91}\right)^2+\frac{2792}{91}}\geq\sqrt{\frac{2792}{91}}\approx 5.54$ since $91\left(\mu+\frac{22}{91}\right)^2\geq 0$ Since 5.54 >5, Griffles does not activate the sensor. <u>Alternative 2 (not recommended):</u> Let F be the foot of perpendicular from E to l_1. $\overrightarrow{OF}=\begin{pmatrix}1\\0\\-3\end{pmatrix}+\mu\begin{pmatrix}3\\1\\-9\end{pmatrix}$ for some $\mu\in\mathbb{R}$ $\overrightarrow{EF}.\begin{pmatrix}3\\1\\-9\end{pmatrix}=0$ $\left[\begin{pmatrix}-4\\-2\\-4\end{pmatrix}+\mu\begin{pmatrix}3\\1\\-9\end{pmatrix}\right].\begin{pmatrix}3\\1\\-9\end{pmatrix}=0$ $91\mu=-22$ $\mu=-\frac{22}{91}$ $\overrightarrow{OF}=\begin{pmatrix}1\\0\\-3\end{pmatrix}-\frac{22}{91}\begin{pmatrix}3\\1\\-9\end{pmatrix}=\frac{1}{91}\begin{pmatrix}25\\-22\\-75\end{pmatrix}$ $\Rightarrow\overrightarrow{EF}=\frac{1}{91}\begin{pmatrix}-430\\-204\\-166\end{pmatrix}$ Shortest distance between E and l_1 is EF	<u>Note 2:</u> Students who solve the inequality $\sqrt{(-4+3\mu)^2+(-2+\mu)^2+(-4-9\mu)^2}<5$ which is equivalent to solving $11+44\mu+91\mu^2<0$ are required to show their working on how they obtain the conclusion, "no real solution". Algebraically, this involves either (a) completing the square or (b) verifying that the discriminant of $11+44\mu+91\mu^2=0$ is negative **<u>and</u>** commenting that the coefficient of μ^2 is positive to conclude that $11+44\mu+91\mu^2>0$ for all real μ, Or if using GC, a sketch of $y=11+44\mu+91\mu^2$ should be included as part of the working with the y-coordinate of the minimum point clearly shown.

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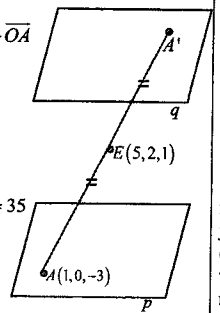
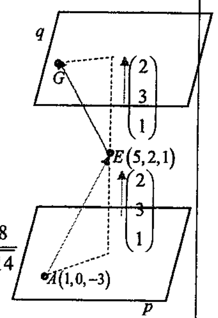
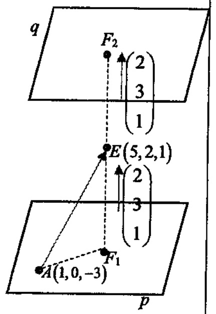
	$=\dfrac{\sqrt{254072}}{91}\approx 5.54$ Since 5.54 >5, Griffles does not activate the sensor.	
(e) [3]	Method 1 (finding a point on q): To begin, we can use any point on p, say $A(1,0,-3)$. Let the point of intersection between line AE and plane q be A'. Then E is midpoint of AA' since E is is equidistant from p and q. $\overrightarrow{OE}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)\Rightarrow\overrightarrow{OA'}=2\overrightarrow{OE}-\overrightarrow{OA}$ $\overrightarrow{OA'}=2\begin{pmatrix}5\\2\\1\end{pmatrix}-\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}9\\4\\5\end{pmatrix}$ Equation of q is $\mathbf{r}.\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}9\\4\\5\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix}=35$  Cartesian equation of q is $2x+3y+z=35$. Method 2 (length of projection): $\overrightarrow{AE}=\begin{pmatrix}4\\2\\4\end{pmatrix}=2\begin{pmatrix}2\\1\\2\end{pmatrix}$  Distance of E to p is $\dfrac{\left\lvert 2\begin{pmatrix}2\\1\\2\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}2\\3\\1\end{pmatrix}\right\rvert}=\dfrac{18}{\sqrt{14}}$ Let G be a point on q where $\overrightarrow{OG}=\mathbf{r}$	Do note that the equation of q needs to be given in cartesian equation. Students who use the formula "distance between 2 planes with equations $\mathbf{r}.\mathbf{n}_1=d_1$ and $\mathbf{r}.\mathbf{n}_2=d_2$ is $\lvert d_1-d_2\rvert$" either forgot the modulus sign, or conveniently remove it without justification and/or chose the wrong form when removing the modulus sign.

	Distance of E to q is $\dfrac{1}{\left\lvert\begin{pmatrix}2\\3\\1\end{pmatrix}\right\rvert}\overrightarrow{EG}.\begin{pmatrix}2\\3\\1\end{pmatrix}$ (Note: $\overrightarrow{AE}.\begin{pmatrix}2\\3\\1\end{pmatrix}>0\Rightarrow\overrightarrow{EG}.\begin{pmatrix}2\\3\\1\end{pmatrix}>0$) Since E is equidistant from p and q, $\dfrac{\overrightarrow{EG}.\begin{pmatrix}2\\3\\1\end{pmatrix}}{\sqrt{14}}=\dfrac{18}{\sqrt{14}}$ $\mathbf{r}.\begin{pmatrix}2\\3\\1\end{pmatrix}=18+\begin{pmatrix}5\\2\\1\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix}=35$ Cartesian equation of q is $2x+3y+z=35$ Method 3 (foot of perpendicular): Let F_1 and F_2 be the foot of perpendicular from E to p and q.  $\overrightarrow{OF_1}=\begin{pmatrix}5\\2\\1\end{pmatrix}+\beta\begin{pmatrix}2\\3\\1\end{pmatrix}$ for some $\beta\in\mathbb{R}$ $\left[\begin{pmatrix}5\\2\\1\end{pmatrix}+\beta\begin{pmatrix}2\\3\\1\end{pmatrix}\right].\begin{pmatrix}2\\3\\1\end{pmatrix}=-1\Rightarrow\beta=-\frac{9}{7}$ $\Rightarrow\overrightarrow{OF_1}=\begin{pmatrix}5\\2\\1\end{pmatrix}-\frac{9}{7}\begin{pmatrix}2\\3\\1\end{pmatrix}=\frac{1}{7}\begin{pmatrix}17\\-13\\-2\end{pmatrix}$ Since $\overrightarrow{OE}=\frac{1}{2}\left(\overrightarrow{OF_1}+\overrightarrow{OF_2}\right)$, $\overrightarrow{OF_2}=2\overrightarrow{OE}-\overrightarrow{OF_1}=\frac{1}{7}\begin{pmatrix}53\\41\\16\end{pmatrix}$	

	Or $\overrightarrow{OF_2} = \begin{pmatrix}5\\2\\1\end{pmatrix} + \frac{9}{7}\begin{pmatrix}2\\3\\1\end{pmatrix} = \frac{1}{7}\begin{pmatrix}53\\41\\16\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}2\\3\\1\end{pmatrix} = \frac{1}{7}\begin{pmatrix}53\\41\\16\end{pmatrix}.\begin{pmatrix}2\\3\\1\end{pmatrix} = 35$ Cartesian equation of q is $2x+3y+z=35$	

5 A discrete random variable X has the following probability distribution:

$$P(X=x)=k(x^2+x) \text{ for } x=1,2,3,4,5, \text{ where } k \text{ is a constant.}$$

(a) Show that $k=\frac{1}{70}$. [1]

(b) Find the exact values of $E(X)$ and $Var(X)$. [3]

(c) Two independent observations X_1 and X_2 are taken of X. Find the probability that the difference between the two observations is at least 3. [3]

	Solution	Comments
(a) [1]	$\sum P(X=x)=1$ $\sum_{x=1}^{5} k(x^2+x)=1$ $2k+6k+12k+20k+30k=1$ $70k=1$ $k=\frac{1}{70}$ (shown)	Most students were able to form the equation, starting from $\sum P(X=x)=1$ specifically.
(b) [3]	$E(X)=\frac{1}{70}[1(2)+2(6)+3(12)+4(20)+5(30)]$ $=\frac{1}{70}(280)$ $=4$ $E(X^2)=\frac{1}{70}[1(2)+4(6)+9(12)+16(20)+25(30)]$ $=\frac{1}{70}(1204)$ $=17.2$ $Var(X)=E(X^2)-(E(X))^2$ $=17.2-16$ $=1.2$	As the question demands "exact values", lifting of GC's 1-Var Stats is not allowed. Students are required to show explicitly the working for $E(X)$ and $E(X^2)$. Apart from using $Var(X)=E(X^2)-(E(X))^2$, students may also use the result $Var(X)=E(X-\mu)^2 = \sum(x-\mu)^2 P(X=x)$

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(c) [3]	$P(\|X_1 - X_2\| \geq 3)$ $= P(X_1 = 1 \text{ and } X_2 = 5 \text{ or vice versa})$ $+P(X_1 = 1 \text{ and } X_2 = 4 \text{ or vice versa})$ $+P(X_1 = 2 \text{ and } X_2 = 5 \text{ or vice versa})$ $= 2\left[\left(\frac{2}{70}\times\frac{30}{70}\right)+\left(\frac{2}{70}\times\frac{20}{70}\right)+\left(\frac{6}{70}\times\frac{30}{70}\right)\right]$ $= \frac{4}{35}$	The intent of the question is to find the difference between two independent observations of X. Students who wrote down the probability distribution earlier were more successful in getting the right answer. It is incorrect to write that $X_1 - X_2$ follows a normal distribution, given that it is a difference of two discrete distributions.

6 Kitty has 5 small, 3 medium and 2 large spherical charms and each of the 10 charms is uniquely designed.

(a) Kitty arranges all the charms in a circle on a corkboard with charms of the same sizes next to each other. How many ways are there for her to do so? [2]

(b) On another occasion, Kitty arranges all the charms in a line at the base of a photo frame with none of the small charms next to each other. How many ways are there for her to do so? [2]

The diameters of all the small, medium and large charms are 0.5 cm, 1 cm and 2 cm respectively. Each of the spherical charms has a hole through its centre that allows it to be threaded through a chain.

(c) Kitty makes a keychain that includes a 6 cm chain with one end attached to a keyring, as shown in **Fig. 1**. It is given that the 6 cm chain is fully threaded with charms, with no gaps between them, and is stretched taut in a straight line. For example, **Fig. 2** shows the 6 cm chain threaded with 2 large and 2 medium charms.

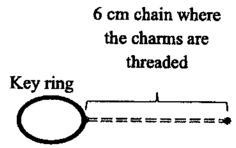


Fig. 1

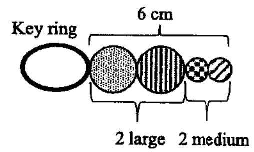


Fig. 2

How many different ways can she make a keychain with at least one charm of each of the three sizes? [4]

	Solution	Comment
(a) [2]	Number of ways to arrange the large charms = 2! Number of ways to arrange the medium charms = 3! Number of ways to arrange the small charms = 5! Number of ways to arrange 3 objects in a circle $=(3-1)! = 2!$ Total number of ways to arrange in a circle $= [(2!)(3!)(5!)](2!) = 2880$	Generally well done. As there are only 3 groups to permutate in a circle, the number of ways to arrange them is $(3-1)!$ instead of $\frac{3!}{10}$
(b) [2]	Number of ways to arrange the big and medium charms = 5! Number of ways to slot the small charms in between the arrangement of the big and medium charms $= \binom{6}{5}\times 5! = 720$ Total number of ways $= 720\times 5! = 86400$	Students who have employed the use of method of slotting were often most successful. A small number of students assumed that charms

	Alternative (not recommended) method using complement Case 1 : SS + S + S + S (only 2 of the small are together) No. of ways = $\left[\binom{5}{2}\times 2!\right]\times 5!\times\binom{6}{4}\times 4! = 864000$ Case 2 : SS + SS + S No. of ways = $\dfrac{\left[\binom{5}{2}\times 2!\right]\times\left[\binom{3}{2}\times 2!\right]}{2!}\times 5!\times\binom{6}{3}\times 3! = 864000$ Case 3 : SS + SSS No. of ways = $\left[\binom{5}{2}\times 2!\right]\times\left[\binom{3}{3}\times 3!\right]\times 5!\times\binom{6}{2}\times 2! = 432000$ Case 4 : SSSS + S No. of ways = $\left[\binom{5}{4}\times 4!\right]\times 5!\times\binom{6}{2}\times 2! = 432000$ Case 5 : SSS + S + S No. of ways = $\left[\binom{5}{3}\times 3!\right]\times 5!\times\binom{6}{3}\times 3! = 864000$ Case 6 : SSSSS No. of ways = $6!\times 5! = 86400$ Required Number = $10!-3(864000)-2(432000)-86400 = 86400$	of the same size were indistinguishable, which is incorrect. Some students incorrectly assumed that the question required the small charms to be alternating between the medium/large charms. None of the students who did the method of complementation arrived at the correct solution.
(c) [4]	Case 1 : 2L, 1M, 2S Number of ways to choose these objects then arrange $=\binom{2}{2}\binom{3}{1}\binom{5}{2}\times 5! = 3600$ Case 2 : 1L, 2M, 4S Number of ways to choose these objects then arrange $=\binom{2}{1}\binom{3}{2}\binom{5}{4}\times 7! = 151200$ Case 3 : 1L, 3M, 2S Number of ways to choose these objects then arrange $=\binom{2}{1}\binom{3}{3}\binom{5}{2}\times 6! = 14400$ Total number of ways = 3600 + 51200 + 14400 = 169200	Students are required to work around the two limitations: (1) Total length must be 6 cm; (2) Must have at least one of each size; (3) Total 5 small, 3 medium and 2 large charms. This meant that an even number of small charms is required, capped at 4. Many students only chose the charms without permutation or vice versa, which is incorrect.

7 **In this question you should state the parameters of any distributions you use.**

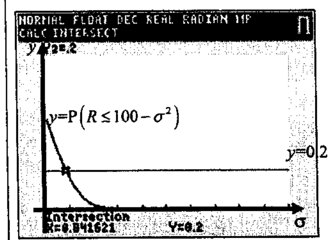
A snack company produces two types of potato chips, Rays and Luffles. The mass, in grams, of a regular packet of Rays follows the distribution $N(100, \sigma^2)$ and the mass, in grams, of a regular packet of Luffles follows the distribution $N(120,16)$. The masses of all regular packets of potato chips are independent of one another.

(a) If the probability of the mass of a randomly chosen packet of Rays not exceeding $(100-\sigma^2)$ grams is less than 0.2, find the possible range of values of σ. [2]

It is given that $\sigma = 3$ for the rest of this question.

(b) Find the probability that the total mass of 2 randomly chosen regular packets of Rays and 3 randomly chosen packets of Luffles is greater than 0.55 kg. [3]

(c) The snack company decides to launch a new product called Mega Jumbo Pack, which consists of $(24-n)$ regular packets of Rays and n regular packets of Luffles. It is given that the probability of the mass of a Mega Jumbo Pack exceeding 20 times the mass of a regular packet of Luffles by more than 500g is at least 0.1. Find the minimum value of n. [4]

	Solution	Comments
(a) [2]	Let R and L be the mass of a regular packet of Rays and Luffles respectively $R \sim N(100,\sigma^2)$, $L \sim N(120,4^2)$ $P(R \le 100-\sigma^2) < 0.2$  From GC, $\sigma > 0.842$. Alternatively, $0.842 < \sigma < 10$	Students are reminded to give their answer correct to 3 s.f. When using the GC to plot graphs, students should also sketch the graphs to support their answers. Note that the GC 'table of values' method is not acceptable, as the solution is not necessarily an integer value. Students who used standardisation method often struggled with the

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	Alternative (Standardisation): $\mathrm{P}\left(R<100-\sigma^2\right)<0.2, \quad R\sim \mathrm{N}\left(100,\sigma^2\right)$ $\mathrm{P}\left(Z<\frac{-\sigma^2}{\sigma}\right)<0.2$ From GC, $\mathrm{P}(Z<-0.84162)=0.2$ $-\sigma<-0.84162$ $\sigma>0.84162$ $\sigma>0.842$ (3sf)	sign change for inequality. Some students did not simplify $\frac{-\sigma^2}{\sigma}$ and wasted time in solving a more complicated quadratic inequality involving σ.
(b) **[3]**	$R_1+R_2+L_1+L_2+L_3\sim \mathrm{N}\left(2(100)+3(120),\ 2(3)^2+3(4)^2\right)$ $R_1+R_2+L_1+L_2+L_3\sim \mathrm{N}(560,66)$ Probability that the total mass of 2 randomly chosen regular packets of Rays and 3 randomly chosen packets of Luffles is greater than 0.55kg $=P(R_1+R_2+L_1+L_2+L_3>550)=0.891$ (3 s.f.)	This part is well attempted by the cohort. Students must show their working clearly when calculating the mean and variance **and** state the distribution of $R_1+R_2+L_1+L_2+L_3$. Quite a number of students wrote the expression incorrectly (i.e. $2R+3L$), even though they calculated the mean and variance correctly.
(c) **[4]**	Let M denote the mass of a Mega Jumbo Pack of 24 regular packets of potato chips in grams. $M=(R_1+R_2+...+R_{24-n})+(L_1+L_2+...+L_n)$ $\mathrm{E}(M)=(24-n)\mathrm{E}(R)+n\mathrm{E}(L)$ $=(24-n)100+120n=2400+20n$ $\mathrm{Var}(M)=(24-n)\mathrm{Var}(R)+n\mathrm{Var}(L)$ $=(24-n)3^2+n(4^2)=216+7n$ $M\sim \mathrm{N}(2400+20n,\ 216+7n)$ $\therefore M-20L\sim \mathrm{N}\left(2400+20n-20(120),\ 216+7n+20^2(4^2)\right)$ $\Rightarrow M-20L\sim \mathrm{N}(20n,\ 6616+7n)$	Once again, students must show their working clearly when calculating the mean and variance **and** state the distribution for $M-20L$. Many students struggled with the calculation of $\mathrm{Var}(M)$. The most common mistake seen was

$\mathrm{P}(M-20L>500)\geq 0.1$

n	$\mathrm{P}(M-20L>500)$
19	$0.0720<0.1$
20	$0.1119>0.1$

Hence the minimum value of n in a Mega Jumbo Pack is 20.

Alternatively,

$\mathrm{P}(M-20L>500)\geq 0.1$

$\mathrm{P}\left(Z>\frac{500-20n}{\sqrt{6616+7n}}\right)\geq 0.1$

$\frac{500-20n}{\sqrt{6616+7n}}\leq 1.28155$

$n\geq 19.733883$

Hence the minimum value of n in a Mega Jumbo Pack is 20.

Comments:

$\mathrm{Var}(M)$
$=(24-n)^2\mathrm{Var}(R)+n^2\mathrm{Var}(L)$
Note that $\mathrm{Var}(X_1+X_2)\neq \mathrm{Var}(2X)$

Students should show their working clearly when determining and justifying the <u>minimum</u> value of n.

8 A small cafe sells one type of coffee. The selling price of a cup of coffee is reviewed and adjusted at the beginning of every year depending on market conditions. Based on sales figures collected over 7 years, the cafe owner, Mr Tay, studied the effect of the selling price of a cup of coffee, $\$x$, on the average number of cups, y cups, sold per day within the year. The data is shown in the table below.

x	2.0	2.2	2.5	3.0	4.0	4.4	4.5
y	280	250	190	150	90	100	70

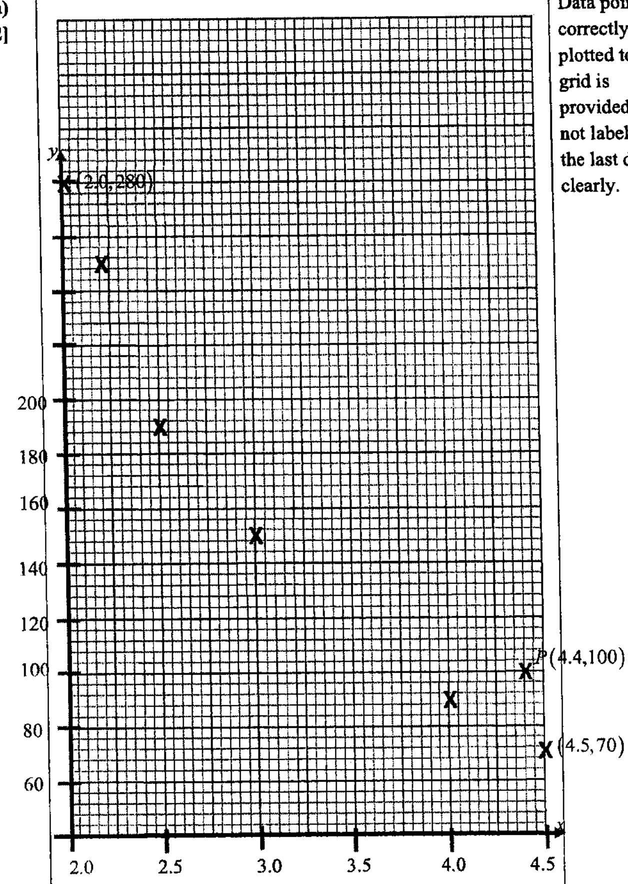
(a) On the grid in the Printed Answer Booklet, draw a scatter diagram of the data. [2]

Mr Tay realised that one of the values of y has been wrongly stated in the data table.

(b) Indicate the corresponding point on your diagram by labelling it P. Explain why the scatter diagram for the remaining points may be consistent with a model of the form $y = a + \frac{b}{x}$, where a and b are constants. [2]

(c) Omitting P, calculate the product moment correlation coefficient and the least squares estimates of a and b for the model $y = a + \frac{b}{x}$. [2]

(d) Use the model $y = a + \frac{b}{x}$ with the values of a and b found in part **(c)** to estimate the value of y that has been wrongly stated in the data table. Give two reasons why you would expect this estimate to be reliable. [3]

	Solution	Comments
(a) **[2]**		Data points MUST be correctly indicated and plotted to scale since a grid is provided. Many did not label the first and the last data point clearly.
(b) **[2]**	As x increases, y decreases but with decreasing amounts. $\therefore$ The scatter diagram for the remaining points may be consistent with a model of the form $y = a + \frac{b}{x}$, where a and b are constants.	It is required to state "y decreases by decreasing amount" to explain why the data points are consistent with the model. However many simply stated "y decreases" or the "graph decreases", which is insufficient because in this case y

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		decreases with decreasing amounts as x increases. The question also clearly indicated to explain from the points on the scatter diagram, hence it is not acceptable to use the value of product moment correlation coefficient (pmcc) to justify choice of model.
(c) **[2]**	Product moment correlation coefficient, $r \approx 0.99711 = 0.997$ (3sf) Least squares estimate of $a = -100.8169522 = -101$ (3sf) and least squares estimate of $b \approx 756.8635473 = 757$ (3sf)	Many did not leave the final answers for r, a and b in 3 s.f.
(d) **[3]**	$y \approx \frac{756.8635473}{x} - 100.8169522$ When $x = 4.4$, $y = \frac{756.8635473}{4.4} - 100.8169522 \approx 71.197$ An estimate of the value of y that was wrongly stated in the table is 71 (nearest integer) or 71.2 (3sf). This estimate is reliable since 4.4 is within the given range of values of x, $[2,4.5]$, for which the model in part (c) was constructed on. Furthermore, the product moment correlation coefficient is 0.997 which is very close to 1 thus suggesting a strong positive linear correlation between $\frac{1}{x}$ and y.	If the final answer for y is left as a final integer, it should be rounded off to 71 instead of 72. For 3s.f. it should be left as 71.2. It is also necessary to state explicitly that x=4.4 lies within the given data range for x values ,i.e. [2,4.5]. However, some students explain the reliability using y values instead. When explaining the reliability using pmcc, students should explicitly state that the value is "close to 1" to describe the strong positive linear correlation.

9 Based on observations over a long period, the mean time taken by male students in Griffles Junior College (GJC) to complete a 2.4 km run is known to be 11.3 minutes with a standard deviation of 2.2 minutes. As part of a review of the current physical training programme, the Physical Education (PE) department in GJC decided to test, at the 2.5% level of significance, whether the mean time taken by male students in GJC to complete a 2.4 km run has changed. The time taken, x minutes, by a random sample of 8 Year 5 male students from GJC to complete a 2.4 km run, are as follows.

11 11.5 10.8 11.2 11.4 11 11.8 12.5

(a) Write down the null and alternative hypotheses for this test, defining any symbols you use. [2]

(b) Stating a necessary assumption, find the critical region for this test. Hence state the conclusion of the test in the context of the question. [6]

The PE department in GJC also conducted a second test, at the 3% level of significance, to determine whether the mean time taken by female students in GJC to complete a 2.4 km run is less than 14.5 minutes. A random sample of n female students from GJC is taken, where n is large. The mean and standard deviation of the time taken by this sample to complete a 2.4km run are found to be 14.2 minutes and 1.5 minutes respectively.

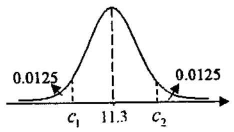
(c) Given that the PE department concludes that the mean time taken by female students to complete a 2.4 km run is less than 14.5 minutes, find the range of values that n can take. [5]

	Solution	Comments
(a) **[2]**	Let μ be the population mean time, in minutes, taken by male students in GJC to complete a 2.4 km run. To test Null Hypothesis $H_0 : \mu = 11.3$ Alternative Hypothesis $H_1 : \mu \neq 11.3$	Do take note of the symbols used and their definition. μ is the **population** mean time.
(b) **[6]**	Let X be the time taken in minutes by a male student in GJC to complete a 2.4km run. Let $\overline{X}$ be the sample mean time. Perform a 2–tail test at 2.5% level of significance.	Since the sample size is small, we are unable to quote Central Limit Theorem for the

Since sample size, $n=8$, is small, we assume that X follows a normal distribution.

From sample, $\bar{x}=11.4$

Under H_0, $\bar{X} \sim N\left(11.3, \frac{2.2^2}{8}\right)$



To find critical region, from the diagram, we have

$P(\bar{X} \le c_1)=0.0125$ and $P(\bar{X} \ge c_2)=0.0125$

Using GC, $c_1 = 9.5566$ and $c_2 = 13.043$

$\therefore$ critical region is $(0, 9.56] \cup [13.0, \infty)$

Since $\bar{x}=11.4$ does not lie in the critical region, we **do not reject** H_0 and conclude that there is **insufficient** evidence, at the 2.5% significance level, *that the population mean time taken by male students in GJC to complete a 2.4 km run has changed.*

Comments: distribution for $\bar{X}$. As such, we need to assume that X follows a normal distribution.

Do read the question carefully. We were given the population variance ie 2.2^2, hence there is no need to find the unbiased estimate of the population variance.

Care should be taken to craft the conclusion. The level of significance for the test must be mentioned and the phrase in italics makes reference to H_1.

(c) [5]

Let Y be the time taken in minutes by a female student in GJC to complete a 2.4 km run, and let μ_Y and $\bar{Y}$ be the population mean time and sample mean time respectively.

From the sample, $\bar{y}=14.2$, $s_y^2 = \frac{n}{n-1}(1.5^2)$

To test $H_0: \mu_Y = 14.5$ vs $H_1: \mu_Y < 14.5$

Under H_0, since n is large,

$$\bar{Y} \sim N\left(14.5, \frac{\frac{n}{n-1}(1.5^2)}{n}\right) \text{ i.e } \bar{Y} \sim N\left(14.5, \frac{1.5^2}{n-1}\right)$$

approximately by Central Limit Theorem.

For H_0 to be rejected,

Comments: Do read the question carefully. We were given the sample variance ie 1.5^2, hence we need to find the unbiased estimate of the population variance using the formula $s^2 = \frac{n}{n-1}(\text{sample variance})$

p-value $= P(\bar{Y} \le 14.2) \le 0.03$

$$P\left(Z \le \frac{14.2-14.5}{\sqrt{\frac{1.5^2}{n-1}}}\right) \le 0.03$$

$$\frac{-0.3\sqrt{n-1}}{1.5} \le -1.8808$$

$$\sqrt{n-1} \ge 9.404$$

$$n \ge 89.435$$

$$\therefore n \ge 90, \text{ where } n \in \mathbb{Z}^+$$

Comment: Note that n is a positive integer value!

<u>Alternative 1 (GC Table of Values):</u>

For H_0 to be rejected,

p-value $= P(\bar{Y} \le 14.2) \le 0.03$

n	$P(\bar{Y} \le 14.2)$
89	$0.0303 > 0.03$
90	$0.0296 < 0.03$

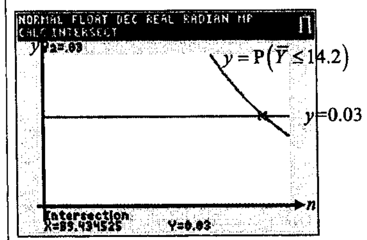
$\therefore n \ge 90$, where $n \in \mathbb{Z}^+$.

Comment: For this method, do provide the information in the table as part of your working.

<u>Alternative 2 (GC graphing):</u>

For H_0 to be rejected,

p-value $= P(\bar{Y} \le 14.2) \le 0.03$



From the above graph, $n \ge 90$, where $n \in \mathbb{Z}^+$.

Comment: For this method, do include the graph as part of your working.

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10 A bag contains 4 identical red and 6 identical blue balls.

(a) In the first game, the balls are randomly picked by a player from the bag, one at a time, without replacement, until there are no balls left in the bag.

(i) Find the probability that the last ball picked is red. [1]

(ii) Find the probability that in the first five picks, exactly 2 blue balls are picked given that at least 3 red balls are picked. [3]

(b) In the second game, each ball picked by a player will be placed back into the bag before the next ball is picked. The player makes a total of 20 random picks from the bag and the colour of each pick is recorded.

(i) Find the probability that the colour red is recorded exactly 4 times. [2]

(ii) Find the probability that the colour red is recorded more than 4 times but not more than 8 times. [2]

(iii) Find the probability that the 8th pick is the 6th time the colour blue is recorded. [2]

(iv) The second game is played by 50 randomly chosen players. Estimate the probability that the average number of times the colour red is recorded is more than 8.5. [4]

	Solution	Comments
(ai) [1]	P(last ball is red) $=\dfrac{\text{n(arrangements where last ball is red)}}{\text{n (arrangements without restrictions)}}=\dfrac{\frac{9!}{3!6!}}{\frac{10!}{4!6!}}=\dfrac{2}{5}$ OR P(last ball is red) $=\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}\times\frac{3}{3}\times\frac{2}{2}\times\frac{9!}{6!3!}=\frac{2}{5}$ OR P(last ball is red) = P(first ball is red) $=\frac{4}{10}=\frac{2}{5}$	Remember to multiply by $\frac{9!}{6!3!}$ for the 2nd Method.
(aii) [3]	P(exactly 2 blue | at least 3 red) $=\dfrac{\text{P(2 blue and 3 red)}}{\text{P(2 blue and 3 red)}+\text{P(1 blue and 4 red)}}$	Show the working for conditional probability clearly.

	$=\dfrac{{}^6C_2\,{}^4C_3}{{}^6C_2\,{}^4C_3+{}^6C_1\,{}^4C_4}=\dfrac{10}{11}$ OR $\dfrac{\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{5!}{2!3!}}{\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{5!}{2!3!}+\frac{6}{10}\times\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times\frac{1}{6}\times\frac{5!}{1!4!}}$ $=\dfrac{\frac{10}{42}}{\frac{10}{42}+\frac{1}{42}}=\dfrac{10}{11}$	
(bi) [2]	Let R be the number of times the colour red is recorded out of 20 picks. $R\sim B\left(20,\frac{2}{5}\right)$ $P(R=4)=0.0350$ (3sf) OR ${}^{20}C_4\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^{16}=0.0350$ (3 s.f.)	Define the Binomial random variable if you intend to use the GC to evaluate $P(R=4)$, else show working.
(bii) [2]	$P(4<R\le 8)=P(R\le 8)-P(R\le 4)$ where $R\sim B\left(20,\frac{2}{5}\right)$ $=0.545$ (3 s.f.)	Be careful whether it is $\le$ or $<$.
(biii) [2]	Let S be the number of times the colour blue is recorded out of 1st 7 picks. $S\sim B\left(7,\frac{3}{5}\right)$ Required probability = P(5 blue balls in first 7 picks) × P(blue ball in 8th pick) $=P(S=5)\times\frac{3}{5}$ OR ${}^7C_5\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^2\times\frac{3}{5}=0.157$ (3 s.f.)	8th pick is the 6th time blue is recorded means there are 5 blue balls in the first 7 picks.
(biv) [4]	$R\sim B\left(20,\frac{2}{5}\right)$, so $E(R)=20\left(\frac{2}{5}\right)=8$ $Var(R)=20\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)=\frac{24}{5}$ or 4.8 Since $n=50$ is large, by Central Limit Theorem, $\bar{R}\sim N\left(8,\frac{4.8}{50}\right)$ approximately, i.e. $\bar{R}\sim N(8, 0.096)$ approximately $P(\bar{R}>8.5)=0.0533$ (3 s.f.) Alternatively, Since $n=50$ is large, by Central Limit Theorem, $R_1+...+R_{50}\sim N(50\times 8, 50\times 4.8)$ approximately, i.e. $R_1+...+R_{50}\sim N(400, 240)$ approximately. $P(R_1+...+R_{50}>8.5\times 50)=0.0533$ (3 s.f.)	Question says "Estimate the probability", i.e. use Central Limit Theorem. For Binomial Dist., mean = np and variance = $np(1-p)$ can be found in MF27.