Solutions for 2025 JC2 H2 Mathematics Preliminary Examination Paper 1

1 Show that the differential equation $y\left(\frac{dy}{dx}+2y\right)=\frac{x}{e^{4x+x^2}}$ can be reduced by the substitution $z=y^2e^{4x}$ to $\frac{dz}{dx}=\frac{2x}{e^{x^2}}$. Hence, find the general solution in the form $y^2=f(x)$. [4]

1	$z=y^2e^{4x}$ $\frac{dz}{dx}=2ye^{4x}\frac{dy}{dx}+4y^2e^{4x}$ $=2ye^{4x}\left(\frac{dy}{dx}+2y\right)$ Since $y\left(\frac{dy}{dx}+2y\right)=\frac{x}{e^{4x+x^2}}$, $\frac{dz}{dx}=2e^{4x}\frac{x}{e^{4x+x^2}}$ $=\frac{2x}{e^{x^2}}$ (shown) $\frac{dz}{dx}=\frac{2x}{e^{x^2}}$ $z=\int 2xe^{-x^2}dx$ $=-e^{-x^2}+c$ $y^2e^{4x}=-e^{-x^2}+c$, where c is an arbitrary constant $y^2=e^{-4x}\left(c-e^{-x^2}\right)$	Differentiate $z=y^2e^{4x}$ wrt x, as $\frac{dy}{dx}$ means "differentiate y wrt x" Since this is a "show" question, clear workings are necessary. Do not skip steps. $z=\int 2xe^{-x^2}dx=-e^{-x^2}+c$ This integral $\int 2xe^{-x^2}dx$ is in the form $\int f'(x)e^{f(x)}dx$. $ce^{-4x}\neq d$ as e^{-4x} varies as x varies.

2 Ronald is writing a novel. He began his new novel on 1 March 2025, writing 23 pages on the first day. On each subsequent day, he writes 90% of the number of pages he has written the previous day.

(a) Find the total number of pages Ronald will have written by 31 March 2025. [1]

Sam is also writing a novel. He began writing his novel on 8 March 2025, writing 2 pages on the first day. On each subsequent day, he writes 1 more page than the day before.

(b) Find the first date on which Sam writes more pages in a day than Ronald. [2]

(c) Find the first date on which Sam's total number of pages written exceeds Ronald's. [2]

2 (a)	Total no. of pages $= \dfrac{23\left(1-0.9^{31}\right)}{1-0.9}$ $= 221\ (3\text{ sf})$	You should round down the answer.
(b)	Let $a_n = 2+\left[(n-7)-1\right](1)$, $b_n = 23(0.9)^{n-1}$ n | a_n | b_n 12 | 6 | 7.22 13 | 7 | 6.50 14 | 8 | 5.85 13 March 2025	Sam began writing his novel 7 days after Ronald has started. Hence the "n" we use for Sam should be $(n-7)$. When $n=1$, the date is 1 March. Hence when $n=13$, the date is 13 March.
(c)	Let $c_n = \dfrac{n-7}{2}\left(2(2)+\left[(n-7)-1\right](1)\right)$, $d_n = \dfrac{23\left(1-0.9^{n}\right)}{1-0.9}$ n | c_n | d_n 26 | 209 | 215.14 27 | 230 | 216.63 28 | 252 | 217.96 27 March 2025	Sam began writing his novel 7 days after Ronald has started. Hence the "n" we use for Sam should be $(n-7)$. When $n=1$, the date is 1 March. Hence when $n=27$, the date is 27 March.

3 **(a)** Find the series expansion of $\dfrac{1}{\sqrt{1+5x}}$, up to and including the term in x^2. State the range of values of x for which the expansion is valid. [3]

(b) Hence, use the substitution $x=\dfrac{1}{20}$ to obtain an approximation of $\sqrt{5}$, expressing your answer as a fraction. [2]

(c) Without any further calculation, explain whether using the substitution $x=-\dfrac{4}{25}$ gives a better approximation of $\sqrt{5}$ than the substitution used in part **(b)**. [1]

3 (a)	$(1+5x)^{-\frac{1}{2}} \approx 1+\left(-\frac{1}{2}\right)(5x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}(5x)^2$ $=1-\frac{5}{2}x+\frac{75}{8}x^2.$ The expansion is valid for $\lvert 5x\rvert<1$. Thus $-\frac{1}{5}<x<\frac{1}{5}$.	This is binomial series expansion, and while you can use the general Maclaurin series formula, you shouldn't as it will be more tedious.
(b)	Sub $x=\frac{1}{20}$ into part **(a)**. **Method 1** LHS: $\left(1+5\left(\frac{1}{20}\right)\right)^{-\frac{1}{2}}=\left(\frac{25}{20}\right)^{-\frac{1}{2}}=\left(\frac{20}{25}\right)^{\frac{1}{2}}=\left(\frac{(4)(5)}{25}\right)^{\frac{1}{2}}=\left(\frac{2}{5}\right)\left(\sqrt{5}\right)$ RHS: $1-\frac{5}{2}\left(\frac{1}{20}\right)+\frac{75}{8}\left(\frac{1}{20^2}\right)=\frac{115}{128}$ $\left(\frac{2}{5}\right)\left(\sqrt{5}\right)=\frac{115}{128}$ $\sqrt{5}=\frac{575}{256}$ **Method 2** LHS: $\left(1+5\left(\frac{1}{20}\right)\right)^{-\frac{1}{2}}=\left(\frac{25}{20}\right)^{-\frac{1}{2}}=\left(\frac{20}{25}\right)^{\frac{1}{2}}=\left(\frac{4}{5}\right)^{\frac{1}{2}}=\left(\frac{2}{\sqrt{5}}\right)$ RHS: $1-\frac{5}{2}\left(\frac{1}{20}\right)+\frac{75}{8}\left(\frac{1}{20^2}\right)=\frac{115}{128}$ $\frac{2}{\sqrt{5}}=\frac{115}{128}$ $\sqrt{5}=\frac{256}{115}$	You must substitute $x=\frac{1}{20}$ to both sides of $(1+5x)^{-\frac{1}{2}}\approx 1-\frac{5}{2}x+\frac{75}{8}x^2$. to approximate $\sqrt{5}$. The LHS may not be $\sqrt{5}$ directly after substitution.

(c)	Since $x=\frac{1}{20}$ is **closer to 0** than $x=-\frac{4}{25}$, hence the substitution $x=-\frac{4}{25}$ does not give a better approximation.	

4 A curve has parametric equations $x=\frac{4t^3}{3}-t$, $y=t-3t^2$.

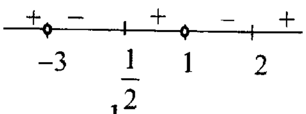
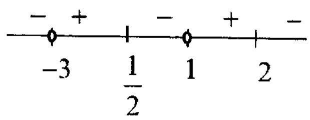
(a) For the part of the curve where $x>0$, find the equation of the tangent to the curve which is parallel to the y-axis. [3]

(b) Find the coordinates of the point where the tangent meets the curve again. [2]

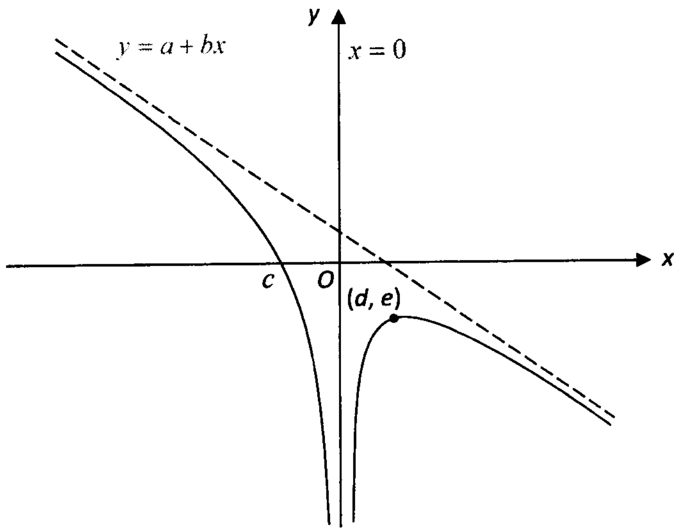
4 (a)	$\frac{dx}{dt}=4t^2-1$, $\frac{dy}{dt}=1-6t$, $\frac{dy}{dx}=\frac{1-6t}{4t^2-1}$. Since tangent is parallel to the y-axis, $\frac{dy}{dx}$ is undefined. $4t^2-1=0$ $t=-\frac{1}{2}$ or $\frac{1}{2}$ (rej as $x>0$) When $t=-\frac{1}{2}$, $x=\frac{4\left(-\frac{1}{2}\right)^3}{3}+\frac{1}{2}=\frac{1}{3}$ Equation of tangent $x=\frac{1}{3}$	All points on that tangent have the same x-value. Thus, the equation is of the form $x=k$. Similarly, when tangent is parallel to x-axis, the equation is of the form $y=k$.
(b)	$x=\frac{4t^3}{3}-t$ $\frac{1}{3}=\frac{4t^3}{3}-t$ $4t^3-3t-1=0$ $t=-\frac{1}{2}$ or 1 At $t=1$, the tangent meets the curve again. i.e. $\left(\frac{1}{3},-2\right)$.	Note that the tangent cuts the curve already at $t=-\frac{1}{2}$, so the other point of intersection will be at $t=1$.

5 **(a)** Without using a calculator, solve the inequality $\dfrac{9x-8}{x^2+2x-3} \geq 2$. [4]

(b) Hence solve the inequality $\dfrac{9\ln x-8}{(\ln x)^2+2\ln x-3} \geq 2$. [2]

5 (a)	$\dfrac{9x-8}{x^2+2x-3} \geq 2$ $\dfrac{9x-8}{x^2+2x-3} - 2 \geq 0$ $\dfrac{9x-8-2\left(x^2+2x-3\right)}{x^2+2x-3} \geq 0$ $\dfrac{-2x^2+5x-2}{x^2+2x-3} \geq 0$ $\dfrac{2x^2-5x+2}{x^2+2x-3} \leq 0$ $\dfrac{(x-2)(2x-1)}{(x+3)(x-1)} \leq 0$  $-3 < x \leq \frac{1}{2}$ or $1 < x \leq 2$	OR $\dfrac{-2x^2+5x-2}{x^2+2x-3} \geq 0$ $\dfrac{-(x-2)(2x-1)}{(x+3)(x-1)} \geq 0$ 	Be mindful of changing the inequality sign when you multiply or divide a negative number throughout the inequality. Check the sign of each region carefully. Exclude the critical values associated with the denominator factors.
(b)	Replace x with $\ln x$, $-3 < \ln x \leq \frac{1}{2}$ or $1 < \ln x \leq 2$ $e^{-3} < x \leq e^{\frac{1}{2}}$ or $e < x \leq e^2$		ln function can be negative.

6 The diagram below shows the graph of $y = \mathrm{f}(x)$. The graph has asymptotes $y = a + bx$ and $x = 0$, intersects the x-axis at the point $(c, 0)$ and has a turning point (d, e).

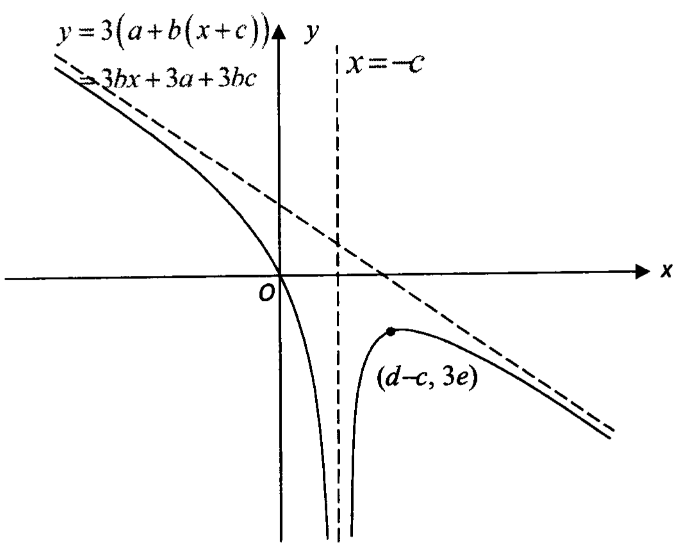


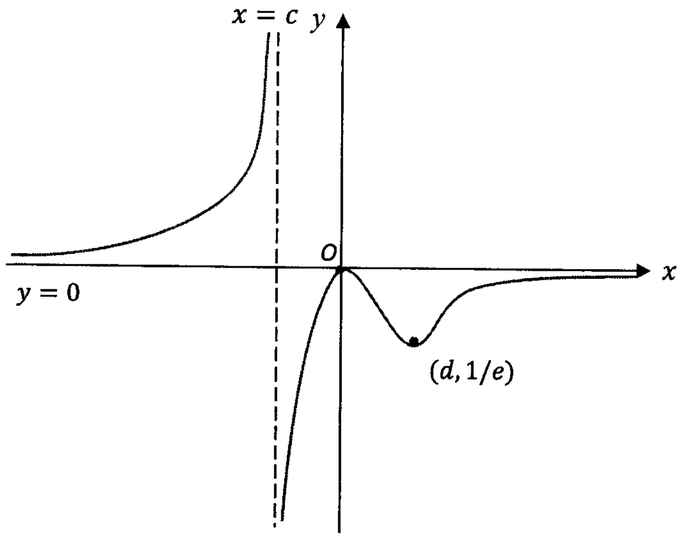
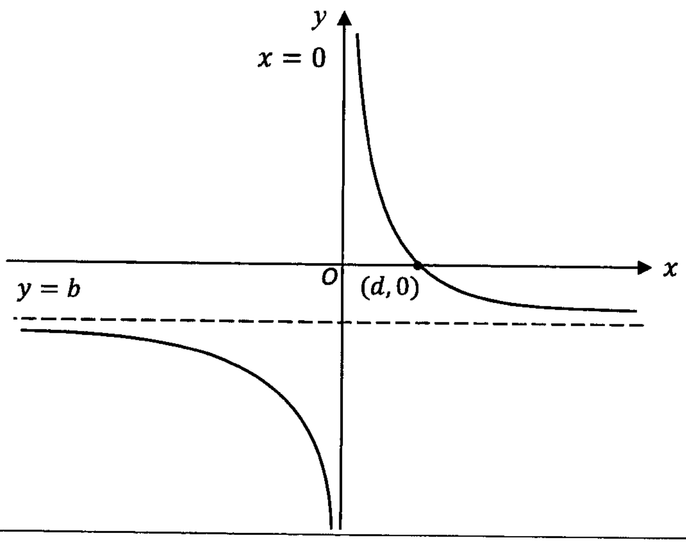
Sketch the following graphs on separate diagrams, labelling the equations of any asymptotes and the coordinates of any points where the graphs cross the axes and of any turning points.

(a) $y = 3\mathrm{f}(x + c)$, [2]

(b) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(c) $y = \mathrm{f}'(x)$. [2]

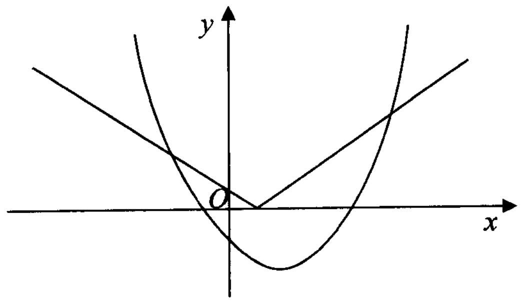
6 (a)		You need to ensure the curve is approaching the asymptote but not parallel to it. The curve should start and end near the edge of axis or asymptotes. It is crucial that c is a negative constant.

(b)		You need to label all the essential features of the sketch: asymptotes, axial intercepts, turning points.
(c)		You need to label all the essential features of the sketch: asymptotes, axial intercepts, turning points.

7 **(a)** On the same diagram, sketch the graphs $y=|2x-1|$ and $y=x^2-4x-2$. [2]

(b) Hence find the set of values of x for which $|2x-1|<x^2-4x-2$. [2]

(c) Without using a calculator, find the range of values of k for which $|2x-1|=x^2-4x-2+k$ has only positive real roots. [4]

7 (a)		You need to zoom out or adjust the window settings to observe the intersection between both graphs in the first quadrant. You have to draw the quadratic curve such that its minimum point is in the 4th quadrant, and that the ends **don't** curve inwards.
(b)	From GC, graphs intersect at $x=-1$ and $x=6.16$ (3sf) Thus $\{x\in\mathbb{R}:\ x<-1 \text{ or } x>6.16\,(3\text{ sf})\}$	You should obtain the intersection points between the 2 graphs using GC, then obtain the inequalities by using the graph above.
(c)	Minimum value of k (graphs have the same y-intercept): $\lvert 2(0)-1\rvert=0^2-4(0)-2+k$ $k=3$ Maximum value of k ($y=2x-1$ intersects $y=x^2-4x-2+k$): $2x-1=x^2-4x-2+k$ $x^2-6x+(k-1)=0$ Have real roots: $\Delta=(-6)^2-4(1)(k-1)\ge 0$ $36-4k+4\ge 0$ $4k\le 40$ $k\le 10$ Hence $3<k\le 10$.	You should note that for $\lvert 2x-1\rvert=x^2-4x-2+k$ to have positive real roots, $y=\lvert 2x-1\rvert$ and $y=x^2-4x-2+k$ must intersect in the first quadrant. The smallest value of k occurs when both graphs intersect at $x=0$ (y-intercept), note that this value of $k=3$ has to be excluded from the final answer as $x=0$ is not a positive real root. The largest value of k can be found by solving for the range of values of k for which $y=2x-1$ (the 'right' side of the modulus graph) and $y=x^2-4x-2+k$ intersect at 1 or 2 points, by using discriminant.

8 **(a)** Find the exact value of $\int_0^{\frac{1}{4}\pi} e^x \cos 2x \, dx$. [4]

(b) **(i)** Show that $\frac{d}{dx}\ln(\tan x) = \frac{k}{\sin 2x}$, where k is a constant to be determined. [2]

(ii) Hence find $\int \text{cosec}2x[\ln(\tan x) + \cos 2x]\, dx$. [2]

8 (a)	$\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx = \left[\cos 2x(e^x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} e^x(-2\sin 2x)\, dx$ $= [0-1] + 2\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx$ $= -1 + 2\left\{\left[\sin 2x(e^x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (e^x)(2\cos 2x)\, dx\right\}$ $= -1 + 2e^{\frac{\pi}{4}} - 4\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx$ $5\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx = 2e^{\frac{\pi}{4}} - 1$ $\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx = \frac{1}{5}\left(2e^{\frac{\pi}{4}} - 1\right)$	Follow LIATE to make working less tedious. Remember to multiply 2 to both terms when doing the 2nd integration by parts. There should be no + C for definite integrals.
(b) (i)	$\frac{d}{dx}\ln(\tan x) = \frac{1}{\tan x}\sec^2 x$ $= \frac{\cos x}{\sin x}\frac{1}{\cos^2 x}$ $= \frac{1}{\sin x \cos x}$ $= \frac{2}{2\sin x \cos x}$ $= \frac{2}{\sin 2x}$	Differentiate directly, keeping chain rule in mind. Since this is a "show" question, clear workings are necessary. Do not skip steps.
(b) (ii)	$\int \text{cosec}2x[\ln(\tan x) + \cos 2x]\, dx$ $= \int \frac{1}{2}\frac{2}{\sin 2x}\ln(\tan x) + \cot 2x \, dx$ $= \frac{1}{2}\frac{[\ln(\tan x)]^2}{2} + \frac{\ln\lvert\sin 2x\rvert}{2} + C$ $= \frac{1}{4}[\ln(\tan x)]^2 + \frac{1}{2}\ln\lvert\sin 2x\rvert + C$	$\ln(\tan x)$ is not in the denominator, so there should not be another ln after integration. 1st integration is of the form $\int f'(x)[f(x)]^n dx$. Remember to put modulus sign for the term inside ln, even if it does not appear in MF27.

9 Do not use a calculator in answering this question.

(a) The complex numbers z and w satisfy the following equations.

$$\mathrm{i}w + z = 3\mathrm{i}$$
$$2w^* - 3z = 12 - \mathrm{i}$$

Find z and w, giving your answers in the form $x + \mathrm{i}y$ where x and y are real numbers. [4]

(b) It is given that $\mathrm{f}(x) = ax^4 + x^3 + bx^2 - 9x + 35$, where a and b are real numbers, and $\mathrm{f}(x) = 0$ has no repeated roots. The graph of $y = \mathrm{f}(x)$ intersects the x-axis exactly once.

(i) Explain why $a = 0$. [2]

One of the roots of $\mathrm{f}(x) = 0$ is $1 + 2\mathrm{i}$.

(ii) Find the other roots and the value of b. [3]

9 (a)	$\mathrm{i}w + z = 3\mathrm{i}$ $z = 3\mathrm{i} - \mathrm{i}w$ Substitute z, $2w^* - 3(3\mathrm{i} - \mathrm{i}w) = 12 - \mathrm{i}$ $2w^* - 9\mathrm{i} + 3\mathrm{i}w = 12 - \mathrm{i}$ Let $w = a + b\mathrm{i}$, $2(a - b\mathrm{i}) + 3\mathrm{i}(a + b\mathrm{i}) = 12 + 8\mathrm{i}$ $(2a - 3b) + (3a - 2b)\mathrm{i} = 12 + 8\mathrm{i}$ Comparing real and imaginary coefficients Real: $2a - 3b = 12 \quad \cdots(1)$ Imaginary: $3a - 2b = 8 \quad \cdots(2)$ $3(1) - 2(2): -9b + 4b = 3(12) - 2(8)$ $-5b = 20$ $b = -4$ $2a - 3(-4) = 12$ $a = 0$ $\therefore w = -4\mathrm{i}$ $z = 3\mathrm{i} - \mathrm{i}(-4\mathrm{i})$ $\therefore z = -4 + 3\mathrm{i}$	Something like $z^* = -3\mathrm{i} + \mathrm{i}w$ will not make sense since w can have both real and imaginary terms inside. The question forbids the use of a calculator, so all working must be shown clearly when solving the simultaneous questions. Otherwise, both method mark and answer mark will be lost.
(b) (i)	As $y = \mathrm{f}(x)$ intersects the x-axis exactly once, so there can only be one real root. Since $\mathrm{f}(x)$ has no repeated roots so we can conclude that $\mathrm{f}(x) = 0$ must have **a unique real root**. As **all the coefficients of $\mathrm{f}(x)$ are real**, by conjugate root theorem, all non-real roots come in conjugate pairs. Therefore $\mathrm{f}(x) = 0$ must have an odd number of roots, hence an **odd degree polynomial** and hence $a = 0$.	Saying there is only 1 real root is insufficient as it may be repeated. Non-real roots in conjugate pairs must be clearly stated.
(b) (ii)	Since all coefficients of $\mathrm{f}(x)$ are real, by conjugate root theorem $1 - 2\mathrm{i}$ is another root of $\mathrm{f}(x)$. $(x - 1 - 2\mathrm{i})(x - 1 + 2\mathrm{i})$ $= (x-1)^2 - (2i)^2$ $= (x^2 - 2x + 1) - (-4)$ $= x^2 - 2x + 5$	"All coefficients real" must be clearly stated. Remove the x^4 term immediately to make working less tedious.

	By observation $x^3+bx^2-9x+35=(x^2-2x+5)(x+7)$ Comparing coefficients of x^2: $b=7-2=5$ Real root: $x=-7$ The other 2 roots are $x=1-2\text{i},-7$ and $b=5$	State all roots clearly. Note that $(x+7)$ is a factor not a root.

10 Planes p and q are perpendicular. Plane p has equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})=6$. Plane q contains the line l with equation $\mathbf{r}=\mathbf{j}-12\mathbf{k}+\lambda(4\mathbf{i}+\mathbf{j}+10\mathbf{k})$ where λ is a parameter. The point A has coordinates $(0,1,-12)$.

(a) Find a cartesian equation of q. [2]

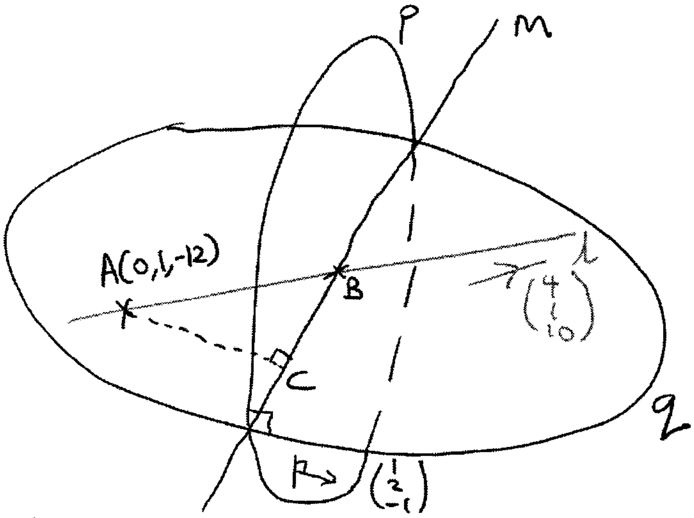
(b) Find a vector equation of the line m, where p and q meet. [2]

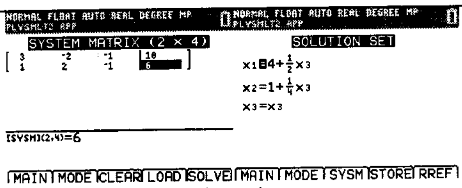
l passes through p at point B. Point C is the foot of the perpendicular of A to p.

(c) Find the position vector of B. [2]

(d) Find the position vector of C. [2]

(e) Find the exact area of triangle ABC. [2]

10 (a)	 Since planes p and q are perpendicular, to find normal of q, $\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}4\\1\\10\end{pmatrix}=\begin{pmatrix}20-(-1)\\-(10-(-4))\\1-8\end{pmatrix}=\begin{pmatrix}21\\-14\\-7\end{pmatrix}=7\begin{pmatrix}3\\-2\\-1\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}3\\-2\\-1\end{pmatrix}=\begin{pmatrix}0\\1\\-12\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\-1\end{pmatrix}=10$ $3x-2y-z=10$	Draw a diagram to help you visualise the objects. For a plane, the most important vector is the normal vector, i.e., a vector perpendicular to the plane. If it can be identified from the diagram, use it. Else, find 2 vectors parallel to the plane, then cross them to get a normal vector. Answer must be simplified.
(b)	a cartesian equation of q is $3x-2y-z=10$ a cartesian equation of p is $x+2y-z=6$	The solution can be found by using the GC to solve the Cartesian equations simultaneously.

	 GC: $\mathbf{r}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0.5\\0.25\\1\end{pmatrix}, \mu\in\mathbb{R}$	Remember to state the range of the parameter: $\mu\in\mathbb{R}$
(c)	<u>Method 1</u>: B is the intersection of l and p $\left[\begin{pmatrix}0\\1\\-12\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\10\end{pmatrix}\right]\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=6$ $14-4\lambda=6$ $\lambda=2$ $\overrightarrow{OB}=\begin{pmatrix}0\\1\\-12\end{pmatrix}+2\begin{pmatrix}4\\1\\10\end{pmatrix}=\begin{pmatrix}8\\3\\8\end{pmatrix}$ <u>Method 2</u>: B is the intersection of l and m $\begin{pmatrix}0\\1\\-12\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\10\end{pmatrix}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\4\end{pmatrix}$ x: $4\lambda-2\mu=4$ y: $\lambda-\mu=0$ $\lambda=2, \mu=2$ $\overrightarrow{OB}=\begin{pmatrix}0\\1\\-12\end{pmatrix}+2\begin{pmatrix}4\\1\\10\end{pmatrix}=\begin{pmatrix}8\\3\\8\end{pmatrix}$	B is the intersection of l and p. So, B satisfies l's and p's equations simultaneously. $\overrightarrow{OB}=\begin{pmatrix}0\\1\\-12\end{pmatrix}+\lambda\begin{pmatrix}4\\1\\10\end{pmatrix}$ $\overrightarrow{OB}\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=6$
(d)	<u>Method 1</u>: C is the foot of perpendicular of A to p Let $\overrightarrow{OC}=\begin{pmatrix}0\\1\\-12\end{pmatrix}+s\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\left[\begin{pmatrix}0\\1\\-12\end{pmatrix}+s\begin{pmatrix}1\\2\\-1\end{pmatrix}\right]\bullet\begin{pmatrix}1\\2\\-1\end{pmatrix}=6$ $14+6s=6$ $s=-\frac{4}{3}$ $\overrightarrow{OC}=\begin{pmatrix}0\\1\\-12\end{pmatrix}-\frac{4}{3}\begin{pmatrix}1\\2\\-1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-4\\-5\\-32\end{pmatrix}$	C is the foot of perpendicular of A to p. 1st find an equation of the line passing through A and C. Since AC is perpendicular to p, we can use the p's normal vector as the direction vector of the line. C will now be the intersection of l_{AC} and p. So, B satisfies l_{AC}'s and p's equations simultaneously.

	<u>Method 2</u>: C is the foot of perpendicular of A to m $\overrightarrow{OC}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\4\end{pmatrix}$ $\overrightarrow{AC}=\begin{pmatrix}4\\1\\0\end{pmatrix}+\mu\begin{pmatrix}2\\1\\4\end{pmatrix}-\begin{pmatrix}0\\1\\-12\end{pmatrix}=\begin{pmatrix}4\\0\\12\end{pmatrix}+\mu\begin{pmatrix}2\\1\\4\end{pmatrix}$ Since $\overrightarrow{AC}$ is perpendicular to m, $\left[\begin{pmatrix}4\\0\\12\end{pmatrix}+\mu\begin{pmatrix}2\\1\\4\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=0$ $56+21\mu=0$ $\mu=-\frac{8}{3}$ $\overrightarrow{OC}=\begin{pmatrix}4\\1\\0\end{pmatrix}-\frac{8}{3}\begin{pmatrix}2\\1\\4\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-4\\-5\\-32\end{pmatrix}$	
(e)	<u>Method 1</u> $\overrightarrow{AB}=2\begin{pmatrix}4\\1\\10\end{pmatrix}$, from part **(c)**; $\overrightarrow{AC}=-\frac{4}{3}\begin{pmatrix}1\\2\\-1\end{pmatrix}$, from part **(d)** Area of ABC $=\frac{1}{2}\left\|2\begin{pmatrix}4\\1\\10\end{pmatrix}\times\left(-\frac{4}{3}\right)\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\|$ $=\frac{1}{2}(2)\left(\frac{4}{3}\right)\left\|-7\begin{pmatrix}3\\-2\\-1\end{pmatrix}\right\|$, cross product done in part **(a)** $=\frac{28}{3}\sqrt{14}$ units^2 <u>Method 2</u> $\overrightarrow{AC}=-\frac{4}{3}\begin{pmatrix}1\\2\\-1\end{pmatrix}$; $\overrightarrow{BC}=-\frac{14}{3}\begin{pmatrix}2\\1\\4\end{pmatrix}$ Area of ABC $=\frac{1}{2}\left\|\left(-\frac{4}{3}\right)\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\|\left\|\left(-\frac{14}{3}\right)\begin{pmatrix}2\\1\\4\end{pmatrix}\right\|$, since $\angle ACB=90°$ $=\frac{1}{2}\left(\frac{4}{3}\right)\left(\frac{14}{3}\right)\sqrt{6}\sqrt{21}=\frac{28}{3}\sqrt{14}$ units^2	<u>Method 1</u> The area can be found using any 2 vectors from the triangle. E.g.: $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$ Simplifying can be easier if we prioritise factorisation before any operation.

11 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{1}{x^2+2x}, \quad x \in \mathbb{R},\ x<-1, x \neq -2.$$

(a) Show that f has an inverse. [1]

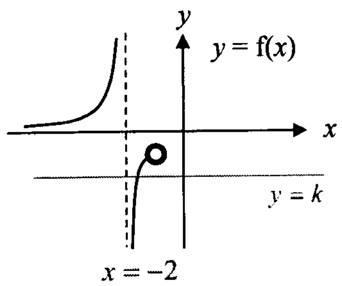
(b) Define f^{-1} in similar form. [4]

(c) The function g is such that $\mathrm{fg}(x) = x^2+1$. Find $\mathrm{g}(x)$. [2]

The function h is defined by

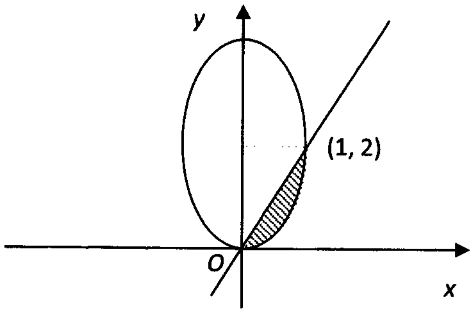
$$\mathrm{h}: x \mapsto \frac{1}{x^2}, \quad x \in \mathbb{R},\ x \neq 0.$$

(d) Only one of the composite functions fh and hf exists. Give a definition (including the domain) of the composite that exists, and explain why the other composite does not exist. [3]

11 (a)	 Since any horizontal line $y=k,\ k\in\mathbb{R}$ cuts the graph of f at most once, the function f is one-one. Hence, f has an inverse.	Graph must be drawn according to the domain. Asymptotes and the line $y=k$ must be drawn. Important points: "$y=k,\ k\in\mathbb{R}$", "cuts graph at most once"
(b)	Let $y=\dfrac{1}{x^2+2x}$, for $x<-1$ $x^2+2x=\dfrac{1}{y}$ $(x+1)^2-1=\dfrac{1}{y}$ $(x+1)^2=\dfrac{1}{y}+1$ $x=-1+\sqrt{\dfrac{1}{y}+1}$ or $x=-1-\sqrt{\dfrac{1}{y}+1}$ (rejected $\because x<-1$) $\therefore\ \mathrm{f}^{-1}: x\mapsto -1-\sqrt{\dfrac{1}{x}+1},\ x\in\mathbb{R},\ x<-1 \text{ or } x>0$ Alternatively, $x^2+2x-\dfrac{1}{y}=0$ $x=\dfrac{-2\pm\sqrt{(2)^2-4(1)\left(-\frac{1}{y}\right)}}{2}$	Let $y=\mathrm{f}(x)$. Make x the subject. Then swap x & y. There must be a $\pm$ after removing the "square". Function is wanted "in similar form", thus f^{-1} must be written in the same way as f in the question. $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}$. As the question did not use set notation for the definition of f, you do not need to do so for f^{-1}.
(c)	$\mathrm{fg}(x)=x^2+1$ $\mathrm{f}^{-1}\mathrm{fg}(x)=\mathrm{f}^{-1}(x^2+1)$ $\mathrm{g}(x)=-1-\sqrt{\dfrac{1}{x^2+1}+1}$	$\mathrm{f}^{-1}\mathrm{f}(x)=x$ for all x. Thus, a neat way to "get rid of f" is to use f^{-1}. Similarly, $\mathrm{ff}^{-1}(x)=x$
(d)	$D_{\mathrm{f}}=(-\infty,-1)\setminus\{-2\}=(-\infty,-2)\cup(-2,-1)$ $R_h=(0,\infty)$ $R_h\not\subseteq D_{\mathrm{f}}$, thus fh does not exist.	Show either $R_h\subseteq D_{\mathrm{f}}$ or $R_h\not\subseteq D_{\mathrm{f}}$ by writing down both sets correctly.

	$R_f = \mathbb{R}\setminus[-1,0] = (-\infty,-1)\cup(0,\infty)$ $D_h = (-\infty,0)\cup(0,\infty)$ $R_f \subseteq D_h$, thus hf exists. $\text{hf}(x) = (x^2+2x)^2 = x^4+4x^3+4x^2$ $D_{hf} = D_f = (-\infty,-1)\setminus\{-2\}$ [You can also define in the same way as f is defined, i.e., $\text{hf}(x) = x^4+4x^3+4x^2,\ x\in\mathbb{R},\ x<-1,\ x\neq -2$]	R_h & D_f are both sets, so they must either be written using set notation or interval notation. We can't use "or" for set notation, e.g., $D_h = (-\infty,0)\text{or}(0,\infty)$. We must use the set notation equivalent "$\cup$".

12 **(a)** Use the substitution $x=\sin\theta$ to show that $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$. [4]

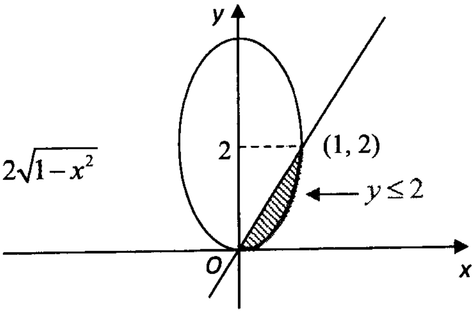


The diagram shows the region R bounded by the ellipse with equation $4x^2+(y-2)^2=4$ and the line $y=2x$. The ellipse and the line intersect at the origin and the point (1, 2).

(b) Find the area of R, giving your answer in terms of π. [4]

(c) Find the volume of solid generated when R is rotated 2π radians about the x-axis. [2]

12 (a)	$x=\sin\theta$ $\frac{dx}{d\theta}=\cos\theta$ When $x=0, \theta=0$ $x=1, \theta=\frac{\pi}{2}$ $\int_0^1\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$ $=\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$ $=\int_0^{\frac{\pi}{2}}\frac{1}{2}+\frac{1}{2}\cos2\theta\,d\theta$ $=\left[\frac{1}{2}\theta+\frac{1}{4}\sin2\theta\right]_0^{\frac{\pi}{2}}$ $=\left[\frac{1}{2}(\frac{\pi}{2})+\frac{1}{4}\sin\pi\right]-\left[\frac{1}{2}(0)+\frac{1}{4}\sin0\right]$	Remember to change the limits when doing integration by substitution. To get rid of the square root, use $1-\sin^2\theta=\cos^2\theta$ (trigo identity) Use MF27 (double angle formula) $\cos2\theta=2\cos^2\theta-1$ to integrate $\cos^2\theta$

	$=\frac{\pi}{4}$	
(b)	$4x^2+(y-2)^2=4$ $(y-2)^2=4-4x^2$ $y-2=\pm\sqrt{4-4x^2}$ $y=2\pm2\sqrt{1-x^2}$ Since $y\le2$, $y=2-2\sqrt{1-x^2}$  Area of R $=\frac{1}{2}(2)(1)-\int_0^1 y\,dx$ $=1-\int_0^1 2-2\sqrt{1-x^2}\,dx$ $=1-\int_0^1 2\,dx+2\int_0^1\sqrt{1-x^2}\,dx$ $=1-[2x]_0^1+2\left(\frac{\pi}{4}\right)$ $=\frac{\pi}{2}-1$ Alternatively Area of R $=\int_0^1 2x-\left(2-2\sqrt{1-x^2}\right)dx$ "upper curve" – 'lower curve"	Observe from the graph that the y values on the curve that we are using to find the area under the curve are less than or equal to 2 Thus, we choose $y=2-2\sqrt{1-x^2}$
(c)	Volume $=\frac{1}{3}\pi(2)^2(1)-\pi\int_0^1\left((2-2\sqrt{1-x^2}\right)^2dx$ $=2.98$ (3 sf) Alternatively Volume $=\pi\int_0^1(2x)^2\,dx-\pi\int_0^1\left((2-2\sqrt{1-x^2}\right)^2dx$	Required volume = Volume of cone – Volume generated by the region under the curve Volume of cone can be found by using $\frac{1}{3}\pi r^2h$ or $\pi\int_0^1(2x)^2\,dx$

13 Around November 2019, there was an outbreak of COVID-19 in Wuhan, China, caused by the coronavirus SARS-CoV-2 virus, which subsequently spread to the rest of the world. On 23 January 2020, there was 1 confirmed case in Singapore. 70 days later, there were 1 049 confirmed cases.

A student is interested to study the spread of the virus in Singapore and uses the model $\frac{dN}{dt} = rN$, where r is a positive constant, N denotes the total number of confirmed cases, and t is the number of days after 23 January 2020.

(a) Solve the differential equation to express N in terms of t. [4]

(b) Use the model to estimate the number of days it will take to reach 1 000 000 cases. [1]

To contain the spread of the virus, Singapore implemented strict circuit breaker lockdown measures starting on 7 April 2020, when the number of confirmed cases stood at 1,481. By the time the measures were lifted 55 days later, the number of confirmed cases had surged to 35 292.

To study the spread of the virus during this period, the student uses a second model $\frac{dN}{du} = sN$, where s is a positive constant, and u is the number of days after 7 April 2020.

(c) Find the value of s. [2]

(d) By comparing the values of r and s, comment on whether the lockdown measures were effective in containing the spread of the virus. [1]

(e) Give a reason why neither model can estimate the actual number of confirmed cases accurately. [1]

On 30 December 2020, Singapore became the first country in Asia to start its COVID-19 vaccination campaign. The student now uses a third model $\frac{dN}{dv} = aN\left(1 - \frac{N}{b}\right)$, where a and b are positive constants, and v is the number of days after 30 December 2020.

(f) Find the maximum rate of change of N in terms of a and b. [3]

13 (a)	$\frac{dN}{dt} = rN$ $\int \frac{1}{N}\,dN = \int r\,dt$ $\ln\lvert N\rvert = rt + C$ $N = \pm e^C e^{rt} = Ae^{rt}$ $t = 0, N = 1:\ A = 1$ $t = 70, N = 1049:\ 1049 = e^{70r}$ $r = \frac{\ln 1049}{70} = 0.099366\ (5\text{ sf})$ $= 0.0994\ (3\text{ sf})$ $N = e^{0.0994t}$	Modulus / stating that $N > 0$ is required. Remove the modulus before substituting initial conditions.
(b)	$1000000 = e^{0.099366t}$ $t = \frac{\ln 1000000}{0.099366}$ $= 139.0366$ Thus 140 days.	Apply ln to solve for t.

(c)	Following part **(a)**, $N = Be^{su}$ $t=0, N=1481:\ B=1481$ $t=55, N=35292:\ 35292=1481e^{55s}$ $s=\frac{1}{55}\ln\frac{35292}{1481}$ $=0.0577\ (3\text{ sf})$	Observing that the expression is similar to that of part (a). Thus, a similar equation can be obtained without solving the DE again.
(d)	Since $s<r$, the lockdown measures were effective in containing the spread of the virus.	An attempt to compare r and s is required by the question.
(e)	The model predicts that the number of confirmed cases will increase indefinitely, this is unlikely as there is a fixed population size in Singapore. Alternatively: The model assumes that r & s are constant, this is unlikely as they will change due to immunity/vaccinations/safe management measures.	Majority talked about unrecorded / undetected cases but note that the context of the question considered only confirmed cases for the model, hence unrecorded / undetected cases are irrelevant. Explain clearly which part of the model is inaccurate (behaviour in the long run / the use of constants r and s) AND specific real-world reasons.
(f)	$\frac{dN}{dv}=aN\left(1-\frac{N}{b}\right)$ $=a\left(N-\frac{1}{b}N^2\right)=-\frac{a}{b}\left(N^2-bN\right)$ $=-\frac{a}{b}\left[\left(N-\frac{b}{2}\right)-\left(\frac{b}{2}\right)^2\right]$ Hence $\frac{dN}{dv}$ is largest when $N=\frac{b}{2}$. Maximum growth rate is $\frac{ab}{4}$	The equation involves $\frac{dN}{dv}$ & N. You are tasked to find the maximum $\frac{dN}{dv}$. This is similar to $y=ax\left(1-\frac{x}{b}\right)$, and tasked to find maximum y. You can thus use any method to find the maximum point of a quadratic graph, such as completing the square. A common error is assuming that this is a typical maxima / minima question. Note that solving for $\frac{dN}{dv}=0$ is not helpful as that helps us find the value of N to maximise N, and not $\frac{dN}{dv}$. Alternatively, you can solve for $\frac{d^2N}{dv^2}=0$ (implicit differentiation is required) to obtain max $\frac{dN}{dv}$.

Solutions for 2025 JC2 H2 Mathematics Preliminary Examination Paper 2

1 The acute angle between two non-zero vectors **a** and **b** is α radians, and the vectors **b** and $2\mathbf{a}-\mathbf{b}$ are perpendicular. Find the value of α if $|\mathbf{b}|=\sqrt{2}|\mathbf{a}|$. [3]

1	**b** is perpendicular to $2\mathbf{a}-\mathbf{b}$ $\mathbf{b}\cdot(2\mathbf{a}-\mathbf{b})=0$ $2\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}=0$ $2|\mathbf{a}||\mathbf{b}|\cos\alpha-|\mathbf{b}|^2=0$ $|\mathbf{b}|=\sqrt{2}|\mathbf{a}|\Rightarrow 2\left(\frac{|\mathbf{b}|}{\sqrt{2}}\right)|\mathbf{b}|\cos\alpha-|\mathbf{b}|^2=0$ $\sqrt{2}\cos\alpha=1$ $\cos\alpha=\frac{1}{\sqrt{2}}$ $\alpha=\frac{\pi}{4}$	Vector dot product is like your scalar (real number) multiplication. Just expand them like how you would do for scalar multiplication. Similar for cross product, although order of multiplication is important for cross product: $\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$. $\mathbf{ab}, \mathbf{b}^2$ etc. are not acceptable notations as there are 2 different types of vector multiplication: dot & cross.

2 **Do not use a calculator in answering this question.**

It is given that $z=\sqrt{3}-\mathrm{i}$ and $w=2\mathrm{i}z$.

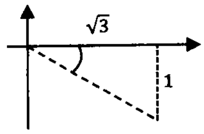
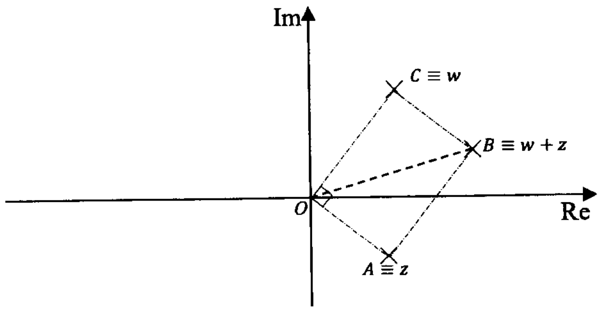
(a) Find the modulus and argument of z. [2]

(b) Deduce the modulus and argument of w. [2]

O, A, B and C represent the complex numbers 0, z, $z+w$ and w respectively.

(c) Illustrate these four points on a single diagram. [2]

(d) Identify the shape of the quadrilateral $OABC$ and state its area. [2]

2 (a)	$\lvert z\rvert=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2$ $\arg(z)=-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$  	Sketch a diagram to see which quadrant the point belongs to. Find the basic angle next, then use the basic angle and the quadrant location to find the argument.
(b)	Using the geometrical relationship between z & $\mathrm{i}z$: $\lvert w\rvert=2\lvert \mathrm{i}z\rvert=2\lvert z\rvert=2\times 2=4$ $\arg(w)=\arg(z)+\frac{\pi}{2}=-\frac{\pi}{6}+\frac{\pi}{2}=\frac{\pi}{3}$	The question asked to deduce so you cannot find w and do like (a) all over again. $\times\mathrm{i}$: Rotation doesn't change the modulus but will add $\frac{\pi}{2}$ to the argument. $\times 2$: This scales the modulus by a factor of 2 but will not change the argument.
(c)	 	Since B represents $w+z$, the four points illustrate the "parallelogram law of addition". From parts (a)&(b), A is in the 4th quadrant, C is in the 1st quadrant, & $OC=2OA$. $\angle AOC=\frac{\pi}{2}$
(d)	$OABC$ is a **rectangle**. Area of rectangle = length × breadth = $2\times 4=8$.	Since O, A, B, C forms a parallelogram and $\angle AOC=\frac{\pi}{2}$, thus OABC is a rectangle.

3 A closed cylindrical tank with radius r m and height h m is to be constructed to hold 1000 m^3 of water. The material for the base and lid costs \$50 per m^2, while the material for the curved surface costs \$30 per m^2.

(a) Show that the total cost C (in dollars) can be expressed as $C = 100\pi r^2 + \dfrac{60000}{r}$. [2]

(b) Use differentiation to find the exact radius of the cylinder for which C is minimum. State the minimum cost. [4]

(c) Sketch the graph showing the total cost as the radius of the tank varies. [2]

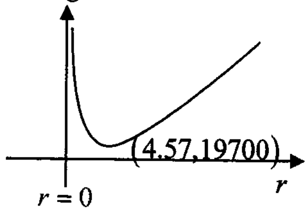
A change to one of the materials used results in the cylinder's volume doubling to 2000 m^3, while maintaining the same minimum cost and radius found in part **(b)**.

(d) State which material's cost has changed and its value. [1]

<table>
<tr><td>3
(a)</td><td>$\pi r^2 h = 1000$
$h = \dfrac{1000}{\pi r^2}$

$C = 50\left(2\pi r^2\right) + 30(2\pi r h)$
$= 100\pi r^2 + 60\pi r\left(\dfrac{1000}{\pi r^2}\right)$
$= 100\pi r^2 + \dfrac{60000}{r}$ (Shown)</td><td></td></tr>
<tr><td>(b)</td><td>$\dfrac{dC}{dr} = 200\pi r - \dfrac{60000}{r^2}$
$\dfrac{dC}{dr} = 0 \Rightarrow 200\pi r - \dfrac{60000}{r^2} = 0$
$200\pi r^3 - 60000 = 0$
$r = \sqrt[3]{\dfrac{300}{\pi}}$
$\dfrac{d^2C}{dr^2} = 200\pi + \dfrac{120000}{r^3} > 0$ since $r > 0$
$\therefore C$ is minimum.

Minimum cost is \$19 690.29</td><td>For minimum, set $\dfrac{dC}{dr} = 0$ not $\dfrac{d^2C}{dr^2} = 0$.

Unless question states that there is no need to prove max or min, you must!</td></tr>
<tr><td>(c)</td><td></td><td>From part (b), there is a minimum point at (4.57, 19700). Use this as a guide to adjust the window setting to see the graph.

Since r is radius, and C is cost, you should only sketch the graph in the 1st quadrant, where both are positive.</td></tr>
</table>

(d)	The material for the curved surface has changed to \$15 per m^2.	To have the same cost and radius, the equation $C=100\pi r^2+\dfrac{60000}{r}$ must be the same. Originally, $C=100\pi r^2+30(2\pi r)\left(\dfrac{1000}{\pi r^2}\right)$. So, to have the same equation, we need $C=100\pi r^2+15(2\pi r)\left(\dfrac{2000}{\pi r^2}\right)$

4 **(a)** A sequence is such that $u_1 = a$ and $u_{n+1} = bu_n$, for $n \geq 1$. Both a and b are constants.

(i) Explain why the sequence is a geometric progression. [1]

(ii) Given that $u_2 = 6$, $u_1 + u_2 + u_3 = 26$, and the sum to infinity exists, find the values of a and b. [4]

(b) It is given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

(i) Find $\sum_{r=1}^{n} r(r+2)$ in terms of n. Leave your answer in factorised form. [2]

(ii) Hence find $\sum_{r=2}^{n} (r+1)(r+3)$ in terms of n. You do not need to leave your answer in factorised form. [3]

4 (a) (i)	$\frac{u_{n+1}}{u_n} = b$, a constant. Hence u_n is a geometric sequence.	To show a GP, you need to show either $\frac{u_{n+1}}{u_n}$ or $\frac{u_n}{u_{n-1}}$ is a constant.
(ii)	$u_2 = ab = 6$ $a = \frac{6}{b} \quad \cdots(1)$ $u_1 + u_2 + u_3 = a + ab + ab^2 = 26 \quad \cdots(2)$ Substitute (1) into (2) $\frac{6}{b}\left(1 + b + b^2\right) = 26$ $3\left(1 + b + b^2\right) = 13b$ $3b^2 - 10b + 3 = 0$ GC: $b = \frac{1}{3}$ or $b = 3$ Since S_∞ exists, $\lvert b \rvert < 1$, thus $b = \frac{1}{3}$. $a\left(\frac{1}{3}\right) = 6 \Rightarrow a = 18$	Use both equations to obtain an equation in **ONE** variable only. The <u>only</u> condition for sum to infinity to exists is $\lvert r \rvert < 1$.
(b) (i)	$\sum_{r=1}^{n} r(r+2)$ $= \sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$ $= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{n}{2}(n+1)$ $= \frac{1}{6}n(n+1)[(2n+1)+6]$ $= \frac{1}{6}n(n+1)(2n+7)$	$\sum_{r=1}^{n} r$ is an AP, short for $1+2+3+\cdots+n$. It is thus expected to use AP $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$ or $S_n = \frac{n}{2}(a+l)$ to obtain $\sum_{r=1}^{n} r = \frac{n}{2}(1+n)$. Factorisation is an answer requirement.

<table>
<tr><td>(ii)</td><td>
<u>Method 1</u>
$\sum_{r=2}^{n}(r+1)(r+3)$
$=3\cdot5+4\cdot6+\cdots+(n+1)(n+3)$
$=\sum_{r=3}^{n+1}r(r+2)$
$=\sum_{r=1}^{n+1}r(r+2)-\sum_{r=1}^{2}r(r+2)$
$=\frac{1}{6}(n+1)(n+2)(2(n+1)+7)-\frac{1}{6}(2)(3)(11)$
$=\frac{1}{6}(n+1)(n+2)(2n+9)-11$

Or let $r=k-1$.
$\sum_{k-1=2}^{k-1=n}(k-1+1)(k-1+3)$
$=\sum_{r=3}^{n+1}k(k+2)$

<u>Method 2</u>
$\sum_{r=2}^{n}(r+1)(r+3)$
$=\sum_{r=2}^{n}\left(r^2+4r+3\right)$
$=\sum_{r=1}^{n}\left(r^2+4r+3\right)-(1+4+3)$
$=\sum_{r=1}^{n}\left(r^2+2r\right)+\sum_{r=1}^{n}(2r+3)-8$
$=\frac{1}{6}n(n+1)(2n+7)+2\frac{n}{2}(n+1)+3n-8$
$=\frac{1}{6}n(n+1)(2n+7)+n(n+1)+3n-8$
</td><td>
The command word here is "hence", thus you must use your <u>final answer in (b)(i)</u>, to obtain the answer.

Method 1a:
If tasked to find another sigma notation via hence, that usually means the current sum can be written using the previous formula, thus we will unpack the sum, then pack it using the previous formula.

Method 1b:
You will need to figure out a substitution to covert the current sum formula, to previous sum formula. Remember to apply the same substitution to the first & last indices.

Method 2:
May not always work for all questions. You can expand and reformulate parts of the sum to use the previous part.
</td></tr>
</table>

5 It is given that $e^y = 1 + e^x$.

(a) Show that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$. [2]

(b) Hence find the first four non-zero terms of the Maclaurin series for y. [5]

(c) Hence find the Maclaurin series for $\frac{e^{2x}}{1+e^{2x}}$, up to and including the term in x^3. [3]

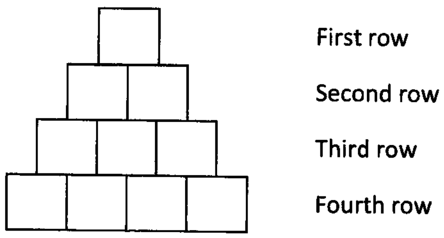
5 (a)	$e^y = 1 + e^x$ $e^y \frac{dy}{dx} = e^x = e^y - 1$ $e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = e^y \frac{dy}{dx}$ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$	There is no need to make y the subject. Perform implicit differentiation. $\ln\left(1+e^x\right) \neq \ln 1 + \ln e^x$ As this is a 'show' question, all working must be clearly presented. If using 'brute force' method, your solution must be presented like this: $\text{LHS} = \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$ $= \cdots$ $= \frac{dy}{dx}$ $= \text{RHS}$
(b)	$\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = \frac{d^2y}{dx^2}$ When $x = 0$, $y = \ln 2, \frac{dy}{dx} = \frac{1}{2}, \frac{d^2y}{dx^2} = \frac{1}{4}, \frac{d^3y}{dx^3} = 0$ For fourth non-zero term: $\frac{d^4y}{dx^4} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 2\left(\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = \frac{d^3y}{dx^3}$ When $x = 0$, $\frac{d^4y}{dx^4} = -\frac{1}{8}$ $y = \ln 2 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \ldots$ $= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots$	The 'hence' in this question forces you to immediately differentiate the previous result using implicit differentiation. After identifying the coefficient of x^3 to be 0, do one more implicit differentiation. Show the working of substituting into the MF27 formula.
(c)	$y = \ln\left(1+e^x\right)$ $\frac{e^x}{1+e^x} = \frac{dy}{dx} = \frac{1}{2} + \frac{1}{4}x - \frac{1}{48}x^3 + \cdots$	The 'hence' in this question implies you should differentiate the earlier result directly. Any form of working involving using the repeated differentiation formula again is unacceptable.

	$\dfrac{e^{2x}}{1+e^{2x}}=\dfrac{1}{2}+\dfrac{1}{4}(2x)-\dfrac{1}{48}(2x)^3+\cdots$ $=\dfrac{1}{2}+\dfrac{1}{2}x-\dfrac{1}{6}x^3+\cdots$	If students opt to replace x with $2x$ first, take note to differentiate the result and then divide by 2.

6 A children's game set consists of 1 red brick, 2 white bricks, 3 orange bricks and 4 blue bricks. All bricks are identical except for their colour.

(a) Find the number of ways to arrange the 10 bricks in a row if all the orange bricks are separated. [2]

A girl plans to arrange the 10 bricks in 4 rows as shown below.



Find the number of arrangements if there are

(b) no restrictions, [1]

(c) exactly two orange bricks and only one white brick in the same row. [3]

All the 10 bricks are now numbered from 1 to 10.

(d) In how many ways can the bricks be arranged in a circle so that the ones with the same colour are next to each other? [2]

6 (a)	No. of ways $= \dfrac{7!}{2!4!} \times {}^{8}C_{3}$ $= 5880$	In this **separation** question, we **choose 3 slots** out of the **8 slots** to insert the orange bricks. Then we arrange the rest of the 7 other bricks, considering the 1 red and 2 **identical** white and 4 **identical** blue bricks.
(b)	No. of ways $= \dfrac{10!}{2!3!4!} = 12600$	Since all the positions are unique, this question is the same as arrangement in a row.
(c)	Case 1: 2 orange and 1 white in 3rd row. No. of ways $= \dfrac{7!}{4!} \times \dfrac{3!}{2!} = 630$ Case 2: 2 orange, 1 white and 1 red in 4th row No. of ways $= \dfrac{4!}{2!} \times \dfrac{6!}{4!} = 360$ Case 3: 2 orange, 1 white and 1 blue in 4th row No. of ways $= \dfrac{4!}{2!} \times \dfrac{6!}{3!} = 1440$ Total no. of ways = 2430	Case 1: Arrange 7 bricks, considering the 1 red, 1 white, 1 orange brick and 4 **identical** blue bricks. Then arrange the 2 orange and 1 white in 3rd row. Case 2: Arrange 6 bricks, considering the 1 white, 1 orange brick and 4 **identical** blue bricks. Then arrange the 2 orange, 1 white and 1 red in 4th row. Case 3: Arrange 6 bricks, considering the 1 white, 1 orange brick, 1 red and 3 **identical** blue bricks. Then arrange the 2 orange, 1 white and 1 blue in 4th row.
(d)	No. of ways = $= (4-1)! \times 2! \times 3! \times 4!$ $= 1728$	There are 4 groups of bricks, each group containing the same-coloured bricks. Arranging them in a circle

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		gives you (4 – 1)!. Then arrange the bricks within each group as they are unique.

7 Customers at a shopping mall participate in a game involving 2 boxes of tokens, labelled A and B. Box A contains n black tokens and 4 gold tokens, while Box B contains 4 black tokens and 4 silver tokens. The tokens are identical in all aspects except for their colour.
In a game, a customer first tosses a fair coin. If a head is obtained, two tokens are drawn at random from Box B. Otherwise, one token is drawn from each box. A black token scores 1 point, a silver token scores 2 points, and a gold token scores 3 points. The total number of points is the customer's score X.

(a) Show that $\mathrm{P}(X=2)=\dfrac{3}{28}+\dfrac{n}{4(n+4)}$. [1]

(b) Given that $\mathrm{P}(X=2)=\dfrac{3}{14}$, determine the probability distribution of X. [4]

(c) Find $\mathrm{P}(X_1=4 \mid X_1+X_2 \geq 9)$, where X_1 and X_2 are independent observations of X. [3]

7(a)	$\mathrm{P}(X=2)=$ P(2 black from B)+P(1 black from each) $=\frac{1}{2}\left(\frac{4}{8}\right)\left(\frac{3}{7}\right)+\frac{1}{2}\left(\frac{n}{n+4}\right)\left(\frac{4}{8}\right)$ $=\frac{3}{28}+\frac{n}{4(n+4)}$ (shown)	You need to show the working clearly. Any redundant numbers that make no sense will result in a deduction of marks.
(b)	$\mathrm{P}(X=2)=\frac{3}{14} \Rightarrow \frac{3}{28}+\frac{n}{4(n+4)}=\frac{3}{14}$ $\Rightarrow \frac{n}{4(n+4)}=\frac{3}{28}$ $\Rightarrow 28n=12(n+4)$ $\Rightarrow n=3$ $\mathrm{P}(X=3)$ = P(1black,1silver from B)+P(black from A,silver from B) $=\frac{1}{2}\left(\frac{4}{8}\right)\left(\frac{4}{7}\right)\times 2+\frac{1}{2}\left(\frac{3}{7}\right)\left(\frac{4}{8}\right)$ $=\frac{11}{28}$ $\mathrm{P}(X=4)$ = P(2 silver from B)+P(gold from A, black from B) $=\frac{1}{2}\left(\frac{4}{8}\right)\left(\frac{3}{7}\right)+\frac{1}{2}\left(\frac{4}{7}\right)\left(\frac{4}{8}\right)$ $=\frac{1}{4}$ $\mathrm{P}(X=5)$ = P(1 gold from A, 1 silver from B) $=\frac{1}{2}\left(\frac{4}{7}\right)\left(\frac{4}{8}\right)$ $=\frac{1}{7}$	Find n using $\frac{3}{28}+\frac{n}{4(n+4)}=\frac{3}{14}$ For $\mathrm{P}(X=3)$ and $\mathrm{P}(X=4)$, consider 2 cases to find the probabilities. For $\mathrm{P}(X=5)$, consider the only case to find the probability.

x	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{3}{14}$	$\frac{11}{28}$	$\frac{1}{4}$	$\frac{1}{7}$

Draw and label the table correctly. Do not write redundant x values (with probability =0), such as 0, 1 or 6.

(c)

$$\mathrm{P}(X_1=4 \mid X_1+X_2\ge 9)$$

$$=\frac{\mathrm{P}\big((X_1=4)\cap(X_1+X_2\ge 9)\big)}{\mathrm{P}(X_1+X_2\ge 9)}$$

$$=\frac{\mathrm{P}(X_1=4,\,X_2=5)}{2\mathrm{P}(X_1=4,X_2=5)+\mathrm{P}(X_1=5,X_2=5)}$$

$$=\frac{\mathrm{P}(X_1=4)\mathrm{P}(X_2=5)}{2\mathrm{P}(X_1=4)\mathrm{P}(X_2=5)+\mathrm{P}(X_1=5)\mathrm{P}(X_2=5)}$$

$$=\frac{\left(\frac{1}{4}\right)\left(\frac{1}{7}\right)}{2\left(\frac{1}{4}\right)\left(\frac{1}{7}\right)+\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)}$$

$$=\frac{7}{18}$$

Conditional probability with numerator intersection of the 2 conditions resulting in finding $\mathrm{P}(X_1=4)\mathrm{P}(X_2=5)$.

Consider 3 cases for the denominator.
$\mathrm{P}(X_1=4)\mathrm{P}(X_2=5)+\mathrm{P}(X_1=5)\mathrm{P}(X_2=4)$
$+\mathrm{P}(X_1=5)\mathrm{P}(X_2=5)$

8 Andy is learning how to solve a Rubik's cube. The table shows his personal record time, y seconds, for solving a Rubik's cube, x days after he started.

x	5	10	15	20	25	30	35	40
y	220	150	100	70	48	35	25	18

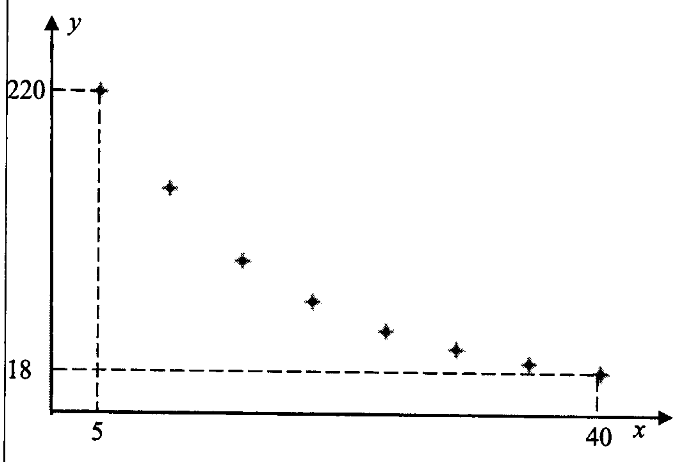
(a) Draw a scatter diagram for these values, labelling the axes. [1]

(b) Use your diagram and the context of the question to explain whether the relationship between y and x should be modelled by $y = a + bx$ or by $y = a + \frac{b}{x}$, where a and b are constants. [2]

(c) Find the product moment correlation coefficient and the equation for the chosen model in part **(b)**. [2]

(d) Use your equation to estimate his personal record time, 28 days after he started. Explain whether your estimate is reliable. [2]

(e) Without further calculations, explain whether the product moment correlation coefficient calculated in part **(c)** would be different if y was recorded in minutes instead. Re-write the equation for the chosen model in this case. You do not need to simplify your answer. [2]

8(a)		Draw the points such that there is a **fixed** horizontal distance between each point. The vertical distance between each point should be decreasing as x increases. Label the axes, and the min/max of both variables on the graph.
(b)	It should be modelled by $y = a + \frac{b}{x}$. The points on the scatter diagram shows that as x increases, y decreases at a decreasing rate, hence $y = a + \frac{b}{x}$ is appropriate. (OR the points on the scatter diagram shows a non-linear relationship, hence $y = a + bx$ is not appropriate). Also, a linear model would mean that the personal record time will continue to decrease linearly eventually reaching a time that is negative, which is not possible. Hence a model of the form $y = a + \frac{b}{x}$ is more appropriate.	It is important to state that 'y decreases at a decreasing rate' as opposed to just 'y decreases' as both models satisfy the trend of 'y decreases', whereas only the model $y = a + \frac{b}{x}$ exhibits the behaviour of 'y decreases at a **decreasing rate**'.
(c)	Using GC,	

	$r \approx 0.971$ $y = 4.0496 + 1165.6\left(\frac{1}{x}\right)$ (5 sf) $y = 4.05 + 1170\left(\frac{1}{x}\right)$ (3 sf)	
(d)	$y = 4.0496 + 1165.6\left(\frac{1}{28}\right)$ $= 45.7$ seconds (3 sf) The estimate is reliable as $r = 0.971$ is close to 1, and $x = 28$ is within the data range of x.	You should use at least 4 s.f. coefficient values for the regression line equation. Note that for an estimate is reliable, both conditions must be satisfied.
(e)	It will not be different as the product moment correlation coefficient is not affected by scaling. $y = \frac{1}{60}\left(4.05 + 1170\left(\frac{1}{x}\right)\right)$	You should state that the r-value is independent of units / not affected by scaling / linear transformation.

9 **In this question, you should state the parameters of any normal distributions you use.**

A confectionary bakes egg tarts for sale. The masses of these egg tarts have a normal distribution and 26% of them have a mass greater than 65 grams. An egg tart is equally likely to have a mass less than 50 grams as it is to have a mass greater than 70 grams.

(a) Find the mean and variance of this distribution. [3]

The confectionary also bakes donuts and buns. The masses in grams of the donuts have the distribution $\mathrm{N}(80, 10^2)$ and the masses in grams of the buns have the distribution $\mathrm{N}(70, 6^2)$.

(b) Find the probability that the mass of a randomly chosen donut is within 5 grams from its mean. [1]

(c) Find the probability that the mass of a randomly chosen bun is less than the mass of a randomly chosen donut. [2]

The confectionery also tops its donuts and buns with chocolate. A chocolate topping increases the mass of a donut by 10% and the mass of a bun by 20%.

(d) Find the probability that the total mass, with chocolate topping, of 3 randomly chosen donuts and 5 randomly chosen buns exceeds 700 grams. [4]

9 (a)	Let X be the mass (grams) of a randomly chosen egg tart. $X \sim \mathrm{N}(\mu, \sigma^2)$ $\mathrm{P}(X < 50) = \mathrm{P}(X > 70)$ By symmetry, $\mu = \frac{50+70}{2} = 60$ $\mathrm{P}(X > 65) = 0.26$ $\mathrm{P}\left(Z > \frac{65-60}{\sigma}\right) = 0.26$	You should use symmetry to obtain the value of μ. Note that the following working is **wrong**. You can sketch out the regions to see why. $\mathrm{P}\left(Z < \frac{50-\mu}{\sigma}\right) = \mathrm{P}\left(Z > \frac{70-\mu}{\sigma}\right)$ $\Rightarrow \frac{50-\mu}{\sigma} = \frac{70-\mu}{\sigma}$

	$\frac{5}{\sigma}=0.643345021$ $\sigma=7.771874927$ $\sigma^2=60.4$ (3 sf)	
(b)	Let A and B be the mass (grams) of a randomly chosen donut and bun respectively. $A\sim N(80,10^2)$ $\quad B\sim N(70,6^2)$ $P(75\le A\le 85)=0.383$ (3 sf)	
(c)	$E(B-A)=70-80=-10$ $Var(B-A)=6^2+10^2=136$ $B-A\sim N(-10,136)$ $P(B-A<0)=0.804$ (3 sf)	Variance is never subtracted. You must write the distribution of $B-A$.
(d)	$T=1.1(A_1+\cdots+A_3)+1.2(B_1+\cdots+B_5)$ $E(T)=1.1(3)(80)+1.2(5)(70)=684$ $Var(T)=1.1^2(3)(10^2)+1.2^2(5)(6^2)=622.2$ $T\sim N(684,622.2)$ $P(T>700)=0.261$ (3 sf)	The required variable is $1.1(A_1+\cdots+A_3)+1.2(B_1+\cdots+B_5)$ as opposed to $1.1(3A)+1.2(5B)$ because $3A$ represents 3 times the mass of a donut, **NOT** the total mass of 3 donuts.

10 A company advertises that its rechargeable batteries can run non-stop for an average of 505 minutes after a full charge. 100 batteries are randomly selected, and their running time after a full charge, x minutes, are measured. The data obtained is summarised by

$$\sum(x-500)=320, \quad \sum(x-500)^2=8416.$$

(a) Suggest a reason why, in this context, the given data is summarised in terms of $(x-500)$ rather than x. [1]

(b) Calculate unbiased estimates of the population mean and variance of the running time of the batteries. [2]

(c) A hypothesis test is carried out at the 5% level of significance. Determine whether the company is overestimating the average running time of the batteries. [4]

(d) State, giving a reason, whether it is necessary to assume a normal distribution for the test to be valid. [1]

(e) After a change is made in the manufacturing process, another test at the 5% significance level is carried out using a new random sample of size 50. The sample standard deviation is found to be 8.5 minutes. Find the range of values of the sample mean in order to conclude that the average running time of the batteries after a full charge is more than 505 minutes. [4]

10 (a)	Because the data points are near 500. Hence using $(x-500)$ will result in smaller recorded/summarised values.	
(b)	Unbiased estimate of population mean, $\bar{x}=\frac{320}{100}+500=503.2$ (exact) Unbiased estimate of population variance, $s^2=\frac{1}{99}\left[8416-\frac{(320)^2}{100}\right]$ $=74.667$ (5 sf) $=74.7$ (3 sf)	+500 is required since the given value is $\sum(x-500)$ and not $\sum x$. $\bar{x}$ needs to be defined with words. Final answers left in 3 s.f. is incorrect since the value is exact. Formula list in MF27 can be referred to for s^2
(c)	$H_0: \mu=505$ $\quad$ $H_1: \mu<505$ Under H_0, $\bar{X} \sim N\left(505, \frac{74.667}{50}\right)$ approximately by Central Limit Theorem as $n=100$ is large At 5% sig. level, Reject H_0 if $p<0.05$ Using GC, p-value = 0.0186 < 0.05. reject H_0 Conclude at 5% sig. level that there is sufficient evidence to suggest that the company is overestimating the average running time of the batteries.	"Determine whether the company is overestimating" hence $H_1: \mu<505$
(d)	No, the sample size 100 is large enough, so the distribution of the sample mean is approximately normal by Central Limit Theorem.	The use of Central Limit Theorem (CLT) is NOT an assumption. The distribution of sample mean being approximately normal due to CLT is also NOT an assumption. Since CLT can be applied (large n), there is no need to assume normal for the distribution of sample mean.

(e)	Unbiased estimate of the population variance, $s^2=\frac{50}{49}\left[8.5^2\right]=73.724$ $H_0:\mu=505$ $H_1:\mu>505$ Under H_0, $\bar{X}\sim N\left(505,\frac{73.724}{50}\right)$ approximately by Central Limit Theorem as $n=50$ is large In order to reject H_0 at 5% sig. level, $\frac{\bar{x}-505}{\sqrt{\frac{73.724}{50}}}\geq 1.6449$ $\bar{x}-505\geq 1.9974$ $\bar{x}\geq 507\ (3\text{ sf})$	8.5 is the **standard deviation** of the **sample**, hence s^2 needs to be calculated first using $s^2=\frac{n}{n-1}\times$sample variance Note also that the sample size has changed to 50 instead of 100 in part (a). "conclude that the average running time of the batteries after a full charge is more than 505 minutes" suggests that $H_1:\mu>505$ and that we want H_0 to be rejected. 1.6449 is the critical value, obtained using invNorm. Note also that critical value is INCLUDED in the critical region (i.e. H_0 rejected)

11 In a large shipment of second-hand T-shirts, 4% of the T-shirts are torn. The T-shirts are sold in boxes of 25 pieces each. Let X denote the number of torn T-shirts in a box.

(a) State, in the context of the question, two assumptions needed to model X by a binomial distribution. [2]

You are now given that X can be modelled by a binomial distribution.

(b) A box is randomly chosen. Find the probability that a box contains at least 2 torn T-shirts. [2]

(c) A box is deemed to be of inferior quality if it contains at least 2 torn T-shirts. Find the probability that, in a random sample of 10 boxes of T-shirts, more than 3 boxes are of inferior quality. [2]

A distributor purchases a batch of 60 boxes of T-shirts.

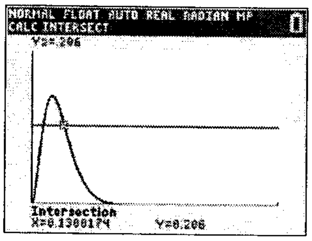
(d) Find the probability that the average number of torn T-shirts per box is more than 1.1. [3]

(e) Find the most probable number of boxes of inferior quality in the batch. [2]

In a large shipment of second-hand blouses, a proportion p of the blouses is torn. The blouses are also sold in boxes of 25 pieces each. The number of torn blouses in a box follows a binomial distribution. It is known that the probability of a box containing exactly 2 torn blouses is 0.206.

(f) Write down an equation satisfied by p. Hence, find the value of p, given that $p > 0.1$. [2]

11 (a)	Assumptions The probability of a randomly chosen T-shirt being torn is the same for every T-shirt. Whether a T-shirt is torn or not is independent of any other T-shirts.	It is important to state the same probability **for all trials** Independence is **on the event** and not on the probability.
(b)	$X \sim \mathrm{B}(25, 0.04)$ $\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(X \leq 1)$ $= 0.26419$ (5 sf) $= 0.264$ (3 sf)	$\mathrm{P}(X \leq 1)$ is evaluated using **binomcdf** from GC
(c)	Let Y be number of boxes with at least 2 torn T-shirts, out of 10 boxes. $Y \sim \mathrm{B}(10, 0.26419)$ $\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \leq 3)$ $= 0.258$ (3 sf)	Remember to define the new distribution Be very careful on stating what is the complement $\mathrm{P}(Y > 3) = 1 - \mathrm{P}(Y \leq 3)$
(d)	Method 1 $\mathrm{E}(X) = 25(0.04) = 1$ $\mathrm{Var}(X) = 25(0.04)(0.96) = 0.96$ Since $n = 60$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1, \frac{0.96}{60}\right)$ approximately $\mathrm{P}(\overline{X} > 1.1) = 0.215$ (3 sf) Method 2 Let T be the number of torn T-shirts, out of 1500 shirts $T \sim \mathrm{B}(1500, 0.04)$	Check that **n is large** before applying Central Limit Theorem

	$\mathrm{P}(T > 1.1(60))$ $= \mathrm{P}(T > 66)$ $= 1 - \mathrm{P}(T \le 66)$ $= 0.194\ (3\text{ sf})$	
(e)	Let A be number of inferior quality boxes, out of 60 $A \sim \mathrm{B}(60, 0.26419)$ $\mathrm{P}(A = 15) = 0.115$ $\mathrm{P}(A = 16) = 0.116$ $\mathrm{P}(A = 17) = 0.108$ Most probable number is 16.	Remember to redefine the new distribution You need to show **3 lines of table** before concluding the mode. Mean and mode are different.
(f)	Let W be number of torn blouses, out of 25. $W \sim \mathrm{B}(25, p)$ $\mathrm{P}(W = 2) = 0.206$ $\binom{25}{2} p^2 (1-p)^{23} = 0.206$ $300 p^2 (1-p)^{23} = 0.206$ Using GC, $p = 0.130\ (3\text{ sf})$ 	Remember to define and state the new distribution Writing down the equation satisfied by p is required by question. You can **use GC to evaluate** the value of p

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