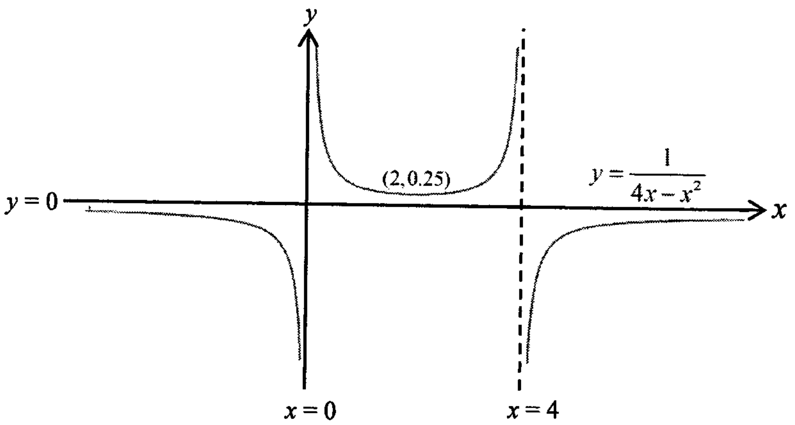
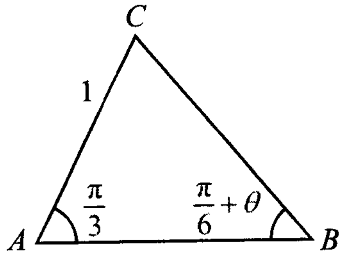
2025 TJC Prelim Exam H2 Math Paper 1 Solution

1(a)	$\frac{v_n}{v_{n-1}}=\frac{9}{u_n}\div\frac{9}{u_{n-1}}=\frac{9}{a\left(\frac{3}{2}\right)^{n-1}}\times\frac{a\left(\frac{3}{2}\right)^{n-2}}{9}=\frac{2}{3}$ (constant) for all positive integers n. Therefore $\{v_n\}$ is a geometric sequence.
1(b)	$v_2=u_2 \Rightarrow \left(\frac{9}{a}\right)\left(\frac{2}{3}\right)=a\left(\frac{3}{2}\right)$ $\Rightarrow a^2=4 \Rightarrow a=2$ (rej -2 since a is positive)

2	Differentiate $xy=1+(x-y)^3$ w.r.t. x, we have $x\frac{dy}{dx}=3(x-y)^2\left(1-\frac{dy}{dx}\right)$ At $y=0,\ x=-1$ Gradient of tangent at $A\,(-1,0)$ is $-\frac{dy}{dx}=3\left(1-\frac{dy}{dx}\right) \Rightarrow \frac{dy}{dx}=\frac{3}{2}$

3(a)	$y=\frac{a}{2bx-x^2}=\frac{a}{b^2-(x-b)^2}$ (Completing Square) ↓ replace x by $x+b$ $y=\frac{a}{b^2-x^2}$ Translation of b units in the negative direction of the x-axis.
3(b)	

4(a)	Using Sine Rule, $\dfrac{BC}{\sin\frac{\pi}{3}} = \dfrac{1}{\sin\left(\frac{\pi}{6}+\theta\right)}$ $\dfrac{BC}{\sin\frac{\pi}{3}} = \dfrac{1}{\sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta}$ $\dfrac{BC}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta}$ $BC = \dfrac{\sqrt{3}}{\cos\theta + \sqrt{3}\sin\theta}$ 
4(b)	$BC \approx \dfrac{\sqrt{3}}{\left(1-\frac{\theta^2}{2}\right)+\sqrt{3}\theta}$ (since θ is small) $= \sqrt{3}\left[1+\left(\sqrt{3}\theta - \frac{\theta^2}{2}\right)\right]^{-1}$ $\approx \sqrt{3}\left[1-\left(\sqrt{3}\theta - \frac{\theta^2}{2}\right)+\left(\sqrt{3}\theta - \frac{\theta^2}{2}\right)^2\right]$ $\approx \sqrt{3}\left[1-\left(\sqrt{3}\theta - \frac{\theta^2}{2}\right)+3\theta^2\right]$ $= \sqrt{3}\left(1-\sqrt{3}\theta + \frac{7\theta^2}{2}\right)$

5(a)

$$\frac{2x-25}{x^2-2x-3} \le 2$$

$$\frac{2x-25-2\left(x^2-2x-3\right)}{x^2-2x-3} \le 0$$

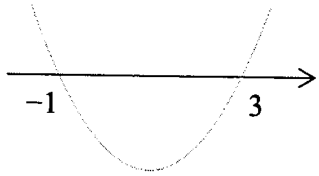
$$\frac{-2x^2+6x-19}{x^2-2x-3} \le 0$$

$$\frac{-2\left(x-\frac{3}{2}\right)^2-\frac{29}{2}}{x^2-2x-3} \le 0 \quad \text{--- (1)}$$

Since the numerator $-2\left(x-\frac{3}{2}\right)^2-\frac{29}{2}<0$ for all real values of x, $x^2-2x-3>0$.

$(x-3)(x+1)>0$

$x<-1$ or $x>3$



5(b)

Replace x by $|x+1|$,

from **(a)**, $|x+1|<-1$ or $|x+1|>3$

$|x+1|<-1$ has no solution

$|x+1|>3 \Rightarrow x+1>3$ or $x+1<-3$

$x>2$ or $x<-4$

6(a)	Using ratio theorem, $\overrightarrow{OX}=\frac{3\mathbf{a}+\mathbf{b}}{4}$
6(b)	$\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\lambda^2$
6(c)	$\overrightarrow{OY}=\left(\overrightarrow{OX}\cdot\mathbf{b}\right)\mathbf{b}=\left(\frac{1}{4}(3\mathbf{a}+\mathbf{b})\cdot\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right)\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$ $=\frac{1}{4}\left(\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)(3\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b})$ $=\frac{1}{4}\left(\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\left(3\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{\pi}{6}+\lvert\mathbf{b}\rvert^2\right)$ $=\frac{1}{4}\left(\frac{\mathbf{b}}{\lambda^2}\right)\left(3\lambda^2\left(\frac{\sqrt{3}}{2}\right)+\lambda^2\right)$ $=\frac{2+3\sqrt{3}}{8}\mathbf{b}$ (Shown) **Alternative solution:** $\overrightarrow{OY}=\mu\mathbf{b}\quad\Rightarrow\overrightarrow{XY}=\mu\mathbf{b}-\frac{3\mathbf{a}+\mathbf{b}}{4}=-\frac{3}{4}\mathbf{a}+\left(\mu-\frac{1}{4}\right)\mathbf{b}$ $\overrightarrow{XY}\bullet\overrightarrow{OB}=0$ $\left[-\frac{3}{4}\mathbf{a}+\left(\mu-\frac{1}{4}\right)\mathbf{b}\right]\bullet\mathbf{b}=0$ $-\frac{3}{4}\mathbf{a}\bullet\mathbf{b}+\left(\mu-\frac{1}{4}\right)\mathbf{b}\bullet\mathbf{b}=0$ $-\frac{3}{4}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{\pi}{6}+\left(\mu-\frac{1}{4}\right)\lvert\mathbf{b}\rvert^2=0$ $-\frac{3}{4}\lvert\mathbf{b}\rvert^2\left(\frac{\sqrt{3}}{2}\right)+\left(\mu-\frac{1}{4}\right)\lvert\mathbf{b}\rvert^2=0$ $\mu-\frac{1}{4}=\frac{3\sqrt{3}}{8}\Rightarrow\mu=\frac{2+3\sqrt{3}}{8}$ $\therefore\overrightarrow{OY}=\frac{2+3\sqrt{3}}{8}\mathbf{b}$, (Shown)

6(d)	Area of $\Delta OXY = \frac{1}{2}\left\lvert \overrightarrow{OX} \times \overrightarrow{OY} \right\rvert$ $= \frac{1}{2}\left\lvert \left(\frac{3\mathbf{a}+\mathbf{b}}{4}\right) \times \left(\frac{\left(2+3\sqrt{3}\right)}{8}\mathbf{b}\right) \right\rvert$ $= \frac{\left(2+3\sqrt{3}\right)}{64}\left\lvert (3\mathbf{a}+\mathbf{b}) \times \mathbf{b} \right\rvert$ $= \frac{\left(2+3\sqrt{3}\right)}{64}\left\lvert 3\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{b} \right\rvert$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\left\lvert \mathbf{a}\times\mathbf{b} \right\rvert$ where $\mathbf{b}\times\mathbf{b} = \mathbf{0}$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\left(\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\frac{\pi}{6}\right)$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\lvert\mathbf{b}\rvert^2\left(\frac{1}{2}\right)$ $\left(\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lambda = \text{radius of circle}\right)$ $= \frac{3\left(2+3\sqrt{3}\right)}{128}\lambda^2$ units^2

7(a)	For Mabel:
	Amount of money Mabel saves in a year $= \$(101 + 102 + \ldots + 112)$ $= \$\frac{12}{2}(101+112) = \$1278$
	Amount of money Mabel saves in 5 year $= \$1278 \times 5 = \$6390$
	For Janice:

nth mth	Amount at start of month (\$)	Amount at end of month (\$)
1 Jan26	100	$100(1.003)$
2	$100+100(1.003)$ $=100(1+1.003)$	$100(1+1.003)(1.003)$ $=100\left(1.003+1.003^2\right)$
$\vdots$		$\vdots$
n		$100\left(1.003+1.003^2+\cdots+1.003^n\right)$ $=\frac{100(1.003)\left(1.003^n-1\right)}{1.003-1}$

At Dec 2030, n =60.

Therefore, Amount Janice saves in 5 years

$$=\frac{100(1.003)\left(1.003^{60}-1\right)}{1.003-1}=\$6582.85 \quad (2 \text{ d.p.}) \quad [A1]$$

$\therefore$ Janice will have more money in her savings account than Mabel has in her piggy bank.

7(b)	At end of 4th year, Mabel will have \$1278 x 4 = \$5112 Janice will have $\frac{100(1.003)(1.003^{48}-1)}{1.003-1}=\$5169.97 > \$5112$ At end of 3rd year, Mabel will have \$1278 x 3 = \$3834 Janice will have $\frac{100(1.003)(1.003^{36}-1)}{1.003-1}=\$3806.97 < \$3834$ Therefore, the year when Janice's saving is more than Mabel's saving happen in 4th year (i.e. 2029). Let it be the k^{th} month. Then set Janice's saving > Mabel's saving $\Rightarrow \frac{100(1.003)(1.003^{36+k}-1)}{1.003-1} > 3834+\frac{k}{2}\left[2(100)+(k-1)(1)\right]$ $\Rightarrow \left(\frac{100003}{3}\right)(1.003^{36+k}-1)-3834-\frac{k}{2}[199+k]>0$ Using GC, we have $k \geq 4$. Therefore, the month and year is Apr 2029.

8(a)

$2z+|w|=2-4\text{i}$ --- (1)

$\text{i}z-w=2\text{i}$ --- (2)

From (1) : $z=\dfrac{2-4\text{i}-|w|}{2}$

Substitute into (2):

$\text{i}\left(\dfrac{2-4\text{i}-|w|}{2}\right)-w=2\text{i}$

$2\text{i}+4-\text{i}|w|-2w=4\text{i}$

Let $w=a+\text{i}b,\ \ a,b\in\mathbb{R}$,

$2\text{i}+4-\text{i}\sqrt{a^2+b^2}-2(a+\text{i}b)=4\text{i}$

$4-2a+\text{i}(2-\sqrt{a^2+b^2}-2b)=4\text{i}$

Compare real and imaginary parts:

$4-2a=0 \Rightarrow a=2$ --- (3)

$2-\sqrt{a^2+b^2}-2b=4$ --- (4)

Substitute (3) into (4):

$-2-2b=\sqrt{4+b^2}$ --- (5)

$\Rightarrow (-2-2b)^2=4+b^2$

$4+8b+4b^2=4+b^2$

$b(3b+8)=0$

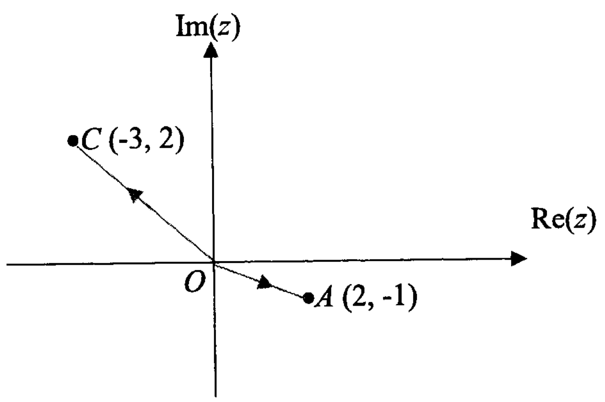
$\Rightarrow b=0$ (rejected since $\sqrt{4+b^2}\neq -2$ from (5))

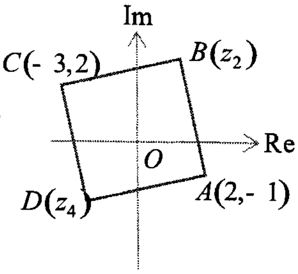
or $b=-\dfrac{8}{3}$

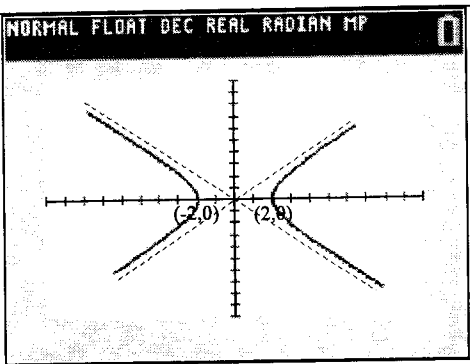
$\therefore\ w=2-\dfrac{8}{3}\text{i}$

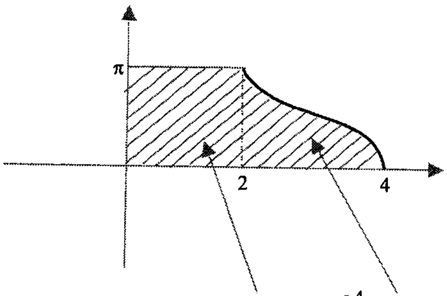
$z=\dfrac{2-4\text{i}-(-2-2b)}{2}=2-2\text{i}+\left(-\dfrac{8}{3}\right)=-\dfrac{2}{3}-2\text{i}$

8(b)

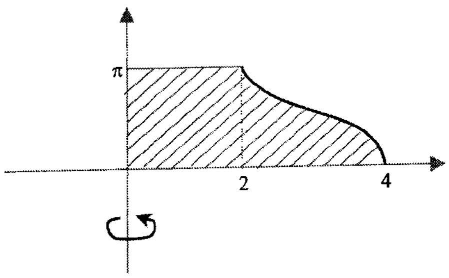


	$\overrightarrow{BA}$ represents the complex number $2-\mathrm{i}-z_2$ and $\overrightarrow{BC}$ represents the complex number $-3+2\mathrm{i}-z_2$. Since $\angle ABC=\frac{\pi}{2}$, A is obtained by rotating C through 90° in anti-clockwise direction about B.  $\mathrm{i}(-3+2\mathrm{i}-z_2)=2-\mathrm{i}-z_2$ $-3\mathrm{i}-2-\mathrm{i}z_2=2-\mathrm{i}-z_2$ $(1-\mathrm{i})z_2=4+2\mathrm{i}$ $z_2=\frac{4+2\mathrm{i}}{1-\mathrm{i}}\cdot\frac{1+\mathrm{i}}{1+\mathrm{i}}=\frac{2+6\mathrm{i}}{2}=1+3\mathrm{i}$ $\overrightarrow{AB}=\overrightarrow{DC}$ $1+3\mathrm{i}-(2-\mathrm{i})=-3+2\mathrm{i}-z_4$ $z_4=-3+2\mathrm{i}+1-4\mathrm{i}=-2-2\mathrm{i}$

9(a)	
9(b)	$x=t+\frac{1}{t} \Rightarrow \frac{dx}{dt}=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}$ $y=t-\frac{1}{t} \Rightarrow \frac{dy}{dt}=1+\frac{1}{t^2}=\frac{t^2+1}{t^2}$ $\frac{dy}{dx}=\frac{t^2+1}{t^2-1}$ Tangent parallel to y-axis $\Rightarrow \frac{dy}{dx}$ is undefined $\Rightarrow t^2-1=0 \Rightarrow t=\pm 1$
(c)	At $t=2$, $\frac{dy}{dx}=\frac{5}{3}$, $x=\frac{5}{2}$, $y=\frac{3}{2}$ Equation of normal at $t=2$ is $y-\frac{3}{2}=-\frac{3}{5}\left(x-\frac{5}{2}\right)$ or $y=-\frac{3}{5}x+3$
(d)	$x=t+\frac{1}{t}$, $y=t-\frac{1}{t}$ ----- (1) $y=-\frac{3}{5}x+3$ ----- (2) Sub (1) into (2): $t-\frac{1}{t}=-\frac{3}{5}\left(t+\frac{1}{t}\right)+3$ $\Rightarrow t^2-1=-\frac{3}{5}t^2+3t-\frac{3}{5}$ $\Rightarrow \frac{8}{5}t^2-3t-\frac{2}{5}=0$ $\Rightarrow t=-\frac{1}{8}$ or $t=2$ (this is already used) So coordinates of Q are $\left(-\frac{65}{8},\frac{63}{8}\right)$

10(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1-x^2}=\frac{-2x}{2\sqrt{1-x^2}}=-\frac{x}{\sqrt{1-x^2}}$ $\int_{-1}^{1}\cos^{-1}x\,\mathrm{d}x=\left[x\cos^{-1}x\right]_{-1}^{1}-\int_{-1}^{1}x\left[-\frac{1}{\sqrt{1-x^2}}\right]\mathrm{d}x$ $=1(0)-(-1)(\pi)-\int_{-1}^{1}\frac{1}{2}\cdot\frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $=\pi-\left[\sqrt{1-x^2}\right]_{-1}^{1}$ $=\pi-0$ $=\pi$
10(bi)	Translate the graph of $y=\cos^{-1}(x-3)$ three units in the negative direction of x-axis to obtain the graph of $y=\cos^{-1}x$
10(bii)	 Area of the region $R=(2)(\pi)+\int_{2}^{4}\cos^{-1}(x-3)\,\mathrm{d}x$ $=2\pi+\int_{-1}^{1}\cos^{-1}u\,\mathrm{d}u$ $=3\pi$ Alternative Method: Area of the region $R=\int_{0}^{\pi}3+\cos y\,\mathrm{d}y$ $=\left[3x+\sin y\right]_{0}^{\pi}$ $=3\pi$

10(c)



Volume generated when rotated completely about y-axis

$$= \int_0^{\pi} \pi x^2 \, dy$$

$$= \pi \int_0^{\pi} (\cos y + 3)^2 \, dy$$

$$= \pi \int_0^{\pi} (\cos^2 y + 6\cos y + 9) \, dy$$

$$= \pi \int_0^{\pi} \left[\frac{1}{2}(\cos 2y + 1) + 6\cos y + 9 \right] dy$$

$$= \pi \left[\frac{1}{4}\sin 2y + 6\sin y + \frac{19}{2} y \right]_0^{\pi}$$

$$= \frac{19}{2}\pi^2$$

11(a)	$\begin{pmatrix}1\\1\\\alpha\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1+\alpha\\-1+2\alpha\\-3\end{pmatrix}$
11(b)	xy-plane $\Rightarrow z=0$ $x+y=6 \quad ...(1)$ $2x-y=4 \quad ...(2)$ $(1)+(2): 3x=10 \Rightarrow x=\frac{10}{3}$ $\therefore y=6-\frac{10}{3}=\frac{8}{3}$ $\therefore \overrightarrow{OC}=\frac{1}{3}\begin{pmatrix}10\\8\\0\end{pmatrix}$ Hence coordinates of C is $(\frac{10}{3},\frac{8}{3},0)$. Equation of the ridge line: $\underset{\sim}{r}=\frac{1}{3}\begin{pmatrix}10\\8\\0\end{pmatrix}+\mu\begin{pmatrix}1+\alpha\\-1+2\alpha\\-3\end{pmatrix}, \mu\in\mathbf{R}$
11(c)	$\cos 60° = \dfrac{\left\lvert\begin{pmatrix}1\\1\\\alpha\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\rvert}{\sqrt{2+\alpha^2}\sqrt{6}} \Rightarrow \frac{1}{2}=\frac{\lvert 1+\alpha\rvert}{\sqrt{2+\alpha^2}\sqrt{6}}$ $\sqrt{2+\alpha^2}\sqrt{6}=2\lvert 1+\alpha\rvert$ $6\left(2+\alpha^2\right)=4\left(1+\alpha\right)^2$ $3\left(2+\alpha^2\right)=2\left(\alpha^2+2\alpha+1\right)$ $\alpha^2-4\alpha+4=0$ $\left(\alpha-2\right)^2=0$ $\Rightarrow \alpha=2$
11(d)	$\overrightarrow{OE}=\begin{pmatrix}2\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ for some $\lambda\in\mathbf{R}$ $\overrightarrow{OE}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ $\begin{pmatrix}2+2\lambda\\3-\lambda\\1+\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ $2\left(2+2\lambda\right)-\left(3-\lambda\right)+\left(1+\lambda\right)=4$ $\lambda=\frac{1}{3}$

	$\overrightarrow{OE}=\begin{pmatrix}\frac{8}{3}\\ \frac{8}{3}\\ \frac{4}{3}\end{pmatrix}$ i.e. $E\left(\frac{8}{3},\frac{8}{3},\frac{4}{3}\right)$

12(a)	Differentiate $w=x^2y$ with respect to x, $\frac{dw}{dx}=2xy+x^2\frac{dy}{dx}$ ---------- (1) Multiply x to given D.E., we have $2xy+x^2\frac{dy}{dx}=\frac{x^5y^2}{\sqrt{x^2+1}}$ ------ (2) Subst. (1) into (2) and replace x^4y^2 by w^2 : $\frac{dw}{dx}=\frac{w^2x}{\sqrt{x^2+1}}$ $\int\frac{1}{w^2}\,dw=\int\frac{x}{\sqrt{x^2+1}}dx$ $\int w^{-2}\,dw=\frac{1}{2}\int 2x\left(x^2+1\right)^{-\frac{1}{2}}dx$ $\frac{w^{-1}}{-1}=\frac{1}{2}\frac{\left(x^2+1\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+C$ where C is arb. constant $-\frac{1}{w}=\sqrt{x^2+1}+C$ $-\frac{1}{x^2y}=\sqrt{x^2+1}+C$ $y=-\frac{1}{x^2\left(\sqrt{x^2+1}+C\right)}$
(b)(i)	$\frac{dx}{dt}=kx(P-x)$ where $k>0$ Given that $x=4$ and $\frac{dx}{dt}=P-4$, $P-4=4k(P-4)$ $k=\frac{1}{4}$ $\frac{dx}{dt}=\frac{1}{4}x(P-x)$

(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{4}x(P-x)$$

$$\int \frac{1}{x(P-x)}\,\mathrm{d}x=\int \frac{1}{4}\,\mathrm{d}x$$

$$\frac{1}{P}\int \frac{1}{x}+\frac{1}{P-x}\,\mathrm{d}x=\int \frac{1}{4}\,\mathrm{d}t$$

$\ln|x|-\ln|P-x|=\frac{P}{4}t+C$ where C is arbitrary constant

$\ln\frac{x}{P-x}=\frac{P}{4}t+C$ where $P-x>0$ and $x>0$

$\frac{x}{P-x}=A\mathrm{e}^{\frac{P}{4}t}$ where $A=\mathrm{e}^{C}$

$$\frac{P-x}{x}=\frac{1}{A}\mathrm{e}^{-\frac{P}{4}t}$$

$\frac{P}{x}-1=B\mathrm{e}^{-\frac{P}{4}t}$ where $B=\frac{1}{A}$

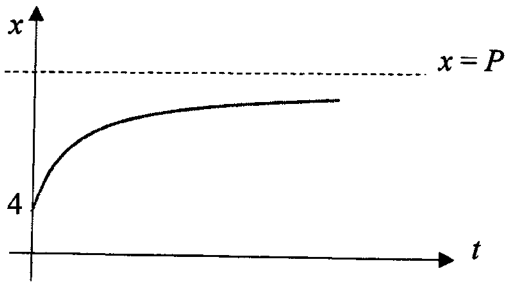
$$x=\frac{P}{1+B\mathrm{e}^{-\frac{P}{4}t}}$$

When $t=0$, $x=4$, $\frac{P}{4}-1=B$

$$x=\frac{P}{1+\left(\frac{P-4}{4}\right)\mathrm{e}^{-\frac{P}{4}t}}=\frac{4P}{4+(P-4)\mathrm{e}^{-\frac{P}{4}t}}$$

(iii)

$$x=\frac{4P}{4+(P-4)\mathrm{e}^{-\frac{P}{4}t}}$$



As $t\to\infty$, $\mathrm{e}^{-\frac{P}{4}t}\to 0$. $\therefore x\to P$

In the long term, all the durians in the plantation will be infected and become bad durians.

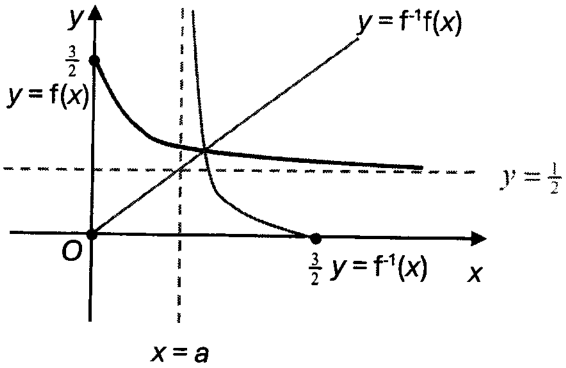
Section A: Pure Mathematics Solutions

1(a)	$y=\frac{1}{2}\ln\left(1+e^{3x}\right)$ $e^{2y}=1+e^{3x}$ Diff. wrt x, $2e^{2y}\frac{dy}{dx}=3e^{3x}$ (Shown) -- (1) <u>Alternative solution:</u> $y=\ln\sqrt{1+e^{3x}}$ $\frac{dy}{dx}=\frac{3e^{3x}\left(1+e^{3x}\right)^{\frac{-1}{2}}}{2\left(1+e^{3x}\right)^{\frac{1}{2}}}=\frac{3e^{3x}}{2\left(1+e^{3x}\right)}$ $2\left(1+e^{3x}\right)\frac{dy}{dx}=3e^{3x}$ $2e^{2y}\frac{dy}{dx}=3e^{3x}$ $\left(\begin{array}{l}\because y=\ln\sqrt{1+e^{3x}}=\frac{1}{2}\ln\left(1+e^{3x}\right)\\ \text{i.e. } 2y=\ln\left(1+e^{3x}\right)\Rightarrow e^{2y}=1+e^{3x}\end{array}\right)$
(b)	Diff. wrt x, $2e^{2y}\frac{d^2y}{dx^2}+4e^{2y}\left(\frac{dy}{dx}\right)^2=9e^{3x}$ ---- (2) When $x=0, y=\frac{1}{2}\ln(1+1)=\frac{1}{2}\ln 2$, $e^{2y}=e^{\ln 2}=2$ Sub into (1): $\frac{dy}{dx}=\frac{3}{4}$ Sub into (2): $2(2)\frac{d^2y}{dx^2}+8\left(\frac{3}{4}\right)^2=9$ => $\frac{d^2y}{dx^2}=\frac{9}{8}$ $y=\frac{1}{2}\ln 2+\frac{3}{4}x+\frac{9}{16}x^2+\ldots$
(c)	$\ln\left(\frac{\sqrt{1+e^{-3x}}}{2}\right)=\ln\left(\sqrt{1+e^{3(-x)}}\right)-\ln 2$ $=-\ln 2+\frac{1}{2}\ln 2+\frac{3}{4}(-x)+\frac{9}{16}(-x)^2+\ldots$ $=-\frac{1}{2}\ln 2-\frac{3}{4}x+\frac{9}{16}x^2+\ldots$

2(a)	$\sum_{r=2}^{2n} \frac{2}{(2r-1)(2r+1)}$ Replace r by $r+1$, $= \sum_{r+1=2}^{r+1=2n} \frac{2}{(2(r+1)-1)(2(r+1)+1)}$ $= \sum_{r=1}^{r=2n-1} \frac{2}{(2r+1)(2r+3)}$ $= \sum_{r=2}^{2n-1} \frac{2}{(2r+1)(2r+3)} + \frac{2}{(2+1)(2+3)}$ $= \frac{1}{5} - \frac{1}{2(2n-1)+3} + \frac{2}{15}$ $= \frac{1}{3} - \frac{1}{4n+1}$
(b)(i)	$x_{n+1} = \sqrt{4+5x_n}$ for $n = 1, 2, 3, \ldots$ Since sequence converges to L, as $n \to \infty$, $x_n \to L$, $x_{n+1} \to L$. $L = \sqrt{4+5L}$ $L^2 = 4+5L$ --- (*) $L^2 - 5L - 4 = 0$ $L = \frac{5 \pm \sqrt{41}}{2}$ Since x_n are positive numbers, $L = \frac{5+\sqrt{41}}{2}$.
(ii)	$x_{n+1}{}^2 - L^2 = (4+5x_n) - (5L+4)$ (fr (*)) $= 5(x_n - L)$
(iii)	If $x_n > L$, $x_{n+1}{}^2 - L^2 = 5(x_n - L) > 0$ $(x_{n+1} - L)(x_{n+1} + L) > 0$ $\Rightarrow x_{n+1} - L > 0$ since $x_{n+1} > 0$, $L > 0$, so $x_{n+1} + L > 0$ $\Rightarrow x_{n+1} > L$

3(a)

$\mathrm{f}: x \mapsto \dfrac{1}{2} + \mathrm{e}^{-x},\ x \geq 0$ and a is a positive constant.

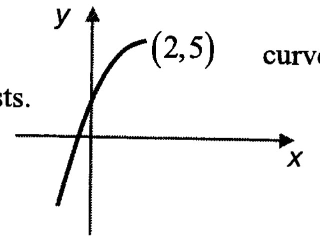


(b)

$\mathrm{g}: x \mapsto 1 + 4x - x^2,\ x \leq 2$

Every horizontal line will cut the curve at most once.

Hence g is 1-1. Therefore, g^{-1} exists.



(c)

$y = \mathrm{f}(x)$

$y = 5 - (x-2)^2$

$(x-2)^2 = 5 - y$

$x = 2 \pm \sqrt{5-y}$

Since $x \leq 2$, $\mathrm{g}^{-1}(x) = 2 - \sqrt{5-x}$

$\mathrm{D}_{\mathrm{g}^{-1}} = \mathrm{R}_{\mathrm{g}} = (-\infty, 5]$

(d)

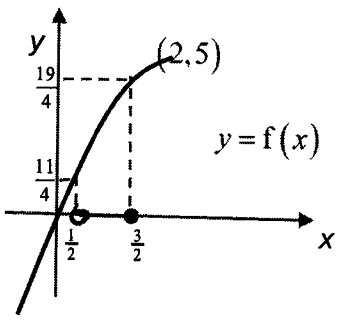
From graph of f in (i), $\mathrm{R_f} = \left(\dfrac{1}{2}, \dfrac{3}{2}\right] \subseteq (-\infty, 2] = D_g$

$\therefore$ gf exists.

(e)

$\mathrm{R_{gf}} = \left(5 - (\tfrac{1}{2} - 2)^2, 5 - (\tfrac{3}{2} - 2)^2\right]$

$= \left(\dfrac{11}{4}, \dfrac{19}{4}\right]$



4(a)	Circumference of base = length of arc AB $2\pi r = a\theta$ $r = \dfrac{a}{2\pi}\theta$
(b)	(i) Let h = height of cone $h^2 = a^2 - r^2$ $= a^2 - \dfrac{a^2}{4\pi^2}\theta^2$ $= \dfrac{a^2}{4\pi^2}\left(4\pi^2 - \theta^2\right)$ $V = \dfrac{1}{3}\pi r^2 h$ $V^2 = \dfrac{1}{9}\pi^2 r^4 h^2$ $= \dfrac{1}{9}\pi^2\left(\dfrac{a^4}{16\pi^4}\right)\cdot\dfrac{a^2}{4\pi^2}\left(4\pi^2 - \theta^2\right)$ $= \dfrac{a^6}{576\pi^4}\theta^4\left(4\pi^2 - \theta^2\right)$
(c)	**Method 1** $\dfrac{dV^2}{d\theta} = \dfrac{a^6}{576\pi^4}\dfrac{d}{d\theta}\left(4\pi^2\theta^4 - \theta^6\right)$ $= \dfrac{a^6}{576\pi^4}\left(16\pi^2\theta^3 - 6\theta^5\right)$ $= \dfrac{a^6}{576\pi^4}\cdot 2\theta^3\left(8\pi^2 - 3\theta^2\right)$ $\dfrac{dV^2}{d\theta} = 0$ when $\theta^3\left(8\pi^2 - 3\theta^2\right) = 0$ $\theta^2 = \dfrac{8\pi^2}{3} \quad \Rightarrow \quad \theta = 2\pi\sqrt{\dfrac{2}{3}}$ $\dfrac{d^2V^2}{d\theta^2} = \dfrac{a^6}{576\pi^4}\left(48\pi^2\theta^2 - 30\theta^4\right)$ When $\theta = 2\pi\sqrt{\dfrac{2}{3}}$, $\dfrac{d^2V^2}{d\theta^2} = \dfrac{a^6}{576\pi^4}\left(48\pi^2\left(\dfrac{8\pi^2}{3}\right) - 30\left(\dfrac{64\pi^4}{9}\right)\right) = -\dfrac{4a^6}{27} < 0$ Therefore V is max when $\theta = 2\pi\sqrt{\dfrac{2}{3}}$ **Method 2**

	$V^2 = \frac{a^6}{576\pi^4}\theta^4\left(4\pi^2 - \theta^2\right)$ Differentiating w.r.t. θ, $2V\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{a^6}{576\pi^4}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(4\pi^2\theta^4 - \theta^6\right)$ $= \frac{a^6}{576\pi^4}\left(16\pi^2\theta^3 - 6\theta^5\right)$ $= \frac{a^6}{576\pi^4}\cdot 2\theta^3\left(8\pi^2 - 3\theta^2\right)$ $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ when $\theta^3\left(8\pi^2 - 3\theta^2\right) = 0$ $\theta^2 = \frac{8\pi^2}{3} \quad \Rightarrow \theta = 2\pi\sqrt{\frac{2}{3}}$ $(\because \theta > 0)$ $2V\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} + 2\left(\frac{\mathrm{d}V}{\mathrm{d}\theta}\right)^2 = \frac{a^6}{576\pi^4}\left(48\pi^2\theta^2 - 30\theta^4\right)$ When $\theta = 2\pi\sqrt{\frac{2}{3}}$, $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ $2V\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} = \frac{a^6}{576\pi^4}\left(48\pi^2\left(\frac{8\pi^2}{3}\right) - 30\left(\frac{64\pi^4}{9}\right)\right) = -\frac{4a^6}{27} < 0$ Since $V > 0$, $\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2} < 0$ Therefore V is max when $\theta = 2\pi\sqrt{\frac{2}{3}}$
(d)	Note that: $\theta = 2\pi\sqrt{\frac{2}{3}}$ $r = \frac{a}{2\pi}\theta = \frac{a}{2\pi}\left(2\pi\sqrt{\frac{2}{3}}\right) = a\sqrt{\frac{2}{3}}$ $h^2 = a^2 - r^2 = a^2 - \left(a\sqrt{\frac{2}{3}}\right)^2 = \frac{1}{3}a^2 \quad \Rightarrow h = \frac{1}{\sqrt{3}}a$ $\frac{r}{h} = a\sqrt{\frac{2}{3}} \times \frac{\sqrt{3}}{a} = \sqrt{2} \quad \Rightarrow \quad r = \sqrt{2}h$ $W = \frac{1}{3}\pi r_w^2 h_w = \frac{1}{3}\pi\left(\sqrt{2}y\right)^2 y = \frac{2\pi y^3}{3}$
(e)	$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$

	$-\dfrac{\pi}{15} = 2\pi y^2 \times \dfrac{dy}{dt}$ $\dfrac{dy}{dt} = -\dfrac{1}{30y^2}$ When $y = 2$, $\dfrac{dy}{dt} = -\dfrac{1}{120}$ cm s^{-1} Rate of change of y is $-\dfrac{1}{120}$ cm s^{-1}.

Section B: Probability and Statistics [60 marks]

5	**[Solution]**
a)	Required Probability = P(Dave hits 2 targets and Rafael hits 0 or 1)+ P(Dave hits 1 and Rafael 0) $=\left(\frac{7}{10}\right)^2\left[\left(\frac{8}{10}\right)\left(\frac{2}{10}\right)(2)+\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)\right]+\left(\frac{7}{10}\right)\left(\frac{3}{10}\right)(2)\left(\frac{2}{10}\right)\left(\frac{2}{10}\right)$ $=0.1932$
b)	P(Day was bad given Dave wins) $=\frac{(0.3)(0.39)}{(0.7)(0.1932)+(0.3)(0.39)}$ $=0.464$ (3s.f.)

6	**[Solution]**
(a)(i)	Required number of ways = $\binom{12}{5}\binom{5}{1}$=3960
(a)(ii)	Required number of ways = $\binom{6}{2}\binom{6}{3}\binom{5}{1}=1500$ **Alternative method:** Case 1: 2 not identical at 3rd storey $\binom{6}{2}\times 2!\binom{6}{3}=600$ Case 2: 2 identical at 3rd storey $\binom{6}{2}\binom{6}{3}\frac{3!}{2!}=900$
(a)(iii)	Case 1: 2 in 1st storey, 1 in 2nd storey, 2 in 3rd storey Number of ways = $\binom{3}{2}\binom{3}{1}\binom{6}{2}\binom{5}{1}=675$ Case 2: 1 in 1st storey, 2 in 2nd storey, 2 in 3rd storey Number of ways = $\binom{3}{1}\binom{3}{2}\binom{6}{2}\binom{5}{1}=675$ Case 3: 1 in 1st storey, 1 in 2nd storey, 3 in 3rd storey Number of ways = $\binom{3}{1}\binom{3}{1}\binom{6}{3}\binom{5}{1}=900$ Total number of ways = 2250
(b)	Number of ways to arrange the 10 participants = $(5-1)!(2!)^5=768$ Number of ways to slot in the 2 game masters = $\binom{5}{2}$2!=20 Required number = $768\times 20=15360$

7	**Solution**
(a)	Let X be the number of faulty bowls in a box of 20 bowls. $X \sim B(20, 0.08)$ $P(X > 2) = 1 - P(X \le 2)$ $= 1 - 0.78795$ $= 0.21205$ (5 s.f.) $= 0.212$ (3 s.f.)
(b)	Let W be the number of faulty bowls in a carton (of 12 boxes). $W \sim B(12 \times 20, 0.08)$, i.e. $W \sim B(240, 0.08)$ $P(W < 15) = P(W \le 14) = 0.12933$ (5 s.f.) Required Probability $= {}^3C_2 (0.12933)^2 (1 - 0.12933)$ $= 0.0437$ (3 s.f.) **OR** Let A be the number of cartons out of 3 contains fewer than 15 faulty bowls each. $A \sim B(3, 0.12933)$ Required Probability $= P(A = 2) = 0.0437$ (3 s.f.)
(c)	$Y \sim B(12, 0.21205)$ $E(Y) = np = 12 \times 0.21205 = 2.5446 = 2.54$ (3 s.f.) $Var(Y) = np(1-p) = 12 \times 0.21205 \times (1 - 0.21205)$ $= 2.0050$ (5 s.f.) $= 2.01$ (3s.f.)
(d)	$Y \sim B(12, 0.21205)$ Let $T = Y_1 + \ldots + Y_{35}$. Since sample size 35 is large, by Central Limit Theorem, $T \sim N(35 \times 2.5446, 35 \times 2.0050)$ approximately i.e. $T \sim N(89.061, 70.175)$ approximately Required probability $= P(T \le 85) = 0.314$ (3 s.f.)

[Solutions]

8(a) Let W and L be the marks obtained in the written paper and lab-based practical respectively.

$W \sim \mathrm{N}(62,\ \sigma^2)$ and $L \sim \mathrm{N}(56,\ 12^2)$

$\mathrm{P}(W \le 85) = 0.95$

$\mathrm{P}\left(Z \le \dfrac{85-62}{\sigma}\right) = 0.95$

Using GC, $\dfrac{85-62}{\sigma} = 1.64485$

$\sigma = 13.983$ (5 s.f.)

(b)

$W - L \sim \mathrm{N}(62-56,\ 13.983^2 + 12^2)$

$W - L \sim \mathrm{N}(6,\ 339.524)$

$\mathrm{P}(|W-L| \ge 9) = 1 - \mathrm{P}(-9 < W - L < 9) = 0.643$ (3 s.f.)

(c)

$T = 0.6W + 0.4L \sim \mathrm{N}\left(0.6\times62 + 0.4\times56,\ 0.6^2\times13.983^2 + 0.4^2\times12^2\right)$

$T \sim \mathrm{N}(59.6,\ 93.42874)$

$\mathrm{P}(T < 60) = 0.517$

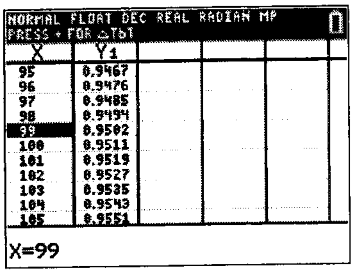
(d)

$\bar{T} = \dfrac{T_1 + T_2 + \ldots + T_n}{n} \sim \mathrm{N}\left(59.6,\ \dfrac{93.42874}{n}\right)$

$\mathrm{P}(\bar{T} \ge 58) \ge 0.95$

Method 1: From GC,

n	$\mathrm{P}(\bar{T} \ge 58)$
98	$0.9494 < 0.95$
99	$0.9502 > 0.95$



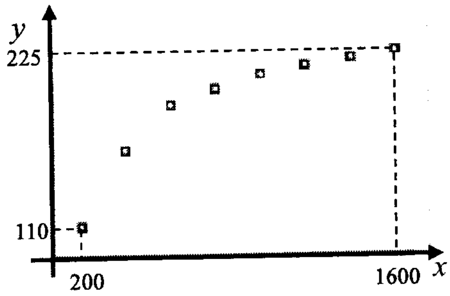
Therefore, the least n is 99.

Method 2 (More tedious)

	$P(\bar{T} \geq 58) \geq 0.95$ $P\left(Z \geq \frac{58-59.6}{\sqrt{\frac{93.42874}{n}}}\right) \geq 0.95$ $\Rightarrow \frac{58-59.6}{\sqrt{\frac{93.42874}{n}}} \leq -1.64485$ $\Rightarrow \frac{-1.6\sqrt{n}}{\sqrt{93.42874}} \leq -1.64485$ $\Rightarrow \frac{1.6\sqrt{n}}{\sqrt{93.42874}} \geq 1.64485$ $\Rightarrow n \geq 98.746$ Therefore, the least n is 99.
(e)	The assumption is that a candidate's mark for the written paper and lab-based practical are independent. This may not be valid in practice as the a candidate who does well for the written paper is more likely to do better in the lab-based practical too.

9(a)	Assume the queuing time follows normal distribution.
(b)	Using GC, $\bar{x} = 4.24$ (exact) Let μ be the population mean queuing time and Y be the queuing time of a randomly chosen student. Null hypothesis $H_0 : \mu = 4.5$ Alternative hypothesis $H_1 : \mu < 4.5$ (manager's claim) At 5 % significance level. Under H_0, $\bar{X} \sim N\left(4.5, \frac{0.48}{10}\right)$ Test statistic: $Z = \frac{\bar{X} - 4.5}{\sqrt{\frac{0.48}{10}}} \sim N(0,1)$ From GC, p-value $= 0.118 > 0.05$ $\therefore$ do not reject H_0 There is insufficient evidence at 5% level of significance to conclude that the manager's claim is justified.
(c)	The sample is not random because (one of the following reasons): • Not all the students from the school were given equal chance to be selected. • Students who are friends or from the same class may visit the same food stall together. Thus they are not selected independently.
(d)	$s^2 = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$ $= \frac{1}{49}\left(1068 - \frac{(4.6 \times 50)^2}{50}\right)$ $= 0.20408 \approx 0.204$
(e)	Assumption in (a) is no longer necessary because sample size 50 is large, by Central Limit Theorem, the sample mean queuing time now follows normal distribution approximately.
(g)	Null hypothesis $H_0 : \mu = \lambda$ Alternative hypothesis $H_1 : \mu > \lambda$ (student leader's claim) At 2 % significance level. Under H_0, $\bar{X} \sim N\left(\lambda, \frac{0.20408}{50}\right)$ approximately by Central limit theorem since $n = 50$ large.

	Test statistic: $Z = \dfrac{\overline{X} - \lambda}{\sqrt{\dfrac{0.20408}{50}}} \sim \mathrm{N}(0,1)$ Critical region = $\{z \in \mathbb{R} : z \geq 2.05375\}$ since upper tail test To reject H_0, $\dfrac{4.6 - \lambda}{\sqrt{\dfrac{0.20408}{50}}} \geq 2.05375$ $0 < \lambda \leq 4.47$

10(a)	
(b)	$r = 0.894$ (3 s.f.) Even though the r-value indicates a strong positive linear correlation between x and y, the scatter diagram indicates that a curvilinear correlation exists between x and y. Therefore the linear model may not be the best.
(d)	Model I: As x increases, y increases at an increasing rate. Model II: As x increases, y increases at a decreasing rate. Model III: As x increases, y decreases. Hence, only Model II can be used to model the relationship between x and y as it is the only one matches the trend of data shown in the scatter diagram. <u>Alternative solution:</u> Since the graph of $y = a + b\ln x$ is concave downwards whereas the graph of $y = a + bx^2$ and $y = a + be^{-x}$ are concave upwards, the graph of $y = a + b\ln x$ will be more suitable to describe the scatter diagram of s and y. Hence Model II is more suitable.
(e)	The regression line of y on $\ln x$ is $y = 54.269\ln x - 168.22$, i.e. $y = 54.3\ln x - 168$ (to 3 s.f.)
(f)	$y = 54.269\ln 700 - 168.22 = 187.30 \approx 187$ (3 s.f.) The input data $x = 700$ is within the data range **[200, 1600]** of the sample. Furthermore, there is a strong positive linear correlation as indicated by $r = 0.984$. Therefore the estimation is reliable.
(g)	It will remain the same.