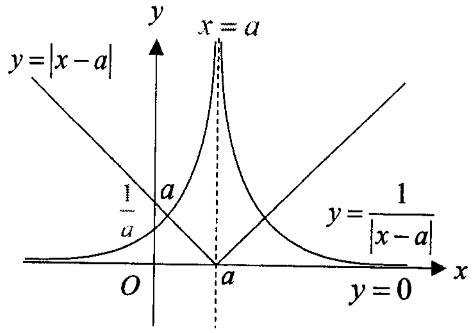
2025 H2 MATH (9758/01) JC 2 PRELIMINARY EXAMINATION

Qn	Solution
1	**Graphing Techniques, Equations and Inequalities**
(a)	
(b)	Points of intersection: $\lvert x-a\rvert=\frac{1}{\lvert x-a\rvert}$ Since $x\neq a$, $\lvert x-a\rvert>0$ $(x-a)^2=1$ $x=a\pm1$ For $\lvert x-a\rvert>\frac{1}{\lvert x-a\rvert}$, from the graph, $x<a-1$ or $x>a+1$

Qn	Solution
2	**Differentiation**
(a)	$y^2 = x + e^{2x}y$ --(1) Differentiate with respect to x, $2y\frac{dy}{dx} = 1 + e^{2x}\frac{dy}{dx} + y(2e^{2x})$ $\frac{dy}{dx}(2y - e^{2x}) = 1 + 2ye^{2x}$ (shown) --(2)
(b)	Subt $x = 0$ into (1) $y^2 = y$ $y^2 - y = 0$ $y(y-1) = 0$ $y = 0$ or $y = 1$ Subt $(0,0)$ into (2) $\frac{dy}{dx} = -1$ $\therefore$ the equation of the tangent is $y = -x$ Subt $(0,1)$ into (2) $\frac{dy}{dx} = 3$ $\therefore$ the equation of the tangent is $y - 1 = 3(x - 0)$ $y = 3x + 1$

Qn	Solution
3	**Techniques of Integration**
(a)	$\int \frac{x^2-3x-1}{x^2-4x+1}\,\mathrm{d}x$ $=\int 1+\frac{x-2}{x^2-4x+1}\,\mathrm{d}x$ $=\int 1\,\mathrm{d}x+\frac{1}{2}\int \frac{2(x-2)}{x^2-4x+1}\,\mathrm{d}x$ $=\int 1\,\mathrm{d}x+\frac{1}{2}\int \frac{2x-4}{x^2-4x+1}\,\mathrm{d}x$ $=x+\frac{1}{2}\ln\left\|x^2-4x+1\right\|+C$
(b)	$\int x\ln(kx)\,\mathrm{d}x$ $=\left[\left(\frac{x^2}{2}\right)(\ln(kx))\right]-\int\left(\frac{x^2}{2}\right)\frac{k}{kx}\,\mathrm{d}x$ $=\frac{x^2}{2}\ln(kx)-\frac{1}{2}\int x\,\mathrm{d}x$ $=\frac{x^2}{2}\ln(kx)-\frac{1}{2}\left(\frac{x^2}{2}\right)+C$ $=\frac{x^2}{2}\ln(kx)-\frac{x^2}{4}+C$
(c)	$x=\cos\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta$ $\int \frac{\sin\theta}{2\cos 2\theta+1}\,\mathrm{d}\theta$ $=\int \frac{\sin\theta}{2(2\cos^2\theta-1)+1}\,\mathrm{d}\theta$ $=\int \frac{1}{4\cos^2\theta-1}(\sin\theta)\,\mathrm{d}\theta$ $=-\int \frac{1}{4x^2-1}\,\mathrm{d}x$ $=\int \frac{1}{1-(2x)^2}\,\mathrm{d}x$ $=\frac{1}{2}\left(\frac{1}{2(1)}\right)\ln\left\|\frac{1+2x}{1-2x}\right\|+C$ $=\frac{1}{4}\ln\left\|\frac{1+2\cos\theta}{1-2\cos\theta}\right\|+C$

Qn	Solution
4	**A&GS, Sequences & Series**

(a)

$a+6d=br^2 \quad -- \quad (1)$

$a+9d=br^5 \quad -- \quad (2)$

$a+10d=br^9 \quad -- \quad (3)$

$(2)-(1)$:

$3d=br^2\left(r^3-1\right) \quad -- \quad (4)$

$(3)-(2)$:

$d=br^5\left(r^4-1\right) \quad -- \quad (5)$

Substitute (5) into (4),

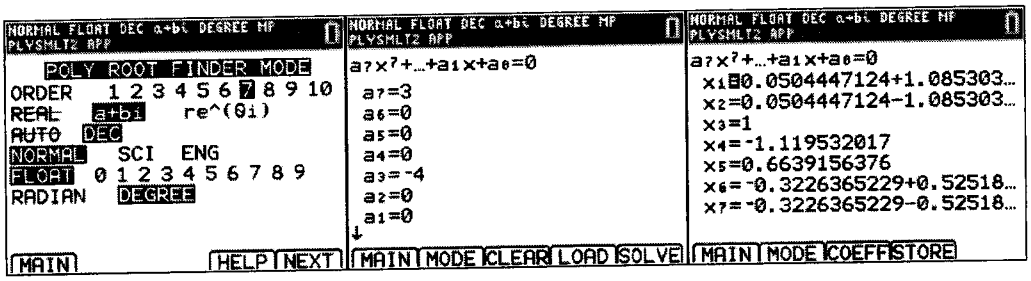
$3br^5\left(r^4-1\right)=br^2\left(r^3-1\right)$

$3r^3\left(r^4-1\right)=\left(r^3-1\right) \quad (\text{since } b,r\neq 0)$

$3r^7-4r^3+1=0 \quad (\text{shown})$

By GC, since the series is convergent, $|r|<1$,

$r=0.6639$ (4 d.p.)



(b)

New Series: $br, br^3, br^5, \ldots$

$\left|\frac{br}{1-r^2}-S\right|\geq 1.78b$

$\left|\frac{br}{1-r^2}-\frac{b}{1-r}\left(1-r^n\right)\right|\geq 1.78b$

$\left|\frac{0.6639b}{1-0.6639^2}-\frac{b}{1-0.6639}\left(1-0.6639^n\right)\right|\geq 1.78b$

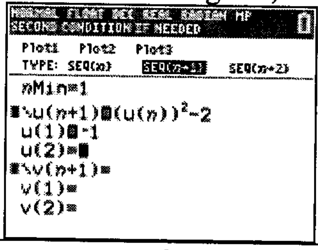
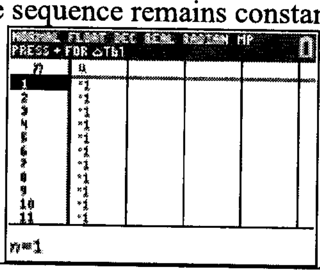
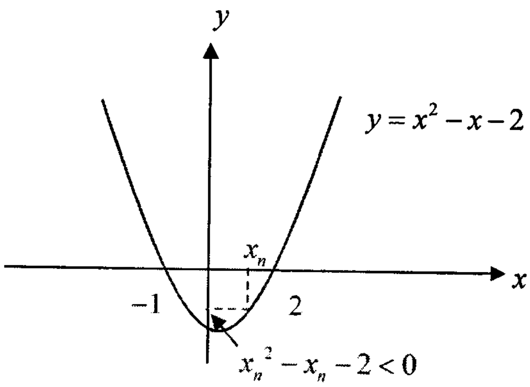
$\left|\frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right|\geq 1.78 \quad (\because b>0)$

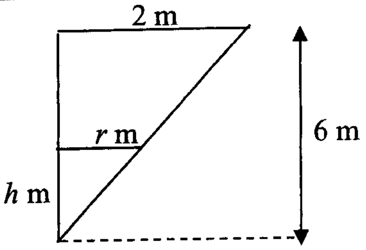
Using GC,

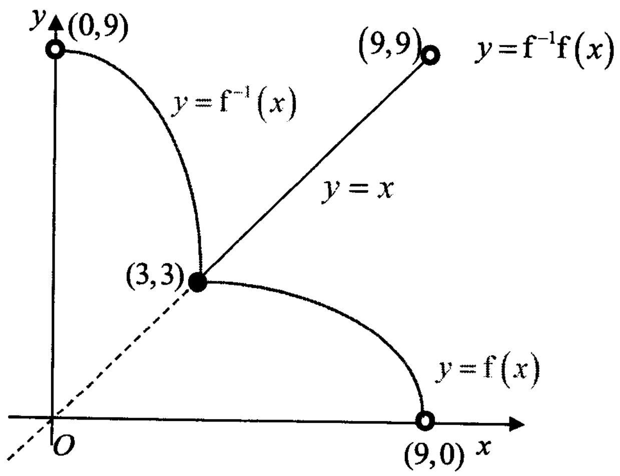
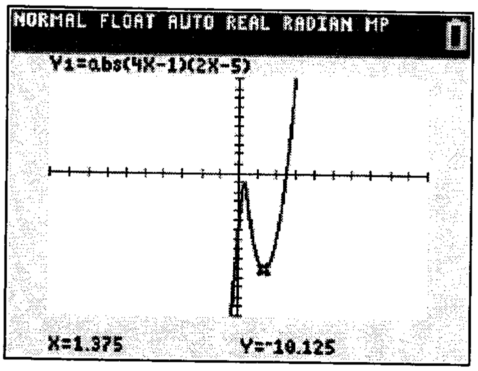
When $n=14$, $\left|\frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right|=1.7785<1.78$

When $n=15$, $\left|\frac{0.6639}{1-0.6639^2}-\frac{1}{1-0.6639}\left(1-0.6639^n\right)\right|=1.7818\geq 1.78$

Therefore, least $n=15$.

Qn	Solution
5	**Recurrence Relations**
(a)	Let $x_n \to L$ as $n \to \infty$. Then $x_{n+1} \to L$. $L = L^2 - 2$ $L^2 - L - 2 = 0$ $(L+1)(L-2) = 0$ $L = -1$ or $L = 2$ (shown)
(b)	**Method 1:** From the limits, Since $L = -1$, the sequence remains constant at -1. **Method 2:** Using GC, the sequence remains constant at -1.  
(c)	$x_{n+1} - x_n = x_n^{\ 2} - 2 - x_n$ $= x_n^{\ 2} - x_n - 2$ If $-1 < x_n < 2$, $x_n^{\ 2} - x_n - 2 < 0$ from the graph.  Hence $x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$.

Qn	Solution
6	**Application of Differentiation**
(a)	 By similar triangles, $\frac{r}{h}=\frac{2}{6}$ $\Rightarrow r=\frac{1}{3}h$ $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h=\frac{1}{3}\pi\left(\frac{1}{9}h^2\right)h=\frac{1}{27}\pi h^3$
(b)	$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-\frac{\mathrm{d}V_{out}}{\mathrm{d}t}$ $=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-0.03$ Since $\frac{\mathrm{d}h}{\mathrm{d}t}=0.1$, $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $=\frac{1}{9}\pi h^2\times 0.1$ $=\frac{1}{9}\pi(3)^2\times 0.1$ $=0.1\pi$ $\therefore \frac{\mathrm{d}V_{in}}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}t}+0.03=0.1\pi+0.03=0.344$ Hence, water is pouring in at a rate of 0.344 m/s

Qn	Solution
7	**Functions Graphing Techniques Equations and Inequalities**
(a)	For f^{-1} to exist, f must be one-one function. Least value of $k = 3$
(b)	Let $y = \frac{1}{2}\sqrt{36-(x-3)^2}$ $2y = \sqrt{36-(x-3)^2}$ $4y^2 = 36-(x-3)^2$ $(x-3)^2 = 36-4y^2$ $x-3 = \pm\sqrt{36-4y^2}$ $x = 3-\sqrt{36-4y^2}$ or $x = 3+\sqrt{36-4y^2}$ (Rejected, $x \geq 3$) $f^{-1}(x) = 3+\sqrt{36-4x^2}$ $= 3+2\sqrt{9-x^2}$
(c)	
(d)	Range of f = (0,3] Domain of g = $(0,\infty)$ Since $R_f \subseteq D_g$, gf exists
(e)	$D_f = [3,9) \xrightarrow{f} R_f = (0,3] \xrightarrow{g} R_{gf} = \left[-\frac{81}{8}, 11\right]$  $R_{gf} = \left[-\frac{81}{8}, 11\right]$

Qn	Solution
8	**Definite Integrals**
(a)	$x = 2t + 3$ $\frac{dx}{dt} = 2$ $y = 5 - 4t^2$ $\frac{dy}{dt} = -8t$ $\frac{dy}{dx} = \frac{-8t}{2} = -4t$ At $(2,4)$, $x = 2t + 3 \Rightarrow 2 = 2t + 3 \Rightarrow t = -\frac{1}{2}$ $\frac{dy}{dx} = -4\left(-\frac{1}{2}\right) = 2$ Equation of normal, N: $y - 4 = -\frac{1}{2}(x-2)$ $y = -\frac{1}{2}x + 5$
(b)	$x = 2t + 3 \Rightarrow t = \frac{x-3}{2}$ Sub t into y: $y = 5 - 4t^2$ $= 5 - 4\left(\frac{x-3}{2}\right)^2$ $= 5 - (x-3)^2$ **Alternatively,** $y = 5 - 4t^2 \Rightarrow t = \pm\sqrt{\frac{5-y}{4}}$ Sub t into x: $x = 2\left(\pm\frac{\sqrt{5-y}}{2}\right) + 3 = 3 \pm \sqrt{5-y}$

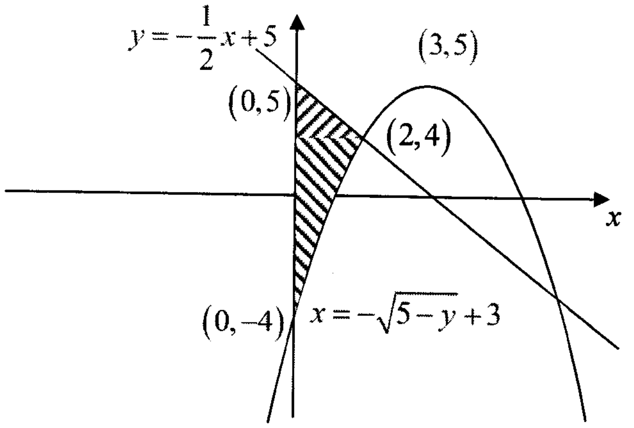
(c)

$y = 5-(x-3)^2$

$x = 3 \pm \sqrt{5-y}$

Since $x \le 3$, $x = 3-\sqrt{5-y}$

$$x^2 = \left(3-\sqrt{5-y}\right)^2$$
$$= 9-6\sqrt{5-y}+5-y$$
$$= 14-y-6\sqrt{5-y}$$



Volume of solid of revolution about y-axis

$$= \pi\int_{-4}^{4} x^2 \, dy + \text{Volume of cone}$$

$$= \pi\int_{-4}^{4} \left(3-\sqrt{5-y}\right)^2 \, dy + \frac{1}{3}\pi(2)^2(1)$$

$$= \pi\int_{-4}^{4} 3^2 - 6\sqrt{5-y} + (5-y) \, dy + \frac{4}{3}\pi$$

$$= \pi\int_{-4}^{4} 14 - y - 6\sqrt{5-y} \, dy + \frac{4}{3}\pi$$

$$= \pi\left[14y - \frac{1}{2}y^2 - \frac{6(5-y)^{\frac{3}{2}}}{\frac{3}{2}(-1)}\right]_{-4}^{4} + \frac{4}{3}\pi$$

$$= \pi\left[14y - \frac{1}{2}y^2 + 4(5-y)^{\frac{3}{2}}\right]_{-4}^{4} + \frac{4}{3}\pi$$

$$= \pi(56-8+4+56+8-108) + \frac{4}{3}\pi$$

$$= \frac{28\pi}{3}$$

Qn	Solution
9	**Complex Numbers**
(a)	**Method 1:** Since the equation has real coefficients, $-2+2\text{i}$ is a root $\Rightarrow -2-2\text{i}$ is also a root. Consider $\left[z-(-2+2\text{i})\right]\left[z-(-2-2\text{i})\right]=\left[(z+2)-2\text{i}\right]\left[(z+2)+2\text{i}\right]$ $=(z+2)^2-[2\text{i}]^2$ $=z^2+4z+4+4$ $=z^2+4z+8$ $z^3+az^2+bz-16\sqrt{2}=\left(z^2+4z+8\right)\left(pz+q\right)$ coeff of z^3 : $p=1$ constant: $q=-2\sqrt{2}$ $z^3+az^2+bz-16\sqrt{2}=\left(z^2+4z+8\right)\left(z-2\sqrt{2}\right)$ coeff of z^2 : $a=-2\sqrt{2}+4$ coeff of z: $b=-8\sqrt{2}+8$ $\therefore a=4-2\sqrt{2},\ b=8-8\sqrt{2}$ and the roots are $-2+2\text{i}, -2-2\text{i}$ and $2\sqrt{2}$. **Method 2:** Since $z=-2+2\text{i}$ is a root, $(-2+2\text{i})^3+a(-2+2\text{i})^2+b(-2+2\text{i})-16\sqrt{2}=0$ $16+16\text{i}+a(-8\text{i})+b(-2+2\text{i})-16\sqrt{2}=0$ $16-2b-16\sqrt{2}+16\text{i}-8a\text{i}+2b\text{i}=0$ Comparing real and imaginary parts, $16-2b-16\sqrt{2}=0$ and $16-8a+2b=0$ $b=8-8\sqrt{2}$ $8a=16+2b$ Subst $b=8-8\sqrt{2}$ into $8a=16+2b$, $8a=16+2\left(8-8\sqrt{2}\right)$ $\therefore a=4-2\sqrt{2}$ $\therefore b=8-8\sqrt{2}$ Since the equation has real coefficients, $-2+2\text{i}$ is a root $\Rightarrow -2-2\text{i}$ is also a root. Consider $\left[z-(-2+2\text{i})\right]\left[z-(-2-2\text{i})\right]=\left[(z+2)-2\text{i}\right]\left[(z+2)+2\text{i}\right]$ $=(z+2)^2-[2\text{i}]^2$ $=z^2+4z+4+4$ $=z^2+4z+8$

$$z^3+\left(4-2\sqrt{2}\right)z^2+\left(8-8\sqrt{2}\right)z-16\sqrt{2}=\left(z^2+4z+8\right)\left(pz+q\right)$$

coeff of z^3 : $p=1$

constant: $q=-2\sqrt{2}$

$$z^3+\left(4-2\sqrt{2}\right)z^2+\left(8-8\sqrt{2}\right)z-16\sqrt{2}=\left(z^2+4z+8\right)\left(z-2\sqrt{2}\right)$$

The roots are $-2+2\text{i}, -2-2\text{i}$ and $2\sqrt{2}$.

(b) Let $z_1=-2+2\text{i} \Rightarrow z_1$ is in 2nd quadrant

$$|z_1|=\sqrt{(-2)^2+(2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\arg(z_1)=\pi-\tan^{-1}\left(\frac{2}{2}\right)=\pi-\tan^{-1}(1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

Let $z_2=-2-2\text{i} \Rightarrow z_2$ is in 3rd quadrant

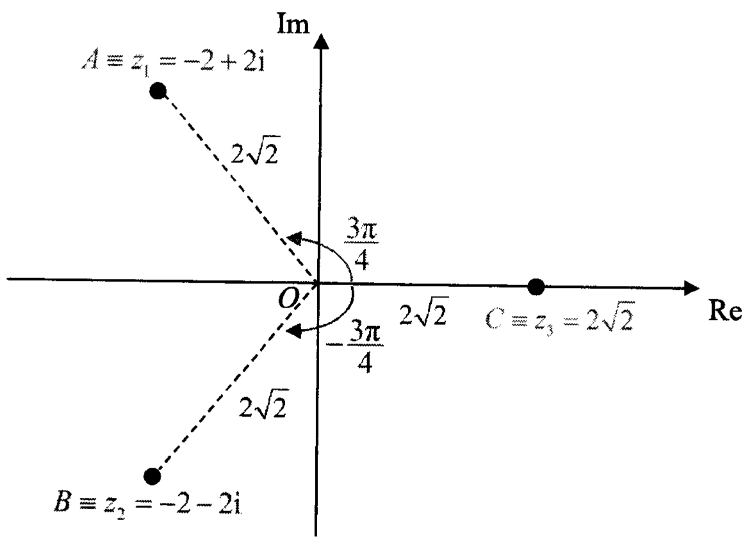
$$|z_2|=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\arg(z_2)=-\pi+\tan^{-1}\left(\frac{2}{2}\right)=-\pi+\tan^{-1}(1)=-\pi+\frac{\pi}{4}=-\frac{3\pi}{4}$$

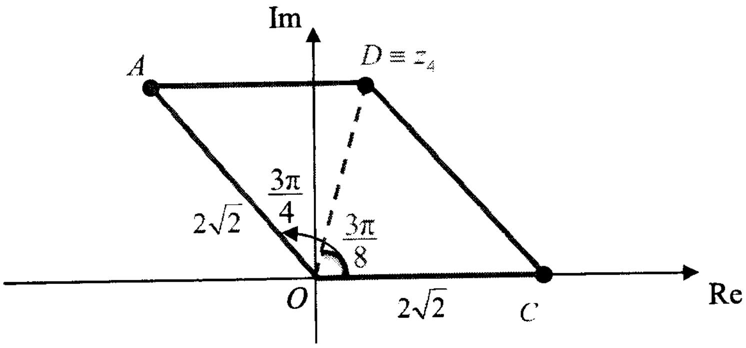
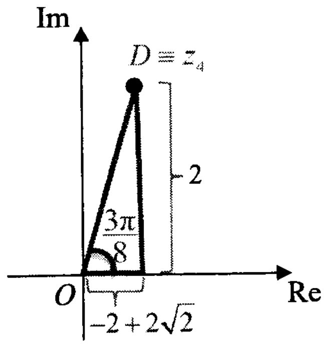
Let $z_3=2\sqrt{2} \Rightarrow z_2$ is in 1st quadrant

$$|z_3|=2\sqrt{2}$$

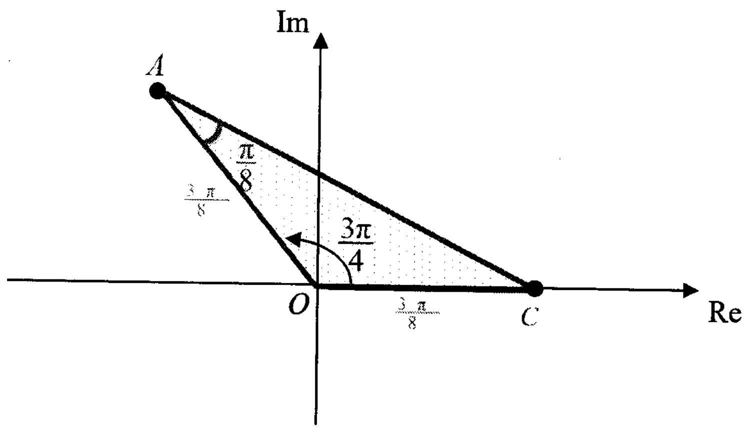
$$\arg(z_3)=0$$



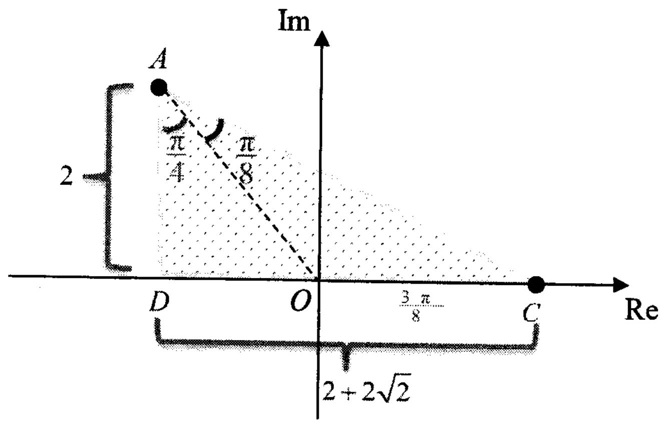
B is the reflection of A **about the real axis**. OR
B is the 90^0 anti-clockwise rotation of A **about O**.

(c)	**Method 1:** Since $OA = OC = 2\sqrt{2}$, we consider D be the point such that $OADC$ is a rhombus.  OD bisects $\angle AOC$ Thus, $\angle DOC = \frac{1}{2}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{8}$ Let $D \equiv z_4$ where $z_4 = -2 + 2i + 2\sqrt{2}$ $\quad = -2 + 2\sqrt{2} + 2i$ Since z_4 lies in 1st quadrant, $\arg(z_4) = \frac{3\pi}{8}$ $\tan\frac{3\pi}{8} = \frac{2}{-2+2\sqrt{2}}$ $= \frac{1}{-1+\sqrt{2}} \times \frac{-1-\sqrt{2}}{-1-\sqrt{2}}$ $= \frac{-1-\sqrt{2}}{(-1)^2 - (\sqrt{2})^2}$ $= 1 + \sqrt{2}$ (shown) 

Method 2:
Consider triangle OAC and observe that triangle OAC is an isosceles triangle as $OA = OC = 2\sqrt{2}$.



$\angle OAC = \frac{1}{2}\left(\pi - \frac{3\pi}{4}\right) = \frac{\pi}{8}$



Let D be the point that represent $-2+0\mathrm{i}$

$\angle OAD = \angle AOD = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

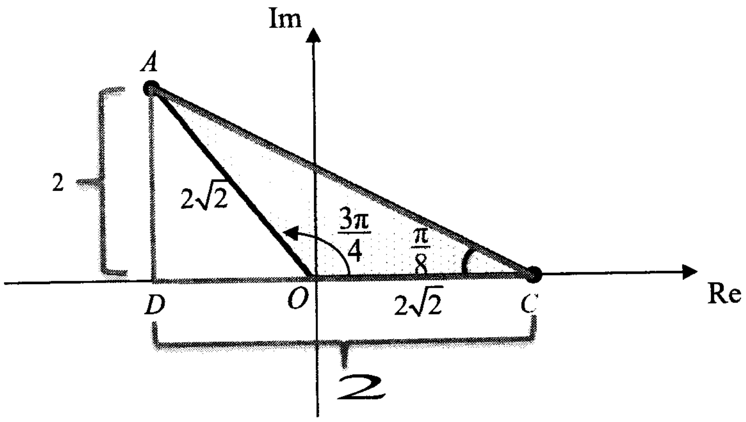
$\angle DAC = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$

Using triangle DAC

$\tan\frac{3\pi}{8} = \frac{2+2\sqrt{2}}{2}$

$= 1+\sqrt{2}$ (shown)

Method 3:
Consider triangle OAC and observe that triangle OAC is an isosceles triangle as $OA = OC = 2\sqrt{2}$.



$\angle OCA = \frac{1}{2}\left(\pi - \frac{3\pi}{4}\right) = \frac{\pi}{8}$

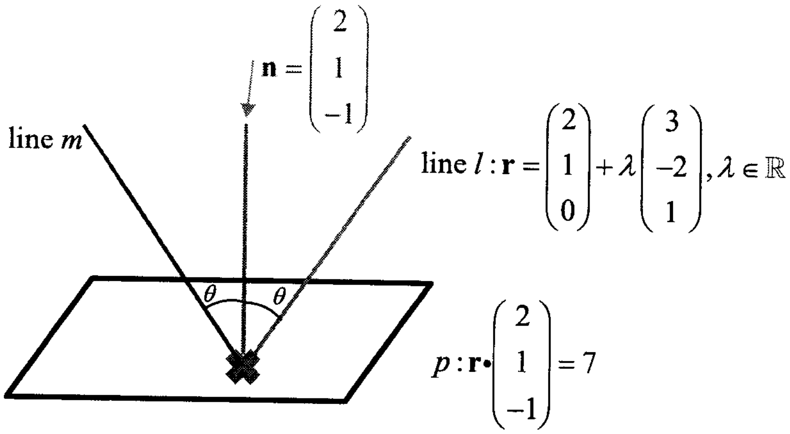
Let D be the point that represent $-2+0\text{i}$

Using triangle DAC

$$\tan\frac{\pi}{8} = \frac{2}{2+2\sqrt{2}}$$
$$= \frac{1}{1+\sqrt{2}}$$

$$\tan\frac{3\pi}{8} = \tan\left(\frac{\pi}{4}+\frac{\pi}{8}\right)$$
$$= \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{8}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{8}}$$
$$= \frac{1+\left(\frac{1}{1+\sqrt{2}}\right)}{1-(1)\left(\frac{1}{1+\sqrt{2}}\right)}$$
$$= \frac{1+\sqrt{2}+1}{1+\sqrt{2}-1}$$
$$= \frac{2+\sqrt{2}}{\sqrt{2}}$$
$$= \frac{2}{\sqrt{2}}+1$$
$$= 1+\sqrt{2} \text{ (shown)}$$

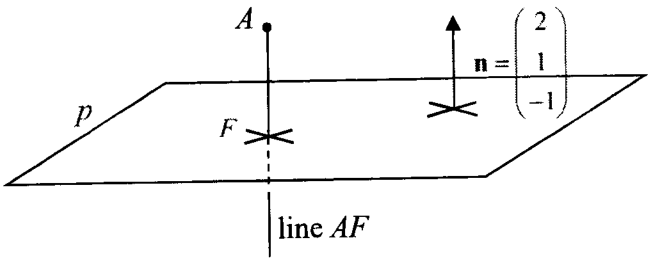
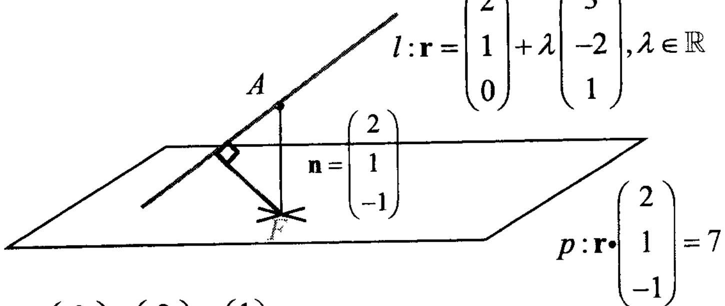
Qn	Solution
10	**3D Vectors**
(a)	Let θ be the acute angle between the direction vector of the flight path and normal vector of the rooftop.



$$\cos\theta=\frac{\left|\begin{pmatrix}3\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}{\sqrt{14}\sqrt{6}}=\frac{3}{\sqrt{14}\sqrt{6}}$$

The angle between the original flight path and the new flight path will be 2θ.

$$\begin{aligned}\cos 2\theta&=2\cos^2\theta-1\\&=2\left(\frac{3}{\sqrt{14}\sqrt{6}}\right)^2-1\\&=-\frac{11}{14}\end{aligned}$$

(b)	Let the foot of perpendicular be F.  $l_{AF}: \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix},\ \mu \in \mathbb{R}$ $\begin{pmatrix} 2+2\mu \\ 1+\mu \\ -\mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 7$ $4+4\mu+1+\mu+\mu=7$ $6\mu+5=7$ $\mu=\frac{1}{3}$ Coordinates of F is $\left(\frac{8}{3}, \frac{4}{3}, -\frac{1}{3}\right)$.
(c)	 $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$ $l_s: \mathbf{r} = \begin{pmatrix} \frac{8}{3} \\ \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix},\ \alpha \in \mathbb{R}$

$$x=\frac{8}{3}+\alpha \Rightarrow \alpha=\frac{3x-8}{3}$$

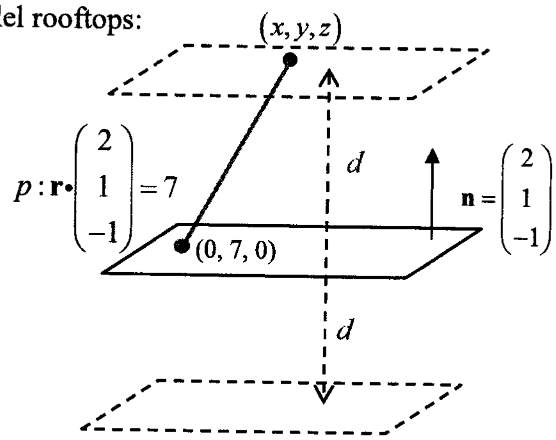
$$y=\frac{4}{3}+5\alpha \Rightarrow \alpha=\frac{3y-4}{15}$$

$$z=-\frac{1}{3}+7\alpha \Rightarrow \alpha=\frac{3z+1}{21}$$

$$\therefore \frac{3x-8}{3}=\frac{3y-4}{15}=\frac{3z+1}{21}$$

Let $(0,7,0)$ be a point on the first rooftop and (x,y,z) be a point on the second rooftop.

Distance between 2 parallel rooftops:



$$\frac{\left|\left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}0\\7\\0\end{pmatrix}\right]\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}{\sqrt{6}}=d$$

$$\frac{\left|\begin{pmatrix}x\\y-7\\z\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|}{\sqrt{6}}=d$$

$$|2x+y-7-z|=\sqrt{6}d$$

$2x+y-7-z=\sqrt{6}d$ or $2x+y-7-z=-\sqrt{6}d$

$2x+y-z=\sqrt{6}d+7$ or $2x+y-z=-\sqrt{6}d+7$

Alternatively,

$$2x+y-z=7$$

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=7$$

$$\mathbf{r}\cdot\frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\sqrt{6}}=\frac{7}{\sqrt{6}}$$

Let the second rooftop be:

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=D$$

$$\mathbf{r}\cdot\frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}}{\sqrt{6}}=\frac{D}{\sqrt{6}}$$

Distance between 2 parallel rooftops:

	$\left\lvert \frac{7}{\sqrt{6}} - \frac{D}{\sqrt{6}} \right\rvert = d$ $\frac{7}{\sqrt{6}} - \frac{D}{\sqrt{6}} = d$ or $-\frac{7}{\sqrt{6}} + \frac{D}{\sqrt{6}} = d$ $D = 7 - \sqrt{6}d$ or $D = 7 + \sqrt{6}d$ $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \sqrt{6}d + 7$ or $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -\sqrt{6}d + 7$ $2x + y - z = \sqrt{6}d + 7$ or $2x + y - z = -\sqrt{6}d + 7$

Qn	Solution
11	**Definite Integrals, Maclaurin Series, Application of Differentiation**
(a)	$\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$ When $x=0$, $\left(\frac{0+y}{2}\right)^2+\left(\frac{y-0}{4}\right)^2=1$ $\frac{5y^2}{16}=1$ $y=\pm\frac{4}{\sqrt{5}}$ Since the region lies in the first quadrant, $y>0$, hence we take $y=\frac{4}{\sqrt{5}}$. **Method 1:** $\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$ $4(x+y)^2+(y-x)^2=16$ $4x^2+8xy+4y^2+y^2-2xy+x^2=16$ $5x^2+6xy+5y^2=16$ Differentiate w.r.t. x, $10x+6y+6x\frac{dy}{dx}+10y\frac{dy}{dx}=0$ Differentiate w.r.t. x, $10+6\frac{dy}{dx}+6\frac{dy}{dx}+6x\frac{d^2y}{dx^2}+10\left(\frac{dy}{dx}\right)^2+10y\frac{d^2y}{dx^2}=0$ $10+12\frac{dy}{dx}+6x\frac{d^2y}{dx^2}+10\left(\frac{dy}{dx}\right)^2+10y\frac{d^2y}{dx^2}=0$ When $x=0$, $y=\frac{4}{\sqrt{5}}$, $10(0)+6\left(\frac{4}{\sqrt{5}}\right)+6(0)\frac{dy}{dx}+10\left(\frac{4}{\sqrt{5}}\right)\frac{dy}{dx}=0$ $\left(\frac{40}{\sqrt{5}}\right)\frac{dy}{dx}=-\frac{24}{\sqrt{5}}$ $\frac{dy}{dx}=-\frac{3}{5}$ $10+12\left(-\frac{3}{5}\right)+6(0)\frac{d^2y}{dx^2}+10\left(-\frac{3}{5}\right)^2+10\left(\frac{4}{\sqrt{5}}\right)\frac{d^2y}{dx^2}=0$ $\left(\frac{40}{\sqrt{5}}\right)\frac{d^2y}{dx^2}=-\frac{32}{5}$ $\frac{d^2y}{dx^2}=-\frac{4\sqrt{5}}{25}$ $y=\frac{4}{\sqrt{5}}-\frac{3}{5}x-\frac{2}{5\sqrt{5}}x^2+\ldots$

Method 2:

$$\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$$

Differentiate w.r.t. x,

$$2\left(\frac{x+y}{2}\right)\left(1+\frac{dy}{dx}\right)\left(\frac{1}{2}\right)+2\left(\frac{y-x}{4}\right)\left(\frac{dy}{dx}-1\right)\left(\frac{1}{4}\right)=0$$

$$\frac{1}{2}(x+y)\left(1+\frac{dy}{dx}\right)+\frac{y-x}{8}\left(\frac{dy}{dx}-1\right)=0$$

Differentiate w.r.t. x,

$$\frac{1}{2}\left(1+\frac{dy}{dx}\right)^2+\frac{1}{2}(x+y)\left(\frac{d^2y}{dx^2}\right)+\frac{1}{8}\left(\frac{dy}{dx}-1\right)^2+\frac{1}{8}(y-x)\left(\frac{d^2y}{dx^2}\right)=0$$

Method 3:

$$\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{4}\right)^2=1$$

$$4(x+y)^2+(y-x)^2=16$$

$$4x^2+8xy+4y^2+y^2-2xy+x^2=16$$

$$5x^2+6xy+5y^2=16$$

$$5y^2+6xy+5x^2-16=0$$

$$y=\frac{-6x\pm\sqrt{(6x)^2-4(5)(5x^2-16)}}{2(5)}$$

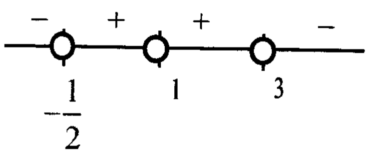
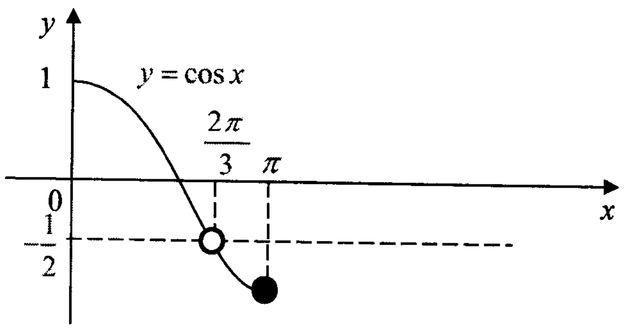
$$=\frac{-6x\pm\sqrt{320-64x^2}}{10}$$

Since $y>0$,

	$y=\frac{-6x+\sqrt{320-64x^2}}{10}$ $=\frac{-6x+8\sqrt{5-x^2}}{10}$ $=\frac{-3x+4\sqrt{5-x^2}}{5}$ $=-\frac{3}{5}x+\frac{4}{5}(5-x^2)^{\frac{1}{2}}$ $=-\frac{3}{5}x+\frac{4\sqrt{5}}{5}\left(1-\frac{x^2}{5}\right)^{\frac{1}{2}}$ $=-\frac{3}{5}x+\frac{4\sqrt{5}}{5}\left(1-\left(\frac{1}{2}\right)\left(\frac{x^2}{5}\right)+...\right)$ $\approx-\frac{3}{5}x+\frac{4\sqrt{5}}{5}-\frac{2\sqrt{5}}{25}x^2$ $\approx\frac{4\sqrt{5}}{5}-\frac{3}{5}x-\frac{2\sqrt{5}}{25}x^2$ Required Estimate $=\int_0^{0.5} y\ \mathrm{d}x$ $\approx\int_0^{0.5}\frac{4}{\sqrt{5}}-\frac{3}{5}x-\frac{2}{5\sqrt{5}}x^2\mathrm{d}x$ $\approx 0.812\mathrm{m}^2\ (3\text{ s.f.})$
(b)	$L=\sqrt{(\cos t-2\sin t)^2+(\cos t+2\sin t)^2}$ $=\sqrt{\cos^2 t-4\sin t\cos t+4\sin^2 t+\cos^2 t+4\sin t\cos t+4\sin^2 t}$ $=\sqrt{2\cos^2 t+8\sin^2 t}$ $=\sqrt{2+6\sin^2 t}$ $\frac{\mathrm{d}L}{\mathrm{d}t}=\frac{12\sin t\cos t}{2\sqrt{2+6\sin^2 t}}$ $=\frac{6\sin t\cos t}{\sqrt{2+6\sin^2 t}}$ Let $\frac{\mathrm{d}L}{\mathrm{d}t}=0$. $6\cos t\sin t=0$ $\cos t=0$ or $\sin t=0$ $t=-\frac{\pi}{2}$ or 0 or $\frac{\pi}{2}$ or π. When $t=\pm\frac{\pi}{2}$, $L=2\sqrt{2}$. When $t=0$ or π, $L=\sqrt{2}$. $\therefore$ Maximum L is $2\sqrt{2}$. $\therefore$ Minimum L is $\sqrt{2}$.

(c)	$\frac{x^2}{8}+\frac{y^2}{2}=1$ **OR** $\frac{x^2}{2}+\frac{y^2}{8}=1$ Required Area $=4\int_0^{\sqrt{8}}\sqrt{2-\frac{x^2}{4}}\ \mathrm{d}x$ $\approx 12.6\mathrm{m}^2\ (3\text{ s.f.})$ **OR** Required Area $=4\int_0^{\sqrt{2}}\sqrt{8-4x^2}\ \mathrm{d}x$ $\approx 12.6\mathrm{m}^2\ (3\text{ s.f.})$

2025 H2 MATH (9758/02) JC 2 PRELIMINARY EXAMINATION

Qn	Solution
1	**Equations and Inequalities**
(a)	$\frac{x+5}{-2x^2+5x+3}<1$ $\frac{x+5}{-2x^2+5x+3}-1<0$ $\frac{x+5-\left(-2x^2+5x+3\right)}{-2x^2+5x+3}<0$ $\frac{x+5+2x^2-5x-3}{-2x^2+5x+3}<0$ $\frac{2x^2-4x+2}{-2x^2+5x+3}<0$ $\frac{2\left(x^2-2x+1\right)}{-2x^2+5x+3}<0$ $\frac{2(x-1)^2}{(3-x)(2x+1)}<0, \quad x\neq -\frac{1}{2}, x\neq 3$  $x<-\frac{1}{2}$ or $x>3$
(b)	$\frac{\cos x+5}{-2\cos^2 x+5\cos x+3}<1$ Replace x by $\cos x$: $\cos x<-\frac{1}{2}$ or $\cos x>3$ (Rej. $\because -1\leq \cos x\leq 1$)  $\therefore \frac{2\pi}{3}<x\leq \pi$

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>2</td><td>Maclaurin Series</td></tr>
<tr><td>(a)</td><td>

Since the curve $f(\theta)=a\theta^2+b\theta+c$ passes through $(0,1)$, $c=1$

Since the curve $f(\theta)=a\theta^2+b\theta+c$ passes through $\left(-\frac{\sqrt{3}}{3},\frac{1}{2}\right)$,

$$a\left(-\frac{\sqrt{3}}{3}\right)^2+b\left(-\frac{\sqrt{3}}{3}\right)+1=\frac{1}{2}\Rightarrow\frac{1}{3}a-\frac{\sqrt{3}}{3}b=-\frac{1}{2}\text{.......(1)}$$

$$f'(\theta)=2a\theta+b$$

Since the turning point is at $\left(-\frac{\sqrt{3}}{3},\frac{1}{2}\right)$, $f'(\theta)=0$

$$2\left(-\frac{\sqrt{3}}{3}\right)a+b=0$$

$$-\frac{2\sqrt{3}}{3}a+b=0$$

$$b=\frac{2\sqrt{3}}{3}a\text{.......(2)}$$

Sub (2) into (1):

$$\frac{1}{3}a-\frac{\sqrt{3}}{3}b=-\frac{1}{2}$$

$$\frac{1}{3}a-\frac{\sqrt{3}}{3}\left(\frac{2\sqrt{3}}{3}a\right)=-\frac{1}{2}$$

$$-\frac{1}{3}a=-\frac{1}{2}\Rightarrow a=\frac{3}{2}$$

$$b=\frac{2\sqrt{3}}{3}a=\frac{2\sqrt{3}}{3}\left(\frac{3}{2}\right)=\sqrt{3}$$

$$a=\frac{3}{2},\ b=\sqrt{3},\ c=1$$

</td></tr>
<tr><td>(b)</td><td>

$$BC^2=AB^2+AC^2-2(AB)(AC)\cos\left(\theta+\frac{\pi}{6}\right)$$

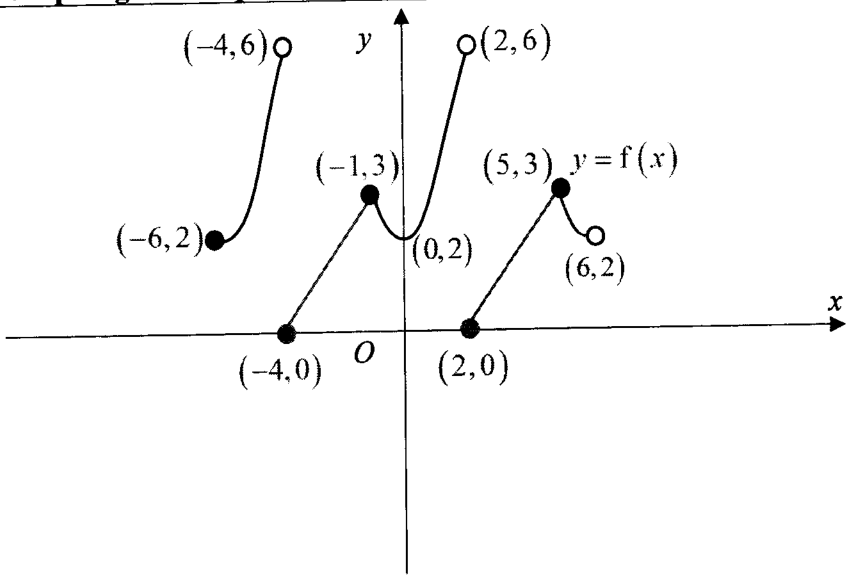
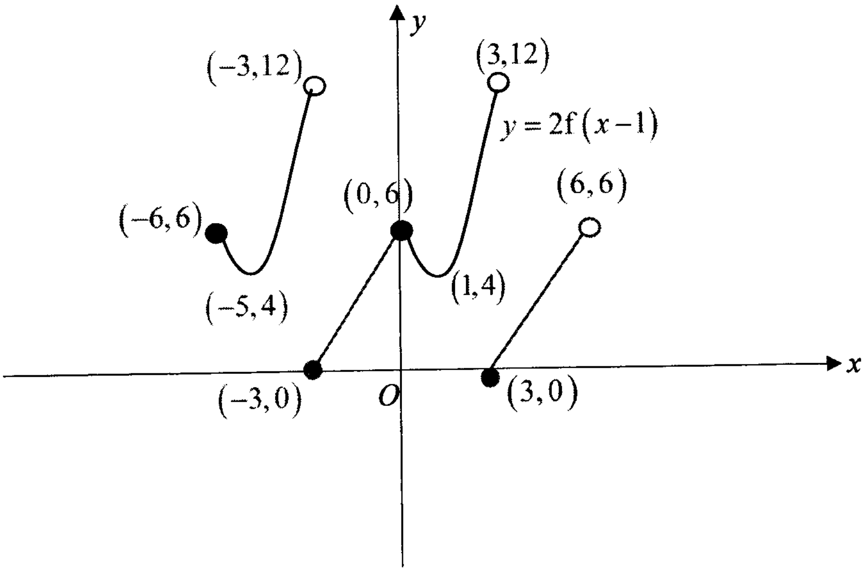
$$=1^2+\left(\sqrt{3}\right)^2-2(1)\left(\sqrt{3}\right)\left(\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}\right)$$

$$=4-2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta\right)$$

$$=4-3\cos\theta+\sqrt{3}\sin\theta$$

</td></tr>
</table>

	$\approx 4-3\left(1-\frac{\theta^2}{2!}\right)+\sqrt{3}\theta$ when θ is small $=4-3+\frac{3\theta^2}{2}+\sqrt{3}\theta$ $=1+\sqrt{3}\theta+\frac{3}{2}\theta^2$ $=\mathrm{f}(\theta)$ (Shown)
(c)	$BC\approx\sqrt{1+\sqrt{3}\theta+\frac{3}{2}\theta^2}$ $\approx\left(1+\sqrt{3}\theta+\frac{3}{2}\theta^2\right)^{\frac{1}{2}}$ $\approx 1+\frac{1}{2}\left(\sqrt{3}\theta+\frac{3}{2}\theta^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\sqrt{3}\theta+\frac{3}{2}\theta^2\right)^2$ $\approx 1+\frac{\sqrt{3}}{2}\theta+\frac{3}{4}\theta^2-\frac{1}{8}\left(3\theta^2\right)$ $=1+\frac{\sqrt{3}}{2}\theta+\frac{3}{8}\theta^2$ (Shown)

Qn	Solution
3	**Graphing Techniques, Transformation of Curves**
(a)	
(b)	

Qn	Solution
4	**Differential Equations**
(a)	$\frac{d^2P}{dt^2}=-0.05\left(\frac{dP}{dt}\right)^2$ Substituting $v=\frac{dP}{dt}$, we get $\frac{dv}{dt}=-0.05v^2$ Using separable variables, $\int\frac{1}{v^2}\,dv=\int -0.05\,dt$ $-\frac{1}{v}=-0.05t+c'$ where c' is an arbitrary constant $v=\frac{1}{0.05t-c'}$ $=\frac{20}{t+C}$ where $C=-20c'$ is a constant (shown)
(b)	$v=\frac{20}{t+C}$ Given $P=0$ and $v=10$ when $t=0$, $10=\frac{20}{C}\Rightarrow C=2$ $v=\frac{20}{t+2}$ $\therefore\frac{dP}{dt}=\frac{20}{t+2}$ $\therefore P=20\ln\lvert t+2\rvert+D$ where D is a constant when $t=0$, $P=0$, $\therefore 0=20\ln\lvert 2\rvert+D$ $D=-20\ln 2$ $P=20\ln\lvert t+2\rvert-20\ln 2$ $=20\ln\left(\frac{t+2}{2}\right)$ since $t>0$
(c)	When $t=30, P=20\ln\left(\frac{30+2}{2}\right)=55.5$ (3 s.f)

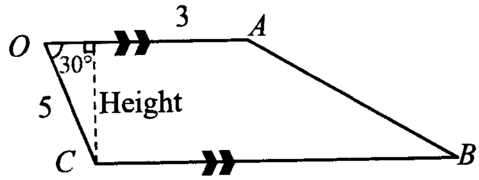
Qn	Solution
5	**Vectors**

(a)

$$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$$
$$\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=\mathbf{0}$$
$$\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}$$

Since $\mathbf{a}\neq\mathbf{0}$ and $\mathbf{b}\neq\mathbf{c}$,
$\therefore \overrightarrow{OA}=\mathbf{a}$ is parallel to $\overrightarrow{CB}=\mathbf{b}-\mathbf{c}$.
$\mathbf{b}-\mathbf{c}=\lambda\mathbf{a}$ (shown)

(b) **Method 1:**



$$\sin 30^\circ=\frac{\text{Height}}{OA}$$
$$\text{Height}=5\sin 30^\circ=\frac{5}{2}$$
$$\text{Area of Trapezium}=\frac{1}{2}(OA+BC)(\text{Height})$$
$$10=\frac{1}{2}(3+BC)\left(\frac{5}{2}\right)$$
$$BC=|\mathbf{c}-\mathbf{b}|=5 \quad \text{(shown)}$$

Method 2:

$$\text{Area of Trapezium}=\frac{1}{2}(|\mathbf{a}|+|\mathbf{c}-\mathbf{b}|)(\text{height of trapezium})$$
$$10=\frac{1}{2}(|\mathbf{a}|+|\mathbf{c}-\mathbf{b}|)\left(\frac{|\mathbf{c}\times\mathbf{a}|}{|\mathbf{a}|}\right)$$
$$10=\frac{1}{2}(3+|\mathbf{c}-\mathbf{b}|)\left(\frac{5\times3\sin 30^\circ}{3}\right)$$
$$20=(3+|\mathbf{c}-\mathbf{b}|)\left(\frac{5}{2}\right)$$
$$BC=|\mathbf{c}-\mathbf{b}|=5 \quad \text{(shown)}$$

Method 3:

$$\text{Area of triangle OAC}=\frac{1}{2}\times|\mathbf{a}|\times|\mathbf{c}|\times\sin 30^\circ=\frac{1}{2}\times3\times5\times\frac{1}{2}=\frac{15}{4}$$
$$\text{Area of triangle ABC}=10-\frac{15}{4}=\frac{25}{4}$$
$$\frac{\text{Area of triangle ABC}}{\text{Area of triangle OAC}}=\frac{\frac{25}{4}}{\frac{15}{4}}=\frac{5}{3}$$

	Area of triangle OAC $=\frac{1}{2}\times\lvert\mathbf{a}\rvert\times h$ Area of triangle ABC $=\frac{1}{2}\times BC\times h$ $\frac{\text{Area of triangle ABC}}{\text{Area of triangle OAC}}=\frac{5}{3}=\frac{BC}{\lvert\mathbf{a}\rvert}=\frac{BC}{3}\Rightarrow BC=5$ (shown)
(c)	$k=\frac{3}{5}$
(d)	$l_{OC}:\mathbf{r}=\lambda\mathbf{c},\quad \lambda\in\mathbb{R}$ $l_{AB}:\mathbf{r}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a}),\quad \mu\in\mathbb{R}$ From **(c)**, $\overrightarrow{OA}=\frac{3}{5}\overrightarrow{CB}\Rightarrow\mathbf{a}=\frac{3}{5}(\mathbf{b}-\mathbf{c})$. At point D, $\lambda\mathbf{c}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$ $l_{AB}:\mathbf{r}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$ $\mathbf{r}=\frac{3}{5}(\mathbf{b}-\mathbf{c})+\mu\left[\mathbf{b}-\frac{3}{5}(\mathbf{b}-\mathbf{c})\right]$ $=\frac{3}{5}\mathbf{b}-\frac{3}{5}\mathbf{c}+\mu\mathbf{b}-\frac{3}{5}\mu\mathbf{b}+\frac{3}{5}\mu\mathbf{c}$ $=\left(\frac{3}{5}-\frac{3}{5}\mu+\mu\right)\mathbf{b}+\left(\frac{3}{5}\mu-\frac{3}{5}\right)\mathbf{c}$ $=\left(\frac{3}{5}+\frac{2}{5}\mu\right)\mathbf{b}+\left(\frac{3}{5}\mu-\frac{3}{5}\right)\mathbf{c}$ At point D, $0\mathbf{b}+\lambda\mathbf{c}=\left(\frac{3}{5}+\frac{2}{5}\mu\right)\mathbf{b}+\left(\frac{3}{5}\mu-\frac{3}{5}\right)\mathbf{c}$ By comparison, $\frac{3}{5}+\frac{2}{5}\mu=0$ $\mu=-\frac{3}{2}$ and $\lambda=\frac{3}{5}\mu-\frac{3}{5}\Rightarrow\lambda=\frac{3}{5}\left(-\frac{3}{2}\right)-\frac{3}{5}=-\frac{3}{2}$ $\therefore\overrightarrow{OD}=-\frac{3}{2}\mathbf{c}$. **Alternatively,** Triangle DOA and triangle DCB are similar triangles.

$$\frac{DC}{CB}=\frac{DO}{OA}$$

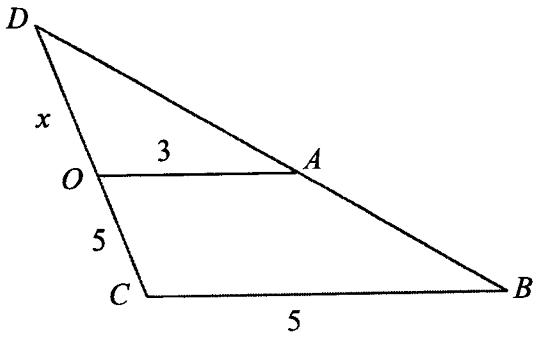
$$\frac{x+5}{5}=\frac{x}{3}$$

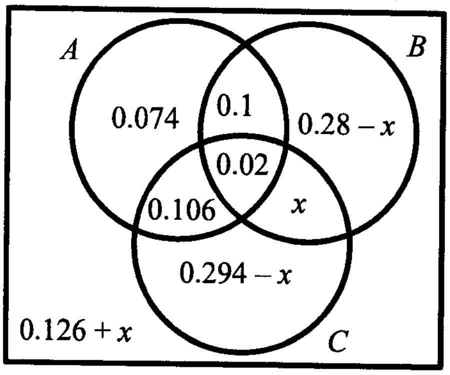
$$3x+15=5x$$

$$x=7.5$$

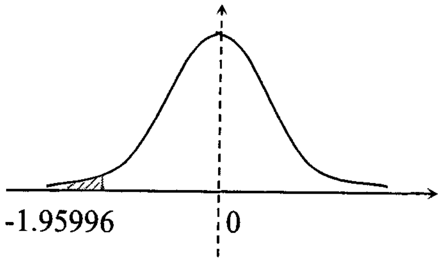
$$\overrightarrow{OD}=-\frac{7.5}{5}\mathbf{c}$$

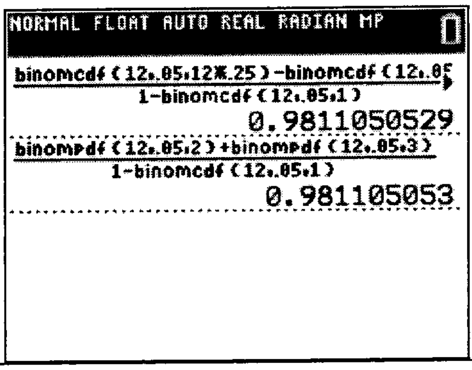
$$=-\frac{3}{2}\mathbf{c}$$

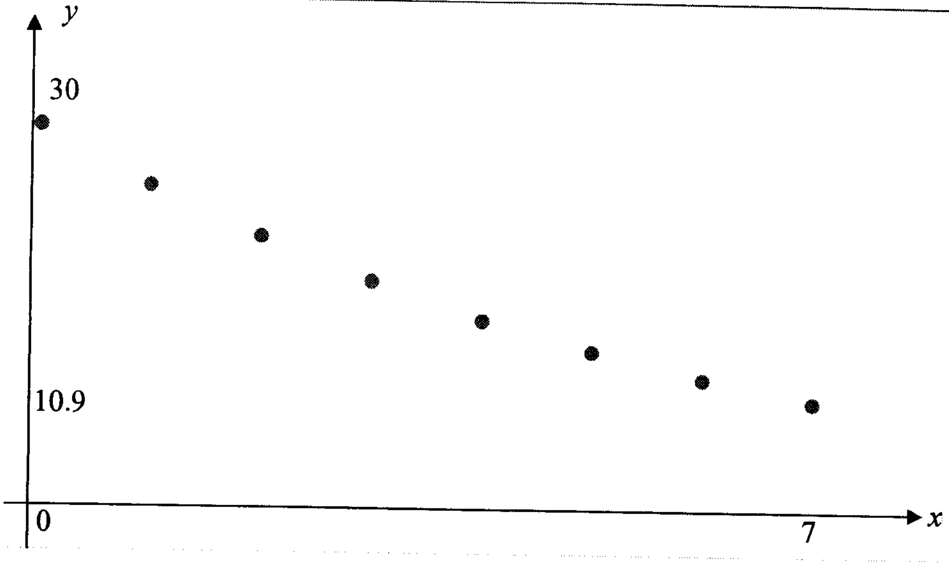


Qn	Suggested Solutions
6	**Probability (Venn diagram)**
(a)	$\text{P}(A\cap B)=\text{P}(A)\times\text{P}(B)$ [since A and B are independent] $=0.3\times0.4$ $=0.12$ $\text{P}(A\cap C)=\text{P}(A)\times\text{P}(C)$ [since A and C are independent] $=0.3\times0.42$ $=0.126$
(b)	Representing values into Venn diagram, 
(c)	From Venn diagram, For minimum x, let $x=0\Rightarrow$ greatest $\text{P}(A'\cap B'\cap C)=0.294$ For maximum x, let $x=0.28\Rightarrow$ least $\text{P}(A'\cap B'\cap C)=0.014$ $\therefore$ least $\text{P}(A'\cap B'\cap C)=0.014$, greatest $\text{P}(A'\cap B'\cap C)=0.294$.

Qn	Solution
7	**Hypothesis Testing**
(a)	Unbiased estimate of μ is $\bar{x}=\frac{\sum x}{n}=\frac{6960}{60}=116$ Unbiased estimate of σ^2 is $s^2=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]$ $=\frac{1}{59}\left[822465-\frac{(6960)^2}{60}\right]$ $=\frac{15105}{59}$ $=256.0169$ $=256$
(b)	Let X be the caffeine content per cup of signature coffee in mg. Let μ denote the population mean caffeine content per cup of signature coffee in mg. $H_0:\mu=120$ $H_1:\mu\neq 120$ Under H_0, since $n=60$ is large, by Central Limit Theorem, $\bar{X}\sim N\left(120,\frac{256.0169}{60}\right)$ approximately. Test Statistic: $Z=\frac{\bar{X}-120}{\sqrt{\frac{256.0169}{60}}}$ Level of significance: 5% Reject H_0 if $p-\text{value}<0.05$ Using G.C, $p-\text{value}=0.0528$ (3 s.f) Since $p-\text{value}=0.0528>0.05$, we do not reject H_0 and conclude that there is insufficient evidence, at the 5% level of significance, that the population mean caffeine content per cup of coffee is not 120 mg. Hence the coffee shop owner's claim is supported by the data at the 5% level of significance.
(c)	Let Y be the caffeine content per cup of premium coffee in mg. Let μ_Y denote the population mean caffeine content per cup of premium coffee in mg. $H_0:\mu_Y=120$ $H_1:\mu_Y<120$ Under H_0, $Y\sim N(120,200)$, $\bar{Y}\sim N\left(120,\frac{200}{n}\right)$ Test Statistic: $Z=\frac{\bar{Y}-120}{\sqrt{\frac{200}{n}}}$ Level of significance: 2.5%

	Reject H_0 if $p-\text{value}<0.025$ Reject H_0 if z-value < -1.95996 $z-\text{value}=\dfrac{116.6-120}{\sqrt{\dfrac{200}{n}}}$  i.e. For H_0 to be rejected at 2.5% level of significance, z-value <-1.95996 $\dfrac{116.6-120}{\sqrt{\dfrac{200}{n}}}<-1.95996$ $-3.4\sqrt{n}<-1.95996\sqrt{200}$ $n>200\left(\dfrac{1.95996}{3.4}\right)^2$ $n>66.461$ Therefore, the set of values is $\{n\in\mathbb{Z}:n\geq 67\}$

Qn	Solution
8	**Probability, Binomial Distribution**
(a)	1) Whether a randomly chosen mystery box contains a seasonal toy is independent of all other mystery boxes. 2) The probability that a randomly chosen mystery box contains a seasonal toy is constant at $\frac{p}{100}$ throughout the sample.
(b)	$S \sim \mathrm{B}\left(n, \frac{p}{100}\right)$ $\mathrm{P}(S=2)=\mathrm{P}(S=3)$ $\binom{n}{2}\left(\frac{p}{100}\right)^2\left(1-\frac{p}{100}\right)^{n-2}=\binom{n}{3}\left(\frac{p}{100}\right)^3\left(1-\frac{p}{100}\right)^{n-3}$ $\frac{n!}{2!(n-2)!}\left(1-\frac{p}{100}\right)=\frac{n!}{3!(n-3)!}\left(\frac{p}{100}\right)$ (since $p>0, 1-p>0$) $\frac{1}{n-2}\left(1-\frac{p}{100}\right)=\frac{1}{3}\left(\frac{p}{100}\right)$ (since $n>0$) $3-3\left(\frac{p}{100}\right)=n\left(\frac{p}{100}\right)-2\left(\frac{p}{100}\right)$ $n\left(\frac{p}{100}\right)=3-\frac{p}{100}$ Since $\mathrm{E}(S)=2.96$, $n\left(\frac{p}{100}\right)=3-\frac{p}{100}=2.96 \Rightarrow p=4$
(c)	Let X be the number of mystery boxes, out of 12, that contains a seasonal toy. $X \sim \mathrm{B}(12, 0.05)$ $12 \times 0.3 = 3.6$ $\mathrm{P}(X \le 3.6 \mid X \ge 2)=\frac{\mathrm{P}(X \le 3 \cap X \ge 2)}{\mathrm{P}(X \ge 2)}$ $=\frac{\mathrm{P}(2 \le X \le 3)}{\mathrm{P}(X \ge 2)}$ $=\frac{\mathrm{P}(X=2)+\mathrm{P}(X=3)}{1-\mathrm{P}(X \le 1)}$ $=0.981$ (3 s.f) 

Qn	Solution
9	**Correlation & Regression**
(a)	
(b)	**(i)** $r = -0.987510 \approx -0.98751$ (5 d.p.) **(ii)** $r = -0.999839 \approx -0.99984$ (5 d.p.)
(c)	From the scatter diagram in (a), it is observed that as x increases, y decreases at a decreasing rate Also, from (b), the product moment correlation coefficient between $\ln y$ and x is closer to -1 than that of y and x. Hence $\ln y = c + dx$ is the better model.
(d)	Using GC, a suitable regression line is $\ln y = 3.3943 - 0.14493x \approx 3.39 - 0.145x$ When $y = 18$, $\ln 18 = 3.3943 - 0.14493x$ $\therefore x = 3.4770 = 3.48$ years
(e)	$\ln y = 3.3943 - 0.14493\left(\dfrac{x}{12}\right)$ $\ln y = 3.3942 - 0.012078x$ $\ln y = 3.39 - 0.0121x$

Qn	Solution
10	**Discrete Random Variables**
(a)	$\text{P}(X=4)=\text{P}(T_4,P_1)+\text{P}(T_2,P_2)$ $=\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}=\frac{1}{6}$ (Shown)
(b)	see tables below

Blouse	Pants		
	P_1 (F)	P_2 (F)	P_3 (C)
T_1 (F)	2	3	3
T_2 (F)	3	4	6
T_3 (C)	3	6	6
T_4 (C)	4	8	7

x	2	3	4	6	7	8
$\text{P}(X=x)$	$\frac{1}{12}$	$\frac{4}{12}=\frac{1}{3}$	$\frac{2}{12}=\frac{1}{6}$	$\frac{3}{12}=\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

(c)

$$\text{E}(X)=2\left(\frac{1}{12}\right)+3\left(\frac{1}{3}\right)+4\left(\frac{1}{6}\right)+6\left(\frac{1}{4}\right)+7\left(\frac{1}{12}\right)+8\left(\frac{1}{12}\right)=\frac{55}{12} \text{ (Shown)}$$

$$\text{E}(X^2)=2^2\left(\frac{1}{12}\right)+3^2\left(\frac{1}{3}\right)+4^2\left(\frac{1}{6}\right)+6^2\left(\frac{1}{4}\right)+7^2\left(\frac{1}{12}\right)+8^2\left(\frac{1}{12}\right)=\frac{293}{12}$$

$$\text{Var}(X)=\frac{293}{12}-\left(\frac{55}{12}\right)^2=\frac{491}{144}$$

Or using G.C.

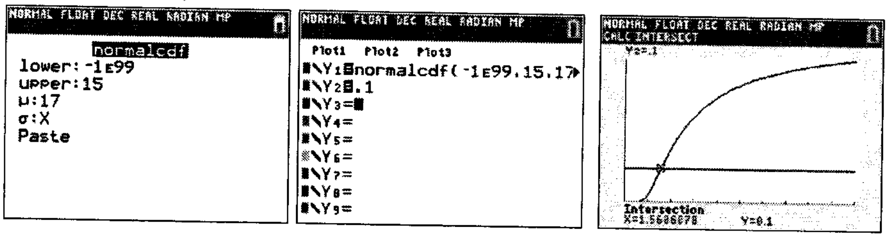
$\text{Var}(X)=3.4097\approx 3.41$

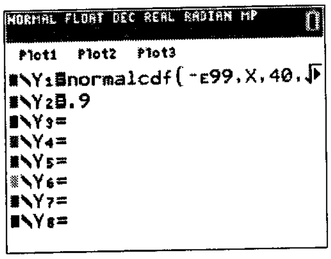
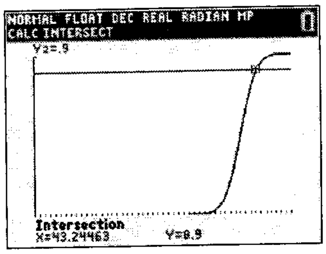
(d)

Required probability $=\left(\frac{2}{4}\times\frac{2}{3}\right)\times\left(\frac{1}{3}\times\frac{1}{2}\right)\times\left(\frac{2}{2}\times\frac{1}{1}\right)\times\frac{3!}{2!}=\frac{1}{6}$

Alternatively,

Required probability $=\frac{4}{12}\times\frac{1}{6}\times\frac{2}{2}\times\frac{3!}{2!}=\frac{1}{6}$

Qn	Solution
11	**Normal, Sampling & Binomial Distribution**
(a)	Let C be the time Anastasia takes to cycle to the train station on a randomly chosen school day (in mins). $C \sim \mathrm{N}\left(8, 0.2^2\right)$ Let T be the time Anastasia takes for a train ride on a randomly chosen school day (in mins). $T \sim \mathrm{N}\left(17, \sigma^2\right)$ Let B be the time Anastasia takes for a bus ride on a randomly chosen school day (in mins). $B \sim \mathrm{N}\left(15, 2.1^2\right)$ **Method 1: Standardisation** $\mathrm{P}(T \leq 15) = 0.1$ $\mathrm{P}\left(Z \leq \frac{15-17}{\sigma}\right) = 0.1$ $\frac{15-17}{\sigma} = -1.28155$ $\sigma = 1.5606$ $= 1.56$ (3 s.f) **Method 2: Graphical** $\mathrm{P}(T \leq 15) = 0.1$ Using GC, $\sigma = 1.5606$ $= 1.56$ (3 s.f) 
(b)	Let J be the time Anastasia takes to travel to school on a randomly chosen school day (in mins). $J = C + T + B$ $\mathrm{E}(J) = 8 + 17 + 15 = 40$ $\mathrm{Var}(J) = 0.2^2 + 1.4^2 + 2.1^2 = 6.41$ $J \sim \mathrm{N}(40, 6.41)$ $\mathrm{P}(35 < J < 50) = 0.97582$ $= 0.976$ (3 s.f)
(c)	If Anastasia leaves house at 6:48 am, she will be late if she reaches school later than 7.30am, ie if the time she takes to travel to school exceed 42 minutes. $\mathrm{P}(J > 42) = 0.21478$ $= 0.215$ (3 s.f)

(d)	Let k be the total duration Anastasia takes to travel to school on a randomly chosen school day (in mins). $\mathrm{P}(\text{not late for school}) \geq 0.9$ $\mathrm{P}(J < k) \geq 0.9$ $k \geq 43.245$ Alternatively,  She needs a **minimum of 44 mins** to travel to school in order be at least 90% confident that she will not be late for school. Thus the latest time to leave house is 6:46 am.
(e)	Aim: To find $\mathrm{P}(C_1 + C_2 + \ldots + C_5 > 2T)$ $\Rightarrow \mathrm{P}(C_1 + C_2 + \ldots + C_5 - 2T > 0)$ Let $X = C_1 + C_2 + \ldots + C_5 - 2T$ $\mathrm{E}(X) = 5\mathrm{E}(C) - 2\mathrm{E}(T) = 5(8) - 2(17) = 6$ $\mathrm{Var}(X) = 5\mathrm{Var}(C) + 2^2\mathrm{Var}(T) = 5(0.2^2) + 2^2(1.4^2) = 8.04$ $X \sim \mathrm{N}(6, 8.04)$ $\mathrm{P}(X > 0) = 0.98283$ $= 0.983$ (3 s.f)
(f)	Let R be the number of days, out of 10, that Anastasia is late for school. $R \sim \mathrm{B}(10, 0.215)$ Let S be the number of days, out of 15, that Anastasia is late for school. $S \sim \mathrm{B}(15, 0.215)$ Required probability $= \mathrm{P}(R = 2)\mathrm{P}(S = 0)\left[0.215 + (0.785)(0.215) + \ldots + (0.785)^4(0.215)\right]$ $= \mathrm{P}(R = 2)\mathrm{P}(S = 0)\left[\dfrac{0.215(1 - 0.785^5)}{1 - 0.785}\right]$ $= 0.0055766$ $= 0.00558$ (3 s.f) **Alternatively,** Let W be the number of days, out of 5, that Anastasia is late for school. $W \sim \mathrm{B}(5, 0.215)$ Required probability

	$=\mathrm{P}(R=2)(1-0.215)^{15}\,\mathrm{P}(W\geq 1)$ $=\mathrm{P}(R=2)(1-0.215)^{15}\left(1-\mathrm{P}(W=0)\right)$ $=0.0055766$ $=0.00558$ (3 s.f)

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