2025 JC2 H2 Math Prelim P1

Solutions

1 Using an algebraic method, solve the inequality $\dfrac{4x-3}{4x^2+3x-1} \le 1$. [3]

Hence, find the set of values of x that satisfy $\dfrac{4\ln x-3}{4(\ln x)^2+3\ln x-1} \le 1$. [2]

<table>
<tr><th>No.</th><th>Solution</th></tr>
<tr><td>1</td><td>

$\dfrac{4x-3}{4x^2+3x-1} \le 1$

$\dfrac{4x-3}{4x^2+3x-1} - 1 \le 0$

$\dfrac{4x-3-4x^2-3x+1}{(x+1)(4x-1)} \le 0$

$\dfrac{-4x^2+x-2}{(x+1)(4x-1)} \le 0$

$\dfrac{4x^2-x+2}{(x+1)(4x-1)} \ge 0$

$4x^2-x+2 = 4\left[\left(x-\dfrac{1}{8}\right)^2 - \left(\dfrac{1}{8}\right)^2\right]+2$

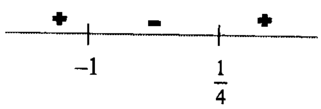
$= 4\left(x-\dfrac{1}{8}\right)^2 + \dfrac{31}{16}$

Since $\left(x-\dfrac{1}{8}\right)^2 \ge 0$ for all real values of x, $4\left(x-\dfrac{1}{8}\right)^2 + \dfrac{31}{16} > 0$

OR

Since coefficient of x^2 is $4>0$ and the discriminant of $4x^2-x+2$ is $(-1)^2-4(4)(2) = -31 < 0$, $4x^2-x+2>0$ for all $x \in \mathbb{R}$.

We have $(x+1)(4x-1) > 0$.



$x < -1$ or $x > \dfrac{1}{4}$

Replace x with $\ln x$:

$\ln x < -1$ or $\ln x > \dfrac{1}{4}$

$0 < x < e^{-1}$ or $x > e^{\frac{1}{4}}$

</td></tr>
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2 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x-1}{x},\ x \in \mathbb{R},\ x \neq 0,\ x \neq 1.$$

(a) Show that $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$. [3]

(b) Find $\mathrm{f}^3(x)$ in simplified form. [1]

(c) Find $\mathrm{f}^{2030}(5)$. [2]

Functions g and h are defined by

$$\mathrm{g}: x \mapsto \frac{x-1}{x},\ x \in \mathbb{R},\ x \geq 1,$$

$$\mathrm{h}: x \mapsto -\sin ax,\ x \in \mathbb{R},$$

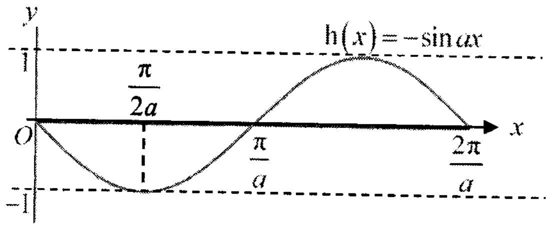
where a is a positive constant.

(d) Find the value of a given that the range of hg is $(-1,0]$. [2]

2a	Let $y = \frac{x-1}{x} = 1 - \frac{1}{x}$ $\frac{1}{x} = 1 - y$ $x = \frac{1}{1-y}$ Hence $\mathrm{f}^{-1}(x) = \frac{1}{1-x}$ $\mathrm{f}^2(x) = \mathrm{f}\left(\frac{x-1}{x}\right)$ $= \frac{\frac{x-1}{x} - 1}{\frac{x-1}{x}}$ $= \frac{x-1-x}{x-1}$ $= \frac{1}{1-x}$ $\therefore \mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$
2b	$\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$ $\mathrm{f}^3(x) = \mathrm{f}\,\mathrm{f}^2(x)$ $= \mathrm{f}\,\mathrm{f}^{-1}(x)$ $= x$
2c	$\mathrm{f}^{2030}(5) = \mathrm{f}^{3(676)+2}(5) = \mathrm{f}^2(5) = \frac{1}{1-5} = -\frac{1}{4}$

2d

Range of g $= [0,1)$ = new domain of h



Given: range of hg is $(-1,0]$

$\frac{\pi}{2a}=1 \Rightarrow a=\frac{\pi}{2}$

3 Referred to the origin *O*, points *A* and *B* have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point *C* lies on *OA*, such that $OC:CA=2:1$. Point *D* lies on *OB*, such that $OD:DB=\lambda:\mu$. It is given that the area of triangle *ABD* is half the area of triangle *ABC*.

(a) Show the area of triangle *ABD* is given by $\frac{\mu}{2(\lambda+\mu)}|\mathbf{a}\times\mathbf{b}|$. Hence find the ratio $\lambda:\mu$. [4]

(b) The point *E* has position vector $\frac{1}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}$. Show that *A*, *E* and *D* are collinear. [3]

It is further given that the angle *AOB* is $\frac{\pi}{4}$ and *O* lies on the perpendicular bisector of the line segment *AB*.

(c) Find the length of projection of $\mathbf{a}$ on $\mathbf{b}$, giving your answer in terms of $|\mathbf{b}|$. Hence find the position vector of the point *F*, the foot of perpendicular from *A* to *OB*. [3]

No.	Solution
3a	Area of triangle *ABD* $=\frac{1}{2}\left\|\overrightarrow{DB}\times\overrightarrow{AB}\right\|$ $=\frac{1}{2}\left\|\frac{\mu}{\lambda+\mu}\underset{\sim}{b}\times(\underset{\sim}{b}-\underset{\sim}{a})\right\|$ $=\frac{\mu}{2(\lambda+\mu)}\left\|\underset{\sim}{b}\times\underset{\sim}{b}-\underset{\sim}{b}\times\underset{\sim}{a}\right\|$ $=\frac{\mu}{2(\lambda+\mu)}\left\|\underset{\sim}{0}+\underset{\sim}{a}\times\underset{\sim}{b}\right\|$ $=\frac{\mu}{2(\lambda+\mu)}\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$ Area of triangle *ABC* $=\frac{1}{2}\left\|\overrightarrow{CA}\times\overrightarrow{AB}\right\|$ $=\frac{1}{2}\left\|\frac{1}{3}\underset{\sim}{a}\times(\underset{\sim}{b}-\underset{\sim}{a})\right\|$ $=\frac{1}{6}\left\|\underset{\sim}{a}\times\underset{\sim}{b}-\underset{\sim}{a}\times\underset{\sim}{a}\right\|$ $=\frac{1}{6}\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$

	Area of triangle ABD = 1/2 (area of triangle ABC) $\frac{\mu}{2(\lambda+\mu)}\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert=\frac{1}{2}\left(\frac{1}{6}\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert\right)$ $\frac{\mu}{2(\lambda+\mu)}=\frac{1}{12}$ $6\mu=\lambda+\mu$ $5\mu=\lambda$ $\lambda:\mu=5:1$
3b	$\overrightarrow{AE}=\overrightarrow{OE}-\overrightarrow{OA}$ $=\frac{1}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}-\mathbf{a}$ $=-\frac{3}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}$ $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$ $=-\mathbf{a}+\frac{5}{6}\mathbf{b}$ $=\frac{4}{3}\left(-\frac{3}{4}\mathbf{a}+\frac{5}{8}\mathbf{b}\right)$ Since $\overrightarrow{AD}=\frac{4}{3}\overrightarrow{AE}$ and A is a common point, A, E and D are collinear.
3c	Length of projection of **a** on **b** $=\lvert\mathbf{a}\cdot\mathbf{b}\rvert$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\frac{\pi}{4}$ $=\lvert\mathbf{a}\rvert\cos\frac{\pi}{4}$ $=\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert$ $\overrightarrow{OF}=\left(\frac{1}{\sqrt{2}}\lvert\mathbf{b}\rvert\right)\mathbf{b}$ $=\frac{1}{\sqrt{2}}\mathbf{b}$

4 **(a)** A sequence is such that $u_1 = p$, where p is a constant and $u_{n+1} = \dfrac{5u_n}{8u_n + 1}$, for $n..1$.

(i) Describe how the sequence behaves when $p = 1$. [2]

(ii) Find the value of p for which $u_6 = \dfrac{3125}{6253}$. [2]

(b) Another sequence v_1, v_2, v_3, ... is such that for all $n..1$, $v_{n+2} - 2v_{n+1} + v_n = k$, where k is a constant.

Let $w_n = v_{n+1} - v_n$ for $n..1$. Explain why the sequence $\{w_n\}$ is an arithmetic progression. [2]

No.	Solution
4ai	When $p = 1$, $u_1 = 1$, $u_2 = \frac{5}{9}$, $u_3 = \frac{25}{49}$, $u_4 = \frac{125}{249}$, $u_5 = \frac{625}{1245}$,... The sequence is a decreasing sequence, and it converges to $\frac{1}{2}$.
4aii	$u_{n+1} = \frac{5u_n}{8u_n + 1} \quad \Rightarrow \frac{1}{u_{n+1}} = \frac{8u_n + 1}{5u_n} = \frac{8}{5} + \frac{1}{5u_n}$ $\frac{1}{5u_n} = \frac{1}{u_{n+1}} - \frac{8}{5} = \frac{5 - 8u_{n+1}}{5u_{n+1}}$ $5u_n = \frac{5u_{n+1}}{5 - 8u_{n+1}}$ $u_n = \frac{u_{n+1}}{5 - 8u_{n+1}}$ Using the GC, $u_1 = \frac{1}{5}$ $\therefore p = \frac{1}{5}$
4b	$w_{n+1} - w_n = v_{n+2} - v_{n+1} - (v_{n+1} - v_n)$ $= v_{n+2} - 2v_{n+1} + v_n$ $= k$ (which is a constant) Hence $\{w_n\}$ is an arithmetic progression.

5 The function f is given by

$$f(x)=\begin{cases}2+\sqrt{2^2-(x-2)^2}, & 0<x\le 4,\\ 2-\sqrt{2^2-(x-6)^2}, & 4<x\le 8.\end{cases}$$

It is given that $f(x)=f(x+8)$ for all real values of x.

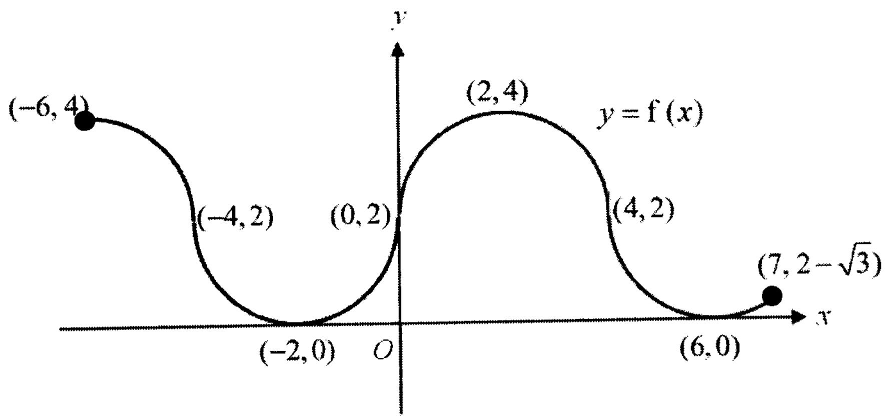
(a) On the diagram in the Printed Answer Book, sketch the graph of $y=f(x)$ for $-6\le x\le 7$, indicating clearly the coordinates of the end points and the points where the graph cuts the axes. [3]

(b) Without integrating, write down the exact area of the region bounded by $y=f(x)$, the line $x=4$, the x-axis and the y-axis. [1]

The curve C has equation $\dfrac{(y-3)^2}{a^2}-(x+2)^2=1$, where a is a positive real constant.

(c) State the equations of the asymptotes of C in terms of a. [1]

(d) Determine the range of values of a if there is at most one intersection between C and the graph of $y=f(x)$. [2]

No.	Solution
5a	
5b	Area = $2\times 4+\frac{1}{2}\pi(2^2)=8+2\pi$
5c	Equations of asymptotes: $\frac{(y-3)^2}{a^2}=(x+2)^2$ $y-3=\pm a(x+2)$ $y=3+a(x+2)$ and $y=3-a(x+2)$
5d	C is a hyperbola centred at $(-2,3)$ with vertices at $(-2,3+a)$ and $(-2,3-a)$. When $a=3$, there is exactly one intersection at $(-2,0)$. Hence there is at most one intersection between C and the graph of $y=f(x)$ when $a\ge 3$.

6 It is given that $\sum_{r=2}^{n} \frac{1}{r(r^2-1)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n}$.

(a) Show that $\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$ is less than $\frac{1}{4}$. [2]

(b) Give a reason why the series $\sum_{r=2}^{\infty} \frac{1}{r(r^2-1)}$ converges, and write down its value. [2]

(c) Find the smallest value of n for which $\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$ differs from $\sum_{r=2}^{\infty} \frac{1}{r(r^2-1)}$ by less than 0.0007. [2]

(d) Find $\sum_{r=m+1}^{N} \frac{1}{(r-m)(r-m+1)(r-m+2)}$, where m and N are integers with $N > m > 0$.
(There is no need to express your answer as a single algebraic fraction.) [2]

No.	Solution
6a	$\frac{1}{2(n+1)} - \frac{1}{2n} = \frac{n-(n+1)}{2n(n+1)} = \frac{-1}{2n(n+1)} < 0$ for all $n \geq 2$ $\sum_{r=2}^{n} \frac{1}{r(r^2-1)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n} < \frac{1}{4}$ **<u>Alternative</u>** $2(n+1) > 2n$ for all $n \geq 2$ $\frac{1}{2(n+1)} < \frac{1}{2n}$ $\frac{1}{2(n+1)} - \frac{1}{2n} < 0$ $\sum_{r=2}^{n} \frac{1}{r(r^2-1)} = \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{2n} < \frac{1}{4}$
6b	As $n \to \infty$, $\frac{1}{2(n+1)} \to 0$ and $\frac{1}{2n} \to 0$, $\therefore \sum_{r=2}^{n} \frac{1}{r(r^2-1)} \to \frac{1}{4}$, a constant. Hence $\sum_{r=2}^{\infty} \frac{1}{r(r^2-1)}$ converges and $\sum_{r=2}^{\infty} \frac{1}{r(r^2-1)} = \frac{1}{4}$.

6c

$$\left|\sum_{r=2}^{n}\frac{1}{r\left(r^2-1\right)}-\sum_{r=2}^{\infty}\frac{1}{r\left(r^2-1\right)}\right|<0.0007$$

$$\left|\frac{1}{4}+\frac{1}{2(n+1)}-\frac{1}{2n}-\frac{1}{4}\right|<0.0007$$

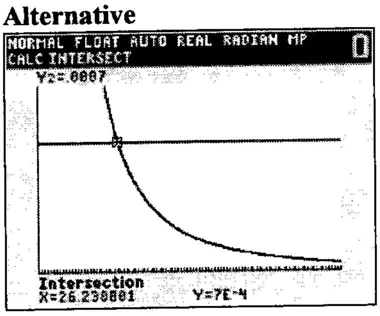
$$\left|\frac{1}{2(n+1)}-\frac{1}{2n}\right|<0.0007$$

Using GC:

n	$\left\lvert\frac{1}{2(n+1)}-\frac{1}{2n}\right\rvert$
26	$0.000712>0.0007$
27	$0.000661<0.0007$

smallest value of $n=27$

Alternative



$n>26.231\Rightarrow$ smallest value of $n=27$

6d

$$\sum_{r=m+1}^{N}\frac{1}{(r-m)(r-m+1)(r-m+2)}$$

$$=\frac{1}{1(2)(3)}+\frac{1}{2(3)(4)}+\ldots+\frac{1}{(N-m)(N-m+1)(N-m+2)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{(r-1)(r)(r+1)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{r\left(r^2-1\right)}$$

$$=\frac{1}{4}+\frac{1}{2(N-m+2)}-\frac{1}{2(N-m+1)}$$

<u>**Alternative**</u>

$$\sum_{r=m+1}^{N}\frac{1}{(r-m)(r-m+1)(r-m+2)}$$

$$=\sum_{r+m-1=m+1}^{r+m-1=N}\frac{1}{(r-1)(r)(r+1)}$$

$$=\sum_{r=2}^{N-m+1}\frac{1}{(r)\left(r^2-1\right)}$$

$$=\frac{1}{4}+\frac{1}{2(N-m+2)}-\frac{1}{2(N-m+1)}$$

7 **Do not use a calculator in answering this question.**

(a) Find the complex number z which satisfies the equation $\frac{4|z|}{15-z^*}=5\text{i}$. [3]

(b) The complex number w is such that $(w-\text{i})^3=-\text{i}$.

(i) Given that one possible value of w is 2i, find the two other possible values of w. Give your answers in cartesian form $a+b\text{i}$. [4]

The points W_1, W_2 and W_3 on the Argand diagram represent the three roots of the equation $(w-\text{i})^3=-\text{i}$, and the point A represents the complex number $k\text{i}$, where k is a positive real number.

(ii) Show that the points W_1, W_2 and W_3 lie on a circle with centre A on the Argand diagram for some value of k, stating the value of k. [2]

No.	Solution
7a	Let $z=a+b\text{i}$, where $a,b\in\mathbb{R}$. $\frac{4\|z\|}{15-z^*}=5\text{i}$ $\frac{4\sqrt{a^2+b^2}}{15-(a-b\text{i})}=5\text{i}$ $4\sqrt{a^2+b^2}=75\text{i}-5a\text{i}-5b=-5b+(75-5a)\text{i}$ Comparing real and imaginary parts: $\begin{cases}4\sqrt{a^2+b^2}=-5b & ---(1)\\ 75-5a=0\Rightarrow a=15 & ---(2)\end{cases}$ Substitute (2) into (1): $4\sqrt{15^2+b^2}=-5b$ $4^2\left(15^2+b^2\right)=25b^2$ $9b^2=3600$ $b^2=400$ $b=-20$ $\quad (\because b<0 \text{ by eqn. } (1))$ $\therefore z=15-20\text{i}$

7bi	$(w-\mathrm{i})^3=-\mathrm{i}$ $w^3+3w^2(-\mathrm{i})+3w(-\mathrm{i})^2+(-\mathrm{i})^3+\mathrm{i}=0$ $w^3-3\mathrm{i}w^2-3w+2\mathrm{i}=0$ Since 2i is a root, $w^3-3\mathrm{i}w^2-3w+2\mathrm{i}=0=(w-2\mathrm{i})(Aw^2+Bw+C)$ By comparing coefficients, $A=1, C=-1$ $C-2\mathrm{i}B=-3 \Rightarrow B=\frac{-1+3}{2\mathrm{i}}=-\mathrm{i}$ $w^2-\mathrm{i}w-1=0$ $w=\frac{\mathrm{i}\pm\sqrt{(-\mathrm{i})^2-4(-1)}}{2}$ $w=\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ or $-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$
7bii	Let W_1, W_2, W_3 and A represent 2i, $\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$, $-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}$ and $k\mathrm{i}$ respectively. For $W_1A=W_2A$, i.e. $\left\|\frac{\sqrt{3}+\mathrm{i}}{2}-k\mathrm{i}\right\|=\|2\mathrm{i}-k\mathrm{i}\|$, $\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}-k\right)^2=(2-k)^2 \Rightarrow \frac{3}{4}+1-k+k^2=4-4k+k^2 \Rightarrow k=1$. So, we have $W_1A=\|2\mathrm{i}-\mathrm{i}\|=1$ and $W_2A=\left\|\frac{\sqrt{3}+\mathrm{i}}{2}-\mathrm{i}\right\|=1$. Check that $W_3A=1$ too: $\left\|\frac{-\sqrt{3}+\mathrm{i}}{2}-\mathrm{i}\right\|=\left\|\frac{-\sqrt{3}-\mathrm{i}}{2}\right\|=\sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2+\left(\frac{-1}{2}\right)^2}=1$ Since the distances are all equal to 1, W_1, W_2 and W_3 lie on a circle with radius 1, centred at A representing the complex number i.

8 **(a)** A curve C with equation $y=\mathrm{f}(x)$ undergoes in succession, the following transformations.

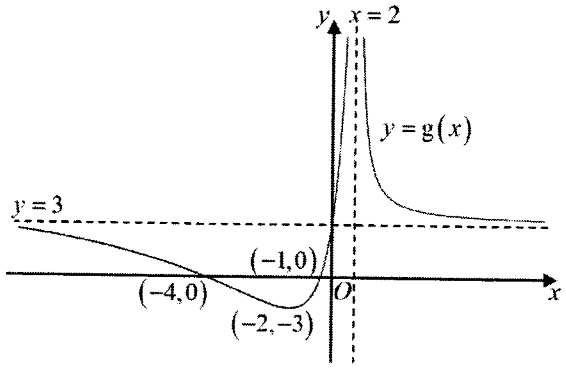
A: A reflection in the x-axis.

B: A stretch parallel to the x-axis with scale factor $\frac{1}{2}$, with the y-axis invariant.

The resulting curve has equation $y=ax^2+\frac{b}{x}$, where a and b are real constants.

Given that $\left(-1,\frac{1}{3}\right)$ is a turning point of $y=\frac{1}{\mathrm{f}(x)}$, find the values of a and b and state the equation of C. [5]

(b) The diagram below shows the curve of $y=\mathrm{g}(x)$. The curve has a minimum point at $(-2,-3)$ and crosses the x-axis at $(-1,0)$ and $(-4,0)$. The line $x=2$ is the vertical asymptote and the line $y=3$ is the horizontal asymptote.



(i) Sketch the graph of $y=\mathrm{g}'(x)$, labelling the coordinates of all relevant point(s) and state the equations of any asymptotes. [2]

(ii) Find the area of the region bounded by the graph of $y=\mathrm{g}'(x)$, the lines $x=-4$, $x=-2$ and the x-axis. [2]

No.	Solution
8a	$y=ax^2+\frac{b}{x}$ ↓ Before B $\left(\text{replace } x \text{ with } \frac{x}{2}\right)$ $y=a\left(\frac{x}{2}\right)^2+\frac{b}{x/2}$ $y=\frac{ax^2}{4}+\frac{2b}{x}$ ↓ Before A (replace y with $-y$) $y=-\frac{ax^2}{4}-\frac{2b}{x}=\mathrm{f}(x)$ $\left(-1,\frac{1}{3}\right)$ turning point on $y=\frac{1}{\mathrm{f}(x)}$ $\Rightarrow$ $(-1,3)$ turning point on $y=\mathrm{f}(x)$: $-\frac{a}{4}+2b=3$ -------------- (1)

$\frac{dy}{dx} = -\frac{1}{2}ax + \frac{2b}{x^2}$

Since when $x = -1$, $\frac{dy}{dx} = 0$, $\frac{1}{2}a + 2b = 0$. -------------- (2)

Solving (1), (2) using a GC, $a = -4$, $b = 1$. Therefore, $y = f(x) = x^2 - \frac{2}{x}$.

Alternative

$y = f(x)$

$\downarrow$ A (replace y with $-y$)

$y = -f(x)$

$\downarrow$ B (replace x with $2x$)

$y = -f(2x) = ax^2 + \frac{b}{x}$

$f(2x) = -\frac{a(2x)^2}{4} - \frac{2b}{2x} \Rightarrow \quad f(x) = -\frac{ax^2}{4} - \frac{2b}{x}$

$y = \frac{1}{f(x)} = \frac{-x}{\frac{a}{4}x^3 + 2b}$

Subst. $\left(-1, \frac{1}{3}\right)$ into $y = \frac{1}{f(x)}$: $\frac{1}{3} = \frac{1}{-\frac{a}{4} + 2b} \Rightarrow -\frac{a}{4} + 2b = 3$ -------------- (1)

$$\frac{dy}{dx} = \frac{\left(\frac{a}{4}x^3 + 2b\right)(-1) - (-x)\left(\frac{3a}{4}x^2\right)}{\left(\frac{a}{4}x^3 + 2b\right)^2}$$

When $x = -1$, $\frac{dy}{dx} = 0$, $\left(\frac{a}{4}(-1)^3 + 2b\right)(-1) - (-(-1))\left(\frac{3a}{4}(-1)^2\right) = 0$.

$\frac{a}{2} + 2b = 0$ -------------- (2)

Solving (1), (2) using a GC, $a = -4$, $b = 1$. Therefore, $y = f(x) = x^2 - \frac{2}{x}$.

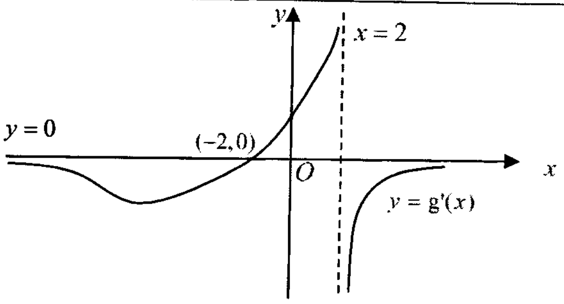
Alternative

Since $\left(-1, \frac{1}{3}\right)$ is a turning point of $y = \frac{1}{f(x)}$, $(-1, 3)$ is a turning point of $y = f(x)$.

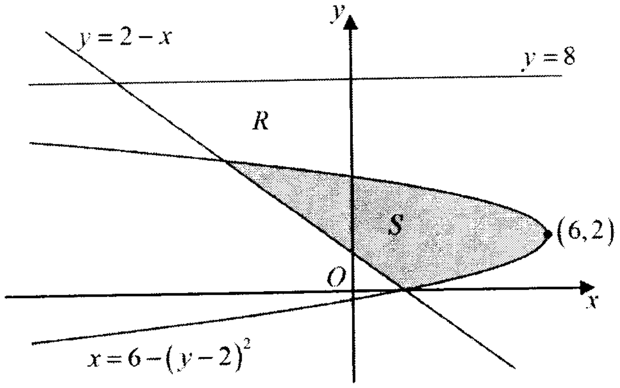
$(-1,3) \xrightarrow{A} (-1,-3) \xrightarrow{B} \left(-\frac{1}{2}, -3\right)$, a turning point on $y = ax^2 + \frac{b}{x}$

$y = ax^2 + \frac{b}{x} \Rightarrow \frac{dy}{dx} = 2ax - \frac{b}{x^2}$

Subst. $\left(-\frac{1}{2}, -3\right)$ into $y = ax^2 + \frac{b}{x}$: $\frac{a}{4} - 2b = -3$ -------------- (1)

	When $x=-\frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$: $-a-4b=0$. --------------- (2) Solving (1), (2) using a GC, $a=-4,\ b=1$. Therefore, the transformed graph has equation $y=-4x^2+\frac{1}{x}$. $y=-4x^2+\frac{1}{x}\xrightarrow{B^{-1}}y=-x^2+\frac{2}{x}\xrightarrow{A^{-1}}C: y=x^2-\frac{2}{x}$
8bi	
8bii	Required area $=-\int_{-4}^{-2}\mathrm{g}'(x)\,\mathrm{d}x=-\left[\mathrm{g}(x)\right]_{-4}^{-2}$ $=-\left[\mathrm{g}(-2)-\mathrm{g}(-4)\right]$ (from graph in the question) $=-\left[-3-0\right]=3$ units^2

9 In the diagram below, the region R is bounded by the curve C with equation $x=6-(y-2)^2$, the lines $y=8$, $y=2-x$ and the y-axis. The shaded region S is bounded by C and the line $y=2-x$.



(a) Find the exact area of region R. [5]

(b) Find the volume of the solid of revolution formed when S is rotated through 360° about the x-axis, leaving your answer to 2 decimal places. [3]

No.	Solution
9a	For point of intersection of C and $y=2-x$: $2-y=6-(y-2)^2$ $2-y=6-y^2+4y-4$ $y^2-5y=0$ $y(y-5)=0$ $y=0$ or $y=5$ When $x=0$, $6-(y-2)^2=0 \Rightarrow y=2\pm\sqrt{6}$. Area of R $=-\left[\int_{2+\sqrt{6}}^{5} 6-(y-2)^2 \, dy+\int_5^8 2-y \, dy\right]$ OR $\dfrac{6+3}{2}\times 3-\int_{2+\sqrt{6}}^{5} 6-(y-2)^2 \, dy$ $=-\left\{\left[6y-\dfrac{(y-2)^3}{3}\right]_{2+\sqrt{6}}^{5}+\left[2y-\dfrac{y^2}{2}\right]_5^8\right\}$ $=-\left[(30-9)-\left(6(2+\sqrt{6})-\dfrac{(\sqrt{6})^3}{3}\right)+\left(16-\dfrac{64}{2}\right)-\left(10-\dfrac{25}{2}\right)\right]$ $=\dfrac{9}{2}+4\sqrt{6}$ units^2 Alternative method: For point of intersection of C and $y=2-x$: $2-y=6-(y-2)^2$ $2-y=6-y^2+4y-4$ $y^2-5y=0$ $y(y-5)=0$ $y=0$ or $y=5$ $\therefore x=-3$ or 2

	When $y=8$, $8=2-x \Rightarrow x=-6$. Area of R $=(6)(8)-\int_{-6}^{-3}(2-x)\,\mathrm{d}x-\int_{-3}^{0}\left(2+\sqrt{6-x}\right)\mathrm{d}x$ $=48-\left[2x-\frac{x^2}{2}\right]_{-6}^{-3}-\left[2x-\frac{(6-x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{-3}^{0}$ $=48-\frac{39}{2}-\left(-\frac{2}{3}(6)^{\frac{3}{2}}-\left(-6-\frac{2}{3}(9)^{\frac{3}{2}}\right)\right)$ $=\frac{57}{2}-\left(24-\frac{2}{3}\left(6\sqrt{6}\right)\right)$ $=\frac{9}{2}+4\sqrt{6}$
9b	$x=6-(y-2)^2$ $y=2\pm\sqrt{6-x}$ Volume of solid $=\pi\int_{-3}^{6}\left(2+\sqrt{6-x}\right)^2\mathrm{d}x-\pi\int_{-3}^{2}(2-x)^2\,\mathrm{d}x-\pi\int_{2}^{6}\left(2-\sqrt{6-x}\right)^2\mathrm{d}x$ $=327.25$ units^3

10 **(a)** **(i)** Express $1+4x$ in the form $A(2x-2)+B$, where A and B are constants to be determined. [1]

(ii) Hence, find $\int \frac{1+4x}{x^2-2x+5}\,dx$. [4]

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} x\sin 2x\,dx$. [3]

No.	Solution
10ai	$1+4x = A(2x-2)+B$ $2Ax-2A+B = 4x+1$ $2A=4 \Rightarrow A=2$ and $B-2A=1 \Rightarrow B=5$ $1+4x = 2(2x-2)+5$
10a ii	$\int \frac{1+4x}{x^2-2x+5}\,dx = \int \frac{2(2x-2)+5}{x^2-2x+5}\,dx$ $= \int \frac{2(2x-2)}{x^2-2x+5}\,dx + \int \frac{5}{x^2-2x+5}\,dx$ $= 2\int \frac{2x-2}{x^2-2x+5}\,dx + 5\int \frac{1}{(x-1)^2+4}\,dx$ $= 2\ln(x^2-2x+5) + \frac{5}{2}\tan^{-1}\frac{x-1}{2} + C$ Note: $x^2-2x+5>0$ for all real x.
10b	$\int_0^{\frac{\pi}{3}} x\sin 2x\,dx = \left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{3}} + \frac{1}{2}\int_0^{\frac{\pi}{3}} \cos 2x\,dx$ $= \left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{3}} + \frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{3}}$ $= \left[-\frac{1}{2}\left(\frac{\pi}{3}\right)\cos\frac{2\pi}{3}\right] + \frac{1}{4}\left[\sin 2\left(\frac{\pi}{3}\right)\right]$ $= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$ Box: $u = x$, $\frac{dv}{dx} = \sin 2x$; $\frac{du}{dx} = 1$, $v = -\frac{1}{2}\cos 2x$

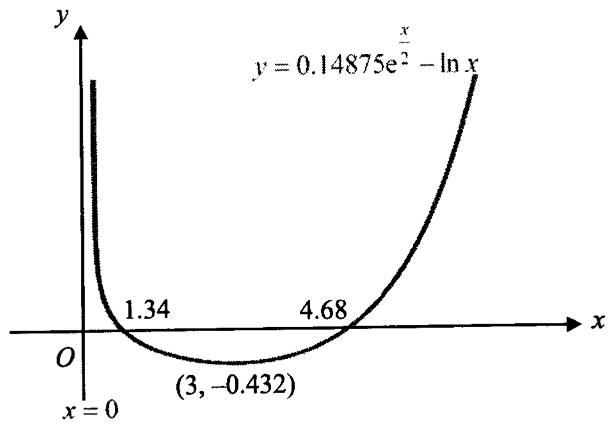
11 It is given that $2\frac{dy}{dx} + \frac{2}{x} - \ln x = y$.

(a) Use the substitution $y = \ln\left(\frac{u}{x}\right)$ to show that the differential equation can be reduced to $\frac{du}{dx} = f(u)$, where the function $f(u)$ is to be found. [3]

(b) Given that y has a minimum value at $x=3$, solve the differential equation $2\frac{dy}{dx} + \frac{2}{x} - \ln x = y$, to find the particular solution for y in terms of x. [5]

(c) Sketch the graph of this particular solution. [2]

No.	Solution
11a	$y=\ln\left(\frac{u}{x}\right)\Rightarrow y=\ln u-\ln x$ --------(1) Differentiating w.r.t. x: $\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}-\frac{1}{x}$ ---(2) Subst. (1) and (2) into given DE $2\frac{dy}{dx}+\frac{2}{x}-\ln x=y$: $2\left(\frac{1}{u}\frac{du}{dx}-\frac{1}{x}\right)+\frac{2}{x}-\ln x=\ln u-\ln x$ $2\frac{1}{u}\frac{du}{dx}=\ln u$ $\frac{du}{dx}=\frac{1}{2}u\ln u$
11b	$\int\frac{\frac{1}{u}}{\ln u}\,du=\int\frac{1}{2}\,dx$ $\ln(\|\ln u\|)=\frac{x}{2}+C$ $\|\ln u\|=Be^{\frac{x}{2}}$, where $B=e^{C}$ $\ln u=Ae^{\frac{x}{2}}$ From (1): $y+\ln x=Ae^{\frac{x}{2}}$ $\Rightarrow$ $y=Ae^{\frac{x}{2}}-\ln x$ $\frac{dy}{dx}=\frac{A}{2}e^{\frac{x}{2}}-\frac{1}{x}$ When $x=3$, $\frac{dy}{dx}=0$: $0=\frac{A}{2}e^{\frac{3}{2}}-\frac{1}{3}$. $A=\frac{2}{3}e^{-\frac{3}{2}}=0.14875$ $\therefore y=0.149e^{\frac{x}{2}}-\ln x$ **<u>Alternative</u>** When $x=3$, $\frac{dy}{dx}=0$. Sub into DE: $y=0+\frac{2}{3}-\ln 3=-0.43195$ Sub into (3): $-0.43195=Ae^{\frac{3}{2}}-\ln 3$ $A=(\ln 3-0.43195)e^{-\frac{3}{2}}=0.14875$ $\therefore y=0.149e^{\frac{x}{2}}-\ln x$

11c	

12 A chemical processing plant uses two types of automated dosing pumps, Pump A and Pump B, to regulate the flow of a catalyst into a reaction chamber. Each pump is removed from the plant and tested over a 10-hour trial period, and the volume of liquid dosed per hour is recorded.

The following data were collected.

- Pump A: The volume of liquid dosed was 4.5 litres in the first hour, and for each subsequent hour, the volume of liquid dosed decreased by a constant percentage of $r\%$.
- Pump B: The volume of liquid dosed was 4.7 litres in the first hour, and for each subsequent hour, the volume of liquid dosed decreased by 0.1 litres.

(a) Show that the total volume of liquid dosed by Pump A at the end of the trial period is

$$\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{k}\right] \text{ litres,}$$

where k is a constant to be determined. [3]

(b) It would be ideal if at the end of the trial period, the volume of liquid dosed by Pump A is equal to that of Pump B. Find the value of r for this ideal situation. [2]

After the trial period, the two pumps were cleaned and reset, and were subsequently installed in the actual plant. Assume that the two pumps perform consistently as what was observed during the trial period.

(c) Pump A is installed for dosing in Reaction Line 1. To prevent underdosing, Pump A will cease to operate if the volume dosed in the next hour falls below 4.0 litres. With the assumption that $r=1$, find the number of complete hours that Pump A should be in operation. [2]

Before operations begin in Reaction Line 2, Pump A breaks down.

(d) To minimise disruption, the company considers buying a new motor for Pump A from a third-party supplier. The supplier claims that with the new motor, Pump A can dose at least 360 litres of liquid in total if it ran continuously without stopping. Determine the range of values of r for which the supplier's claim is invalid. [2]

(e) The company eventually decides to use Pump B in Reaction Line 2. Using an algebraic method, determine if Pump B can dose at least 150 litres of liquid if left to run continuously without stopping. [3]

No.	Solution
12a	Total volume of liquid dosed by Pump A at the end of 10 hours $=4.5+4.5\left(1-\frac{r}{100}\right)+\ldots+4.5\left(1-\frac{r}{100}\right)^9$ $=\frac{4.5\left[1-\left(1-\frac{r}{100}\right)^{10}\right]}{1-\left(1-\frac{r}{100}\right)}$ $=\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{10}\right]$ [Box: $a=4.5$, $r=1-\frac{r}{100}$, $n=10$] Hence $k=10$.
12b	$\frac{450}{r}\left[1-\left(1-\frac{r}{100}\right)^{10}\right]=\frac{10}{2}[2(4.7)+9(-0.1)]=42.5$ By GC, $r=1.2771=1.28$ (to 3 s.f.).
12c	Let n be the number of hours pump A has operated. $4.5(0.99)^{n-1}\geq 4.0$ (table below) It should operate 12 complete hours. **<u>Alternative</u>** $(n-1)\ln(0.99)\geq\ln\left(\frac{4.0}{4.5}\right)$ $n-1\leq\ln\left(\frac{4.0}{4.5}\right)\Big/\ln(0.99)$ $n\leq 12.719$ It should operate 12 complete hours.
12d	For the supplier's claim to be invalid, $\frac{4.5}{1-\left(1-\frac{r}{100}\right)}<360$ $\frac{450}{r}<360$ $r>1.25$

Table in 12c:

n	$4.5(0.99)^{n-1}$
11	4.07
12	4.03
13	3.99

12e	Total volume of liquid by Pump B $=\frac{n}{2}\left[2(4.7)+(n-1)(-0.1)\right]=4.75n-0.05n^2$ $4.75n-0.05n^2\geq 150 \Rightarrow 0.05n^2-4.75n+150\leq 0$ Since Discriminant $=(-4.75)^2-4(0.05)(150)=-7.43<0$ and Coeff of $n^2>0$, there is no real solution for n. Thus it is not possible for Pump B to dose at least 150 litres of liquid if run continuously. **<u>Alternative</u>** Let $4.7+(n-1)(-0.1)=0$ $n=48$ Hence Pump B can only operate at most 47 hours. Total volume dosed at the end of 47 hours $=\frac{47}{2}\left[2(4.7)+(47-1)(-0.1)\right]$ $=112.8<150$ Thus it is not possible for Pump B to dose at least 150 litres of liquid if run continuously.

2025 JC2 H2 Math Prelim P2

Solutions

1 It is given that $\mathrm{f}(x)=\ln(a+x)$, $x\in\mathbb{R}$, $x>-a$, where a is a constant.

(a) Using the standard series from the List of Formulae (MF27), find the series expansion for $\mathrm{f}(x)$, up to and including the term in x^3. [2]

It is given that $a=1$.

(b) Hence, or otherwise, show that the series expansion of $\sin\left[\mathrm{f}(x)\right]$, up to and including the term in x^3 is given by $x-\frac{1}{2}x^2+\frac{1}{6}x^3$. [2]

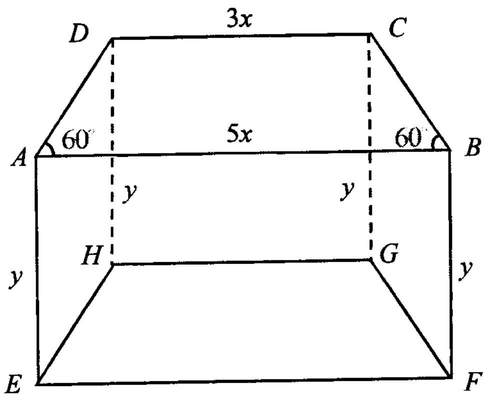
(c) Deduce the Maclaurin series for $\cos\left[\mathrm{f}(x)\right]$ up to and including the term in x^2. [2]

(d) Find $\int_1^3\left(x-\frac{1}{2}x^2+\frac{1}{6}x^3\right)\mathrm{d}x$. Without the use of a calculator or any further calculation, explain, whether this value is a good approximation to the value of $\int_1^3\sin\left[\mathrm{f}(x)\right]\mathrm{d}x$. [2]

1a	$\ln(a+x)=\ln\left[a\left(1+\frac{x}{a}\right)\right]$ $=\ln a+\ln\left(1+\frac{x}{a}\right)$ $=\ln a+\left(\frac{x}{a}-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{1}{3}\left(\frac{x}{a}\right)^3+...\right)$ $=\ln a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}+...$	
1b	$\sin\left[\ln(1+x)\right]=\sin\left(x-\frac{x^2}{2}+\frac{x^3}{3}+...\right)$ $=\left(x-\frac{x^2}{2}+\frac{x^3}{3}+...\right)-\frac{1}{3!}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+...\right)^3+...$ $=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{1}{6}\left(x^3+...\right)$ $=x-\frac{x^2}{2}+\frac{x^3}{6}+...$	
1c	$\sin\left[\ln(1+x)\right]=x-\frac{1}{2}x^2+\frac{1}{6}x^3+.....$ Differentiate w.r.t. x: $\frac{1}{1+x}\cos\left[\ln(1+x)\right]=1-x+\frac{1}{2}x^2+...$ $\cos\left[\ln(1+x)\right]=(1+x)\left(1-x+\frac{1}{2}x^2+...\right)=1-\frac{1}{2}x^2+...$	

1d	$\int_1^3 x-\frac{x^2}{2}+\frac{x^3}{6}\,dx=3$ For $x\in(1,3)\not\subseteq(-1,1]$, the expansion $\sin\left[\ln(1+x)\right]=\sin\left(x-\frac{x^2}{2}+\frac{x^3}{3}+...\right)$ is not valid and hence, the approximation is not a good approximation.	

2 The following diagram shows the dimensions of a trapezoidal prism with fixed volume $4k\sqrt{3}$ units³, with variables x and y.

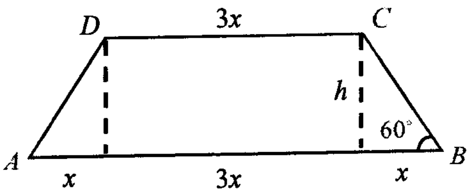


The top surface of the prism, *ABCD*, is an isosceles trapezoid with *AB* of length $5x$ units, *DC* of length $3x$ units, $AD=BC$ and $\angle ABC=\angle BAD=60°$. The rectangular sides *ABFE* and *BCGF* are perpendicular to both the top surface *ABCD* and the bottom surface *EFGH*, with $AE=BF=DH=CG=y$ units.

(a) Show that the total external surface area A of the trapezoidal prism is given by $A=8x^2\sqrt{3}+\frac{12k}{x}$. [4]

(b) Using differentiation, find the value of x in terms of k at which A is a minimum. [4]

(c) It is given instead that the volume of the prism is 1000 units³ and its external surface area is 800 units². Find the two possible values of x. [2]

No.	Solution	Marks
2a	 Let the height of the isosceles trapezoid be h. $h=x\tan 60° =\sqrt{3}x$ and $BC=\frac{x}{\cos 60°}=2x$	

	$V=\frac{1}{2}(5x+3x)\times\sqrt{3}x\times y$ $4\sqrt{3}k=4\sqrt{3}x^2y$ $y=\frac{k}{x^2}$ $A=\frac{1}{2}(5x+3x)\left(\sqrt{3}x\right)\times 2+3xy+5xy+2(2xy)$ $=8\sqrt{3}x^2+12xy$ $=8\sqrt{3}x^2+12x\left(\frac{k}{x^2}\right)$ $=8\sqrt{3}x^2+\frac{12k}{x}$ (shown)	
2b	For minimum A, $\frac{\mathrm{d}A}{\mathrm{d}x}=0$. $16\sqrt{3}x-\frac{12k}{x^2}=0 \Rightarrow x^3=\frac{12k}{16\sqrt{3}} \Rightarrow x=\left(\frac{3k}{4\sqrt{3}}\right)^{\frac{1}{3}}=\left(\frac{\sqrt{3}k}{4}\right)^{\frac{1}{3}}$ $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=16\sqrt{3}+\frac{24k}{x^3}>0 \quad \left(\because k,x>0\Rightarrow\frac{24k}{x^3}>0\right)$ Hence, A is minimum when $x=\left(\frac{\sqrt{3}k}{4}\right)^{\frac{1}{3}}$. **Alternative** (for 2nd derivative test) At $x=\left(\frac{\sqrt{3}}{4}k\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=16\sqrt{3}+\frac{24k}{\frac{\sqrt{3}k}{4}}=48\sqrt{3}>0$. **Alternative** (for 1st derivative test)	

x	$0.75k^{\frac{1}{3}}$	$\left(\frac{\sqrt{3}}{4}k\right)^{\frac{1}{3}}$	$0.76k^{\frac{1}{3}}$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	$-0.549k^{\frac{1}{3}}$	0	$0.286k^{\frac{1}{3}}$
sign (since $k>0$)	–	0	+
slope	\	—	/

2c	When $V=1000$, $k=\dfrac{1000}{4\sqrt{3}}=\dfrac{250}{\sqrt{3}}$. $800=8\sqrt{3}x^2+\dfrac{12}{x}\left(\dfrac{250}{\sqrt{3}}\right)$ From GC, $x=-8.5102$ (N.A. since $x>0$) or $x=6.10$ or $x=2.41$	

3 With reference to the point O as the origin and the x-y plane as a horizontal plane, the pyramid $OPQRV$ has a parallelogram base $OPQR$ and height OV. The position vectors of the points P and R are $-3\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ and $5\mathbf{i}-2\mathbf{j}-\mathbf{k}$ respectively.

(a) Find the coordinates of the point S that lies on the line PR such that the distance from O to S is a minimum. [3]

(b) Find the cartesian equations of the planes such that the perpendicular distance from each plane to the base $OPQR$ is $2\sqrt{86}$ units. [3]

(c) Find the acute angle between OV and the vertical. [2]

(d) Given that QV is parallel to the vector $-5\mathbf{j}+8\mathbf{k}$, find the position vector of the point V. Hence find the exact volume of the pyramid $OPQRV$. [5]

[Volume of a pyramid = $\frac{1}{3}\times$base area$\times$height]

No.	Solution	Marks
3a	$\overrightarrow{PR}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}-\begin{pmatrix}-3\\4\\3\end{pmatrix}=\begin{pmatrix}8\\-6\\-4\end{pmatrix}=-2\begin{pmatrix}-4\\3\\2\end{pmatrix}$ Line PR: $\underset{\sim}{r}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\3\\2\end{pmatrix}$, $\lambda\in\mathbb{R}$ $OS\perp PR$ for OS to be minimum: $\overrightarrow{OS}\bullet\overrightarrow{PR}=0$ Since S lies on line PR, $\left[\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\3\\2\end{pmatrix}\right]\bullet\begin{pmatrix}-4\\3\\2\end{pmatrix}=0$. $(-20-6-2)+\lambda(16+9+4)=0 \Rightarrow \quad \lambda=\dfrac{28}{29}$ $\overrightarrow{OS}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}+\dfrac{28}{29}\begin{pmatrix}-4\\3\\2\end{pmatrix}=\dfrac{1}{29}\begin{pmatrix}33\\26\\27\end{pmatrix}$ Coordinates of S are $\left(\dfrac{33}{29},\dfrac{26}{29},\dfrac{27}{29}\right)$.	

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Alternative

$\overrightarrow{PS} = \left(\dfrac{\overrightarrow{PO} \bullet \overrightarrow{PR}}{\left|\overrightarrow{PR}\right|} \right) \dfrac{\overrightarrow{PR}}{\left|\overrightarrow{PR}\right|}$

$\overrightarrow{PS} = \dfrac{\begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{29}} \dfrac{\begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}}{\sqrt{29}} = \dfrac{-30}{29} \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$

$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix} + \dfrac{-30}{29} \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix} = \dfrac{1}{29} \begin{pmatrix} 33 \\ 26 \\ 27 \end{pmatrix}$

Coordinates of S are $\left(\dfrac{33}{29}, \dfrac{26}{29}, \dfrac{27}{29} \right)$.

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$\overrightarrow{OP} \times \overrightarrow{OR} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}$

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Let the equation of a plane be $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix} = t$, and note that $(t, 0, 0)$ is a point on the plane.

$\dfrac{\left| \left(\begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix} \right|}{\sqrt{1+36+49}} = 2\sqrt{86}$

$|t| = 2(86)$

$t = \pm 172$

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OR

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$\left|\underset{\sim}{r} \cdot \underset{\sim}{\hat{n}}\right| = 2\sqrt{86}$

$\dfrac{\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}}{\sqrt{1+36+49}} = \pm 2\sqrt{86}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix} = \pm 2(86) = \pm 172$

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The planes have equations $x + 6y - 7z = 172$ and $x + 6y - 7z = -172$.

Alternative

$\overrightarrow{OP} \times \overrightarrow{OR} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ -14 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}$

A normal to the plane $OPQR = \begin{pmatrix} 1 \\ 6 \\ -7 \end{pmatrix}$

Let T and U be different points on the two planes such that $OT = 2\sqrt{86} = OU$.

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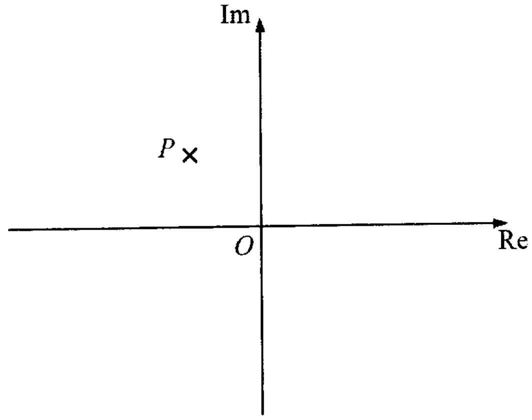
	$\overrightarrow{OT}=2\sqrt{86}\times\frac{\begin{pmatrix}1\\6\\-7\end{pmatrix}}{\sqrt{1+36+49}}=\begin{pmatrix}2\\12\\-14\end{pmatrix}$ $\overrightarrow{OU}=2\sqrt{86}\times\frac{-\begin{pmatrix}1\\6\\-7\end{pmatrix}}{\sqrt{1+36+49}}=\begin{pmatrix}-2\\-12\\14\end{pmatrix}$ The equations of the planes are $\underset{\sim}{r}\bullet\begin{pmatrix}1\\6\\-7\end{pmatrix}=\begin{pmatrix}2\\12\\-14\end{pmatrix}\bullet\begin{pmatrix}1\\6\\-7\end{pmatrix}$ and $\underset{\sim}{r}\bullet\begin{pmatrix}1\\6\\-7\end{pmatrix}=\begin{pmatrix}-2\\-12\\14\end{pmatrix}\bullet\begin{pmatrix}1\\6\\-7\end{pmatrix}$, i.e. $x+6y-7z=172$ and $x+6y-7z=-172$.	
3c	Let the acute angle between OV and the vertical be θ. $\cos\theta=\frac{\left\|\begin{pmatrix}1\\6\\-7\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{86}}=\frac{7}{\sqrt{86}}$ $\theta=41.0^\circ$	
3d	$\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OR}=\begin{pmatrix}-3\\4\\3\end{pmatrix}+\begin{pmatrix}5\\-2\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$ $\begin{pmatrix}2\\2\\2\end{pmatrix}=\overrightarrow{OQ}=\overrightarrow{OV}+\overrightarrow{VQ}=a\begin{pmatrix}1\\6\\-7\end{pmatrix}+b\begin{pmatrix}0\\-5\\8\end{pmatrix}$, for some $a,b\in\mathbb{R}$ By comparing LHS and RHS, $a=2$ (and $b=2$). $\therefore\overrightarrow{OV}=2\begin{pmatrix}1\\6\\-7\end{pmatrix}=\begin{pmatrix}2\\12\\-14\end{pmatrix}$	

	Volume of pyramid $=\frac{1}{3}\times\left\|\overrightarrow{OP}\times\overrightarrow{OR}\right\|\times\left\|\overrightarrow{OV}\right\|$ $=\frac{1}{3}\times\left\|\begin{pmatrix}-3\\4\\3\end{pmatrix}\times\begin{pmatrix}5\\-2\\-1\end{pmatrix}\right\|\times\left\|\begin{pmatrix}2\\12\\-14\end{pmatrix}\right\|$ $=\frac{1}{3}\times\left\|\begin{pmatrix}2\\12\\-14\end{pmatrix}\right\|\times\left\|\begin{pmatrix}2\\12\\-14\end{pmatrix}\right\|$ $=\frac{1}{3}\times(4+144+196)$ $=\frac{344}{3}$ units^3 **<u>Alternative</u> (to find position vector of *V*)** $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{OR}=\begin{pmatrix}-3\\4\\3\end{pmatrix}+\begin{pmatrix}5\\-2\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$ Line QV: $\underset{\sim}{r}=\begin{pmatrix}2\\2\\2\end{pmatrix}+\alpha\begin{pmatrix}0\\5\\-8\end{pmatrix},\ \alpha\in\mathbb{R}$ Line OV: $\underset{\sim}{r}=\beta\begin{pmatrix}1\\6\\-7\end{pmatrix},\ \beta\in\mathbb{R}$ V is the intersection of lines QV and OV: $\beta\begin{pmatrix}1\\6\\-7\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}+\alpha\begin{pmatrix}0\\5\\-8\end{pmatrix}$, for some $\alpha,\beta\in\mathbb{R}$ Solving gives $\beta=2,\ \alpha=2$. $\overrightarrow{OV}=2\begin{pmatrix}1\\6\\-7\end{pmatrix}=\begin{pmatrix}2\\12\\-14\end{pmatrix}$	

4 Do not use a calculator in answering this question.

The complex number z has modulus 1 and argument θ, where $\frac{\pi}{2} < \theta < \pi$, and the complex number w is given by $w = \mathrm{i}\sqrt{3}\,z$. The point P on the Argand diagram represents z.



(a) On the copy of the Argand diagram with origin O in the Printed Answer Booklet, plot the points Q and R to represent w and $z-w$ respectively. Show clearly the geometrical relationship between the points P, Q and R. [3]

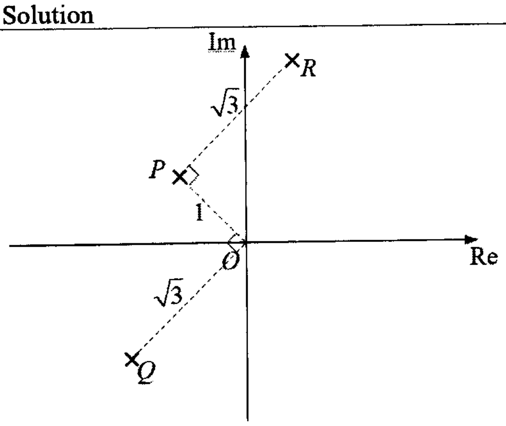
(b) Find the area of the quadrilateral $ORPQ$. [1]

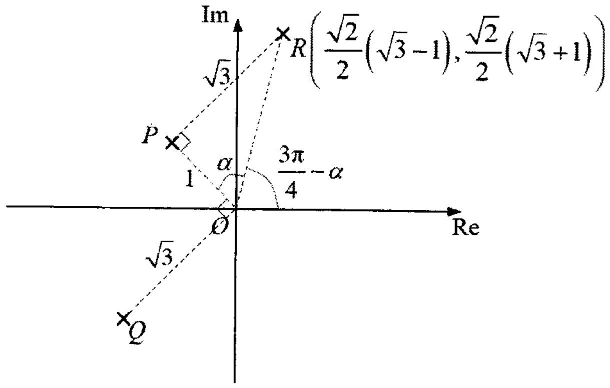
It is given that $\theta = \frac{3\pi}{4}$.

(c) Find z in the form $x+y\mathrm{i}$, where x and y are real numbers. [2]

(d) Show that $z-w=k\left[\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)\mathrm{i}\right]$, where k is a constant to be determined. [2]

(e) Hence show that $\tan\frac{5\pi}{12}=\frac{\sqrt{3}+1}{\sqrt{3}-1}$. [1]

No.	Solution	Marks
4a		
4b	Area $ORPQ = \frac{1}{2}(1)\left(\sqrt{3}+\sqrt{3}\right)=\sqrt{3}$	

4c	Length $x = 1\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ Length $y = 1\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\therefore z = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$	
4d	$z - w = \left(z - \sqrt{3}\mathrm{i}z\right)$ $= \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)\left(1 - \sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}(-1+\mathrm{i})\left(1-\sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}\left(-1+\sqrt{3}+\mathrm{i}+\sqrt{3}\mathrm{i}\right)$ $= \frac{1}{\sqrt{2}}\left(\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)\mathrm{i}\right)$ $k = \frac{1}{\sqrt{2}}$	
4e	 $\alpha = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$ $\tan\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \frac{\frac{\sqrt{2}}{2}\left(\sqrt{3}+1\right)}{\frac{\sqrt{2}}{2}\left(\sqrt{3}-1\right)}$ $\tan\frac{5\pi}{12} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$	

5 A code consists of 10 characters.

The first 5 characters of the code is formed using 5 letters chosen from the set {A, B, C, D, E, F, G, H} and the last 5 characters of the code is formed using 5 digits chosen from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 0}. The code allows for repetitions of letters and/or digits.

(a) Find the number of different codes that can be formed. [1]

(b) Find the probability that a code chosen at random

(i) contains the letter E exactly thrice and the digit 5 exactly once, [2]

(ii) has E as the first character or 4 as its tenth character, but not both. [2]

5a	Number of codes $= 8^5 \times 10^5 = 3276800000$	
5bi	P(E exactly thrice and 5 once) $= \dfrac{{}^5C_3 \times 7^2 \times {}^5C_1 \times 9^4}{8^5 10^5} \approx 0.00491$ **Alternative** P(E exactly thrice and 5 once) $= {}^5C_3\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^2 \times {}^5C_1\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^4$ ≈ 0.00491	
5bii	Required probability $= \frac{1}{8} + \frac{1}{10} - 2\left(\frac{1}{8} \times \frac{1}{10}\right)$ $= 0.2$ **Alternative** Required probability $= \frac{1}{8} \times \frac{9}{10} + \frac{7}{8} \times \frac{1}{10}$ $= 0.2$ **Alternative** Required probability $= \dfrac{8^4 \times 10^5 + 8^5 \times 10^4 - 2\left(8^4 \times 10^4\right)}{8^5 10^5}$ $= 0.2$ **Alternative** P(E first, 4 not tenth character) $= \dfrac{1 \times 8^4 \times 10^4 \times 9}{8^5 10^5}$ or $\frac{1}{8} \times \frac{9}{10} = 0.1125$ P(4 tenth, E not first character) $= \dfrac{10^4 \times 1 \times 7 \times 8^4}{8^5 10^5}$ or $\frac{7}{8} \times \frac{1}{10} = 0.0875$ Required probability $= 0.1125 + 0.0875 = 0.2$	

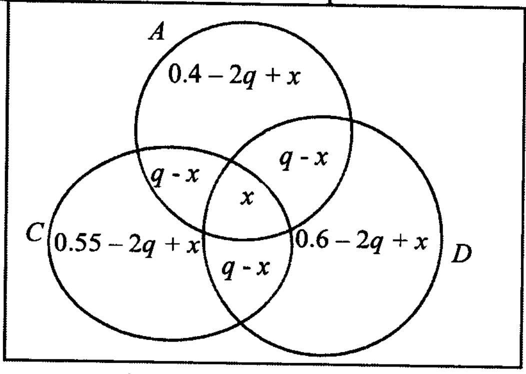
6 Two events A and B are such that $\mathrm{P}(A)=p$ and $\mathrm{P}(B)=\frac{5}{4}p$. It is given that $\mathrm{P}(A\cup B)=\mathrm{P}(A\mid B)=0.6$.

(a) Find the value of p. [3]

(b) Explain whether the events A' and B are independent. [1]

The events C and D are such that $\mathrm{P}(C)=0.55$, $\mathrm{P}(D)=0.6$ and $\mathrm{P}(A\cap C)=\mathrm{P}(A\cap D)=\mathrm{P}(C\cap D)=q$.

(c) Find the value of q that gives the minimum value of $\mathrm{P}(A\cup C\cup D)$ and state the minimum value of $\mathrm{P}(A\cup C\cup D)$. [2]

6a	$\mathrm{P}(A\mid B)=0.6\Rightarrow\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=0.6$ $\frac{\mathrm{P}(A\cap B)}{\frac{5}{4}p}=0.6\Rightarrow\mathrm{P}(A\cap B)=0.75p$ $\mathrm{P}(A\cup B)=0.6$ $\Rightarrow\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.6$ $\Rightarrow p+\frac{5}{4}p-0.75p=0.6$ $\Rightarrow p=0.4$	
6b	$\mathrm{P}(A\mid B)=0.6\neq0.4=\mathrm{P}(A)$ Since A and B are not independent, then A' and B are also not independent. **Alternative** $\mathrm{P}(A'\cap B)=\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.5-0.3=0.2\neq0.6\times0.5=\mathrm{P}(A')\mathrm{P}(B)$ Hence, A' and B are not independent.	
6c	 Let x be $\mathrm{P}(A\cap C\cap D)$. For minimum $\mathrm{P}(A\cup C\cup D)$, $0.4-2q+x=0$ and $q-x=0$, i.e. $q=x$. $\therefore 0.4-2q+q=0\Rightarrow q=0.4$ $\mathrm{P}(A\cup C\cup D)=\mathrm{P}(A)+\mathrm{P}(C\cap A'\cap D')+\mathrm{P}(C\cap A'\cap D)+\mathrm{P}(C'\cap A'\cap D)$ $=0.4+(0.55-2q+x)+(q-x)+(0.6-2q+x)$ $=1.55-3q+x$ $\mathrm{P}(A\cup C\cup D)=1.55-3(0.4)+(0.4)=0.75$	

7 A shop sells apples in bags of 12. The average number of unripe apples in a bag is 2.16.

(a) State two assumptions needed for the number of unripe apples in a bag to be well modelled by a binomial distribution. [2]

A bag of apples is sold at a reduced price if more than 3 apples are unripe.

(b) Find the probability that a randomly selected bag of apples is sold at a reduced price. [2]

(c) Twenty bags of apples are selected at random. Find the probability that fewer than 4 of these bags will be sold at a reduced price. [1]

(d) The shop also sells oranges in bags of n oranges. On average, the proportion of oranges that is unripe is 0.16. It is known that the modal number of unripe oranges in a bag is 2. Find the set of possible values of n. [2]

7a	1. The probability of each apple being unripe is constant. 2. The ripeness of an apple is independent of other apples. OR Unripe apples occur independently of other apples.	
7b	$p = \frac{2.16}{12} = 0.18$ Let X be the number of unripe apples in a bag of 12. $X \sim \text{B}(12, 0.18)$ $\text{P}(X > 3) = 1 - \text{P}(X \le 3) \approx 0.15515 = 0.155$ (3 s.f.)	
7c	Let Y be the number of bags that are sold at a reduced price out of 20 bags. $Y \sim \text{B}(20, 0.15515)$. $\text{P}(Y < 4) = \text{P}(Y \le 3) = 0.623$ (3 s.f.)	
7d	Let W be the number of unripe oranges in a bag out of n. $W \sim \text{B}(n, 0.16)$ Since modal number of unripe oranges in a bag is 2, $\text{P}(W=1) < \text{P}(W=2)$ and $\text{P}(W=3) < \text{P}(W=2)$.	

n	$\text{P}(W=1)$	$\text{P}(W=2)$	$\text{P}(W=3)$
11	0.308	0.293	0.168
12	0.282	0.296	0.188
⋮	⋮	⋮	⋮
17	0.167	0.255	0.243
18	0.149	0.241	0.244

$\{n \in \mathbb{Z} : 12 \le n \le 17\}$

Alternative

$\text{P}(W=1) - \text{P}(W=2) < 0$ and $\text{P}(W=3) - \text{P}(W=2) < 0$

For $\text{P}(W=1) - \text{P}(W=2)$:

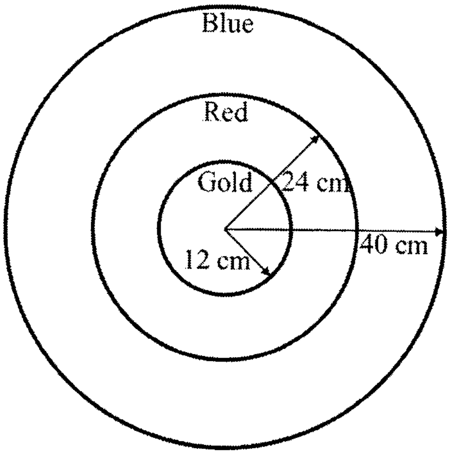
n	$\text{P}(W=1) - \text{P}(W=2)$
11	0.0147
12	−0.0134
13	−0.0367

For $\text{P}(W=3) - \text{P}(W=2)$:

n	$\text{P}(W=3) - \text{P}(W=2)$
16	−0.0297
17	−0.0121
18	0.00382

$\{n \in \mathbb{Z} : 12 \le n \le 17\}$

8 The diagram below shows a circular target board of radius 40 cm, divided by two concentric circles of radii 12 cm and 24 cm, into three regions, Gold, Red and Blue.



Arrows are shot at the target board and the scores obtained for hitting the Gold, Red and Blue regions are 10 points, 6 points and 3 points respectively. Arrows that land outside the target board score 0 points.

Alex shoots one arrow at the target board. The probability that his shot lands on the target board is p, where $p > 0.5$. If his shot lands on the target board, the arrow is equally likely to hit any position on the board. Let X represent the score obtained from a single shot.

You may assume that the arrow **does not land on the boundary** of any region.

(a) Show that the $\text{P}(X = 6) = 0.27p$. [1]

(b) Given that $\text{Var}(X) = 5.464476$, find the value of p. [4]

8a	$\text{P}(X=6)=(p)\left(\dfrac{\pi(24^2)-\pi(12^2)}{\pi(40^2)}\right)=0.27p$	
8b	$\text{P}(X=10)=\dfrac{12^2\pi}{40^2\pi}p=0.09p$ (table below) $\text{E}(X)=(3)(0.64)p+(6)(0.27)p+(10)(0.09)p=4.44p$ $\text{Var}(X)=\left(0+3^2(0.64p)+6^2(0.27p)+10^2(0.09p)\right)-(4.44p)^2$ $5.464476=24.48p-19.7136p^2$ $19.7136p^2-24.48p+5.464476=0$ $p=0.292$ (rej. $\because p>0.5$) or $p=0.95$	

x	0	3	6	10
$\text{P}(X=x)$	$1-p$	$0.64p$	$0.27p$	$0.09p$

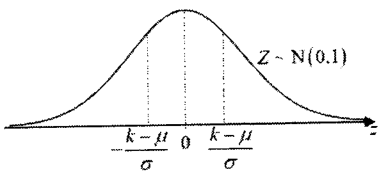
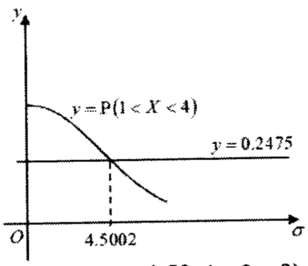
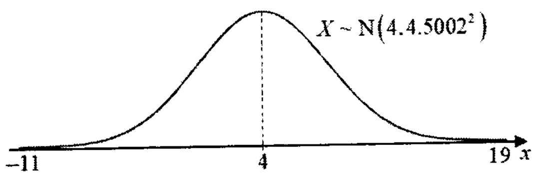
9 The random variable X has distribution $\mathrm{N}(\mu,\ \sigma^2)$.

(a) Given that $\mathrm{P}(X>k)=0.45$, find the value of $\mathrm{P}(X>2\mu-k)$. [2]

It is given that $\mu=4$ and $\mathrm{P}(1<X<4)=0.2475$.

(b) Find the value of σ. [2]

(c) Draw a sketch to show the distribution of X for x between -11 and 19. [2]

9a	$\mathrm{P}(X>k)=0.45 \Rightarrow \mathrm{P}\left(Z>\frac{k-\mu}{\sigma}\right)=0.45$ $\mathrm{P}(X>2\mu-k)=\mathrm{P}\left(Z>\frac{2\mu-k-\mu}{\sigma}\right)$ $=\mathrm{P}\left(Z>\frac{\mu-k}{\sigma}\right)$ $=\mathrm{P}\left(Z<\frac{k-\mu}{\sigma}\right)$ $=1-0.45$ $=0.55$ 	
9b	$\mathrm{P}(1<X<4)=0.2475$ $\mathrm{P}(X<1)=0.5-0.2475=0.2525$ $\mathrm{P}\left(Z<\frac{1-4}{\sigma}\right)=0.2525$ From the GC, $\mathrm{P}(Z<-0.66664)=0.2525$ $\frac{1-4}{\sigma}=-0.66664$ $\sigma=4.50$ (to 3 s.f.) **<u>Alternative</u>**  By GC, $\sigma=4.50$ (to 3 s.f.).	
9c		

10 In this question you should state the parameters of any distributions you use.

An orchard sells kiwis of two varieties, Type A and Type B. The masses, in grams, of Type A and Type B kiwis follow the distributions $\mathrm{N}(110, 6^2)$ and $\mathrm{N}(85, 8^2)$ respectively. It is assumed that these two distributions are independent.

Kiwis are selected randomly and packed into bags. Type A kiwis are packed in bags of 30, while Type B kiwis are packed in bags of 60.

(a) In a randomly chosen bag of Type A kiwis, find the probability that the mean mass of kiwis is between 109 grams and 115 grams. [2]

The selling prices of Type A and Type B kiwis are \$25 per kilogram and \$15 per kilogram respectively.

(b) Find the probability that the average selling price of a kiwi from a bag of Type A kiwis is more than the average selling price of a kiwi from a bag of Type B kiwis by at most \$1.50. [3]

The orchard also produces genetically modified Type G kiwis, which are packed in bags of 50. The masses of Type G kiwis are identically distributed with mean and standard deviation 90 grams and 5 grams respectively.

Each Assortment Box contains one bag of Type A kiwis and one bag of Type G kiwis. A box is fit for sale if the total mass of the kiwis in the box is at least 7.85 kilograms.

(c) Ten Assortment Boxes are selected at random. Find the probability that none of the boxes are fit for sale. [3]

(d) State, in context, the approximation(s) and assumption(s) needed for the distribution(s) used in part **(c)**. [2]

10a	Let A g be the mass of one Type A kiwi. $\bar{A} = \dfrac{A_1 + A_2 + \ldots + A_{30}}{30} \sim \mathrm{N}(110, 1.2)$ $\mathrm{P}(109 < \bar{A} < 115) = 0.819$ (to 3 s.f.)	
10b	Let B g be the mass of one Type B kiwi. $\bar{B} = \dfrac{B_1 + B_2 + \ldots + B_{60}}{60} \sim \mathrm{N}\left(85, \dfrac{16}{15}\right)$ $0.025\bar{A} \sim \mathrm{N}(2.75, 0.00075)$ $0.015\bar{B} \sim \mathrm{N}(1.275, 0.00024)$ $0.025\bar{A} - 0.015\bar{B} \sim \mathrm{N}(1.475, 0.00099)$ $\mathrm{P}(0 < 0.025\bar{A} - 0.015\bar{B} \le 1.50) \approx 0.787$	
10c	Let G g be the mass of one Type G kiwi. Since $n = 50$ is sufficiently large, $T = G_1 + G_2 + \ldots + G_{50} \sim \mathrm{N}(4500, 1250)$ approximately. Together with $S = A_1 + A_2 + \ldots + A_{30} \sim \mathrm{N}(3300, 1080)$, $S + T \sim \mathrm{N}(7800, 2330)$ approximately. Required probability $= \left[\mathrm{P}(A + G < 7850)\right]^{10}$ ≈ 0.197	

	Alternative $30\overline{A} \sim \mathrm{N}(3300,1080)$ Since $n=50$ is sufficiently large, $\overline{G} \sim \mathrm{N}(90,0.5)$ approximately. $30\overline{A}+50\overline{G} \sim \mathrm{N}(7800,2330)$ approx. Required probability $= \mathrm{P}(A+G<7850)^{10}$ ≈ 0.197	
10d	Approximation: By the **Central Limit Theorem**, the **sample sum** of masses of Type G kiwis is **approximately normally distributed**, since the **number of kiwis in each bag, 50, is sufficiently large**. Assumption: The **masses** of kiwis are **independent**.	

11 An energy company is studying the relationship between the average daily temperature, x (in degrees Celsius), and the amount of electricity consumed, y (in megawatt-hours MWh). For a random sample of ten days, the records are shown in the table below.

Average daily temperature (x°C)	20	22	24	26	28	30	32	34	36	38
Electricity consumption (y MWh)	12.5	12.8	13.2	14.0	14.7	16.2	16.2	19.9	22.3	29.5

(a) Draw a scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the electricity consumption and average daily temperature can be modelled by one of the formulae

$$y=a+bx^2, \qquad y=c+dx,$$

where a, b, c and d are constants.

(b) Find the value of the product moment correlation coefficient between

(i) x and y,

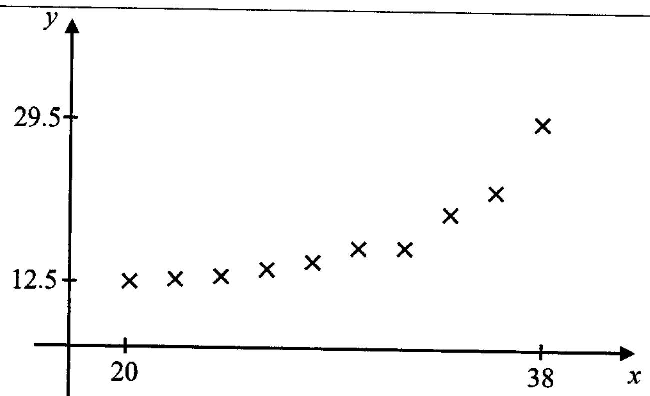
(ii) x^2 and y. [2]

(c) Using your answers in parts **(a)** and **(b)**, explain which of $y=a+bx^2$ and $y=c+dx$ is the better model and find the equation of a suitable regression line for this model. [3]

(d) Use the equation of your regression line to estimate the electricity consumption on a day with an average temperature of $37°\mathrm{C}$. Comment on the reliability of your estimate. [2]

In some regions, temperature is measured in degrees Fahrenheit instead of degrees Celsius. A temperature of C degrees Celsius is equivalent to F degrees Fahrenheit, where $F=\frac{9}{5}C+32$.

(e) Rewrite the equation of your regression line found in part **(c)** in terms of y and F, where F is the average daily temperature in degrees Fahrenheit. [1]

11a		
11bi	From GC, $r = 0.89056 \approx 0.891$.	
11bii	From GC, $r = 0.92229 \approx 0.922$.	
11c	From the scatter diagram, it is observed that as x increases, y increases by increasing amounts, and the product moment correlation coefficient between x^2 and y is 0.922, which is closer to 1 than that between x and y, which is 0.891. Hence $y = a + bx^2$ is the better model. $a = 4.8291 \approx 4.83$ $b = 0.014074 \approx 0.0141$ $y = 4.83 + 0.0141x^2$	
11d	$y = 4.8291 + 0.014074(37)^2$ ≈ 24.1 Since $x = 37$ lies within the given range of x values, $20 \le x \le 38$, and the value of $r \approx 0.922$ is close to 1, the estimated electricity consumption is reliable.	
11e	$F = \frac{9}{5}x + 32 \Rightarrow x = \frac{5}{9}(F - 32)$ Replace x by $\frac{5}{9}(F-32)$: $y = 4.8291 + 0.014074\left[\frac{5}{9}(F-32)\right]^2$ $= 4.8291 + 0.0043438(F-32)^2$ $\approx 4.83 + 0.00434(F-32)^2$	

12 A hospital introduces a new medication regimen to help patients recover from a viral infection more quickly. Previously, the average recovery time was 7.2 days.

After adopting the new regimen, a random sample of 40 patients is observed, and their recovery times, x, in days, are recorded. The results are summarised below.

$$\sum(x-7.2)=-42 \quad \text{and} \quad \sum(x-7.2)^2=648$$

(a) Explain whether the hospital should carry out a one-tailed test or a two-tailed test. [1]

(b) Calculate unbiased estimates of the population mean and variance of the recovery times under the new regimen. [2]

(c) Carry out an appropriate test at the 5% significance level. You should state clearly the hypotheses for your test and define any parameters that you use. [4]

(d) Explain, in the context of the question, the meaning of 'at the 5% significance level'. [1]

(e) Explain whether it would have been sufficient for the hospital to take a random sample of 10 patients and record their recovery times under the new regimen in order to carry out the hypothesis test. [1]

It was discovered that an intern at the hospital had made an error in recording the recovery times. A random sample of another 30 patients was taken and their recovery times under the new regimen was taken and recorded by a senior researcher. The sample standard deviation was found to be 1.7 days and the sample mean time was denoted by k.

(f) A test at the 1% significance level concludes that the average recovery time did not improve from 7.2 days. Find the set of values of k. [3]

12a	The researchers should carry out a one-tailed test as they want to test whether the new regimen has **reduced mean recovery time.**	
12b	Let $y=x-7.2$ Unbiased estimate of population mean, $\bar{x}=\bar{y}+7.2=\frac{-42}{40}+7.2=6.15$ Unbiased estimate of population variance, $s_x^2=s_y^2=\frac{1}{39}\left(648-\frac{(-42)^2}{40}\right)=\frac{603.9}{39}=15.4846\approx 15.5$	
12c	Let X days be the recovery time of a patient and μ days be the population mean of X. $H_0: \mu=7.2$ $H_1: \mu<7.2$ Level of Significance: 5% Test Statistic: Since $n=40$ is large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z=\frac{\bar{X}-7.2}{S/\sqrt{n}}\sim N(0,1)$ approximately. Computation: $\bar{x}=6.15$, $s_x^2\approx 15.4846$, p-value $=0.045744$ (or $z=-1.6876$)	

	Conclusion: Since p-value = 0.0457 < 0.05, H_0 is rejected at 5% level of significance. Hence there is sufficient evidence to conclude that the mean recovery time has reduced under the new regimen.	
12d	A 5% significance level means that there is a probability of 0.05 that the test concludes that the mean recovery time has reduced when it is actually 7.2 days.	
12e	No. Since the distribution of recovery time under the new regimen (X) is unknown, the sample size would have to be sufficiently large for Central Limit Theorem to be applied so that the **sample mean recovery time** is approximately normally distributed. A sample of 10 is not large enough.	
12f	$H_0 : \mu = 7.2$ $H_1 : \mu < 7.2$ Level of Significance: 1% Test Statistic: Since $n = 30$ is large, by Central Limit Theorem, $\overline{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\overline{X} - 7.2}{S/\sqrt{n}} \sim N(0,1)$ approximately. Computation: $s^2 = \dfrac{n}{n-1}(\text{sample variance}) = \dfrac{30}{29}(1.7)^2 = 2.9897$ Rejection Region: $z \le -2.3263$ Since H_0 is not rejected, $\dfrac{k - 7.2}{\sqrt{2.9897}/\sqrt{30}} > -2.3263$ $k > 6.4656$ $\therefore \{k \in \mathbb{R} : k > 6.47\}$	