2016 JC2 Preliminary Examination

MATHEMATICS
HIGHER 1

8864/1

18 AUGUST 2016
THURSDAY 0800h – 1100h

Additional materials :
  Answer paper
  Graph paper
  List of Formulae (MF15)

TIME 3 hours

READ THESE INSTRUCTIONS FIRST

Write your CTG and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless specifically states otherwise. Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [35 marks]

1 Find the set of values of p for which the line $y = px + 4$ does not meet the curve $x^2 - xy = 2$. [3]

2 Using the substitution $u = 2^x$, solve the equation $2^{2x} - 2^{1+x} = 8$. [3]

3 (i) Differentiate

(a) $3\ln(1+3x^2)$, [2]

(b) $\dfrac{1}{\sqrt{1-2x}}$. [2]

(ii) Use a non-calculator method to find the exact value of $\displaystyle\int_1^4 \left(\frac{\sqrt{x}-2}{x}\right) \mathrm{d}x$. [4]

4 The gradient of a curve $y = \mathrm{f}(x)$ at any point (x, y) is given by e^{1-ax}, where a is a constant.

(i) The curve cuts the x-axis at point P with coordinates $(1 - \ln 4, 0)$ and the gradient at P is 4. Show that $a = 1$ and find the equation of the curve. [4]

(ii) Hence, find the equation of the normal at point P, giving your answer in the form $y = mx + c$, where m and c are constants in exact form. [2]

5 A curve C has equation $y = 4 - k^2x^2$ and a line L has equation $y = 2 + kx$, where k is a positive constant.

(i) Find, in terms of k, the x-coordinates of the points where L and C intersect. [2]

(ii) Sketch C and L on the same diagram, labelling clearly the coordinates of the points of intersection with the axes and also between C and L. [3]

(iii) Find, in terms of k, the area of the region bounded by C, L and the x-axis. [4]

6 A closed cylindrical can of radius r cm and height h cm has a volume of 300 cm^3. In order to minimise the use of material, a beverage company wants the external surface area of the can, S cm^2, to be as small as possible.

(i) Find an expression for S in terms of r. [2]

(ii) Use differentiation to find the minimum value for S as r varies, giving your answer correct to 2 decimal places. [4]

Section B: Statistics [60 marks]

7 A certain factory employs 300 general workers, 100 foremen and 40 management personnel. To gather feedback on a new policy, the factory owner wishes to select a representative sample of 22 employees. From the list of all employees with names in alphabetical order, he selects at random one of the first 20 employees and then chooses every 20th employee thereafter.

(i) What is this type of sampling called? [1]

(ii) State a disadvantage of the sampling method used in this context. [1]

(iii) Describe a more appropriate sampling method in this scenario. [2]

8 The masses, in kg, of durians and mangosteens are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard deviation
Durians	1.7	0.2
Mangosteens	μ	0.05

(i) If more than 95% of the mangosteens weigh less than 0.3 kg, find the range of values of μ. [3]

It is given that $\mu = 0.2$.

(ii) Find the probability that the total mass of 2 randomly chosen durians and 4 randomly chosen mangosteens exceeds 4.4 kg. [3]

(iii) Find the probability that the total mass of 4 randomly chosen mangosteens is within ± 0.2 kg of half the mass of a randomly chosen durian. [4]

(iv) Durians are sold at \$8 per kg and mangosteens at \$3 per kg.
Find the probability that the total selling price of 3 randomly chosen durians and 10 randomly chosen mangosteens is less than \$45. [4]

9 The weather in a particular area is being monitored and each day is classified as rainy or dry. The probability that any particular day is a rainy day is 0.3.

(i) Calculate the probability that, in a given week, there are

(a) exactly 3 rainy days, [1]
(b) more rainy days than dry days. [2]

The weather is monitored for a year (52 weeks).

(ii) Find the probability that there are at least 40 weeks with more dry days than rainy days. [3]

(iii) Find the probability that the average number of rainy days per week is more than 2. [4]

(iv) Using a suitable approximation, find the probability that the number of rainy days is between 100 and 120 during the period. [4]

10 A resident claims that the average speed of vehicles along a particular road is greater than the speed limit of 50 km per hour. The traffic police recorded the speed, x km per hour, of 60 randomly selected vehicles and summarised the data collected as follows.

$$\sum(x-50)=75, \qquad \sum(x-50)^2=2016.$$

(i) Find unbiased estimates of the population mean and the variance. [3]

(ii) Test at the 5% level of significance whether there is evidence to support the resident's claim. [4]

(iii) Explain what is meant by the phrase "5% level of significance" in the context of this question. [1]

From past records, it is known that the standard deviation is 14 km per hour. To investigate the resident's claim, the traffic police recorded the speed of 100 randomly selected vehicles and the mean is c km per hour.

(iv) Find the range of values of c such that there is evidence to support the resident's claim at the 5% level of significance. [3]

11 The trainees in a company went through a training programme and they sat for two performance assessments. The scores for 10 trainees in the two performance assessments are shown in the following table.

Trainee	1	2	3	4	5	6	7	8	9	10
Score for assessment 1, x	106	87	103	151	37	114	193	57	186	117
Score for assessment 2, y	152	138	104	190	45	141	184	86	17	142

(i) Give a sketch of the scatter diagram for the data and calculate the product moment correlation coefficient. [3]

(ii) Which data point appears to be incorrect? [1]

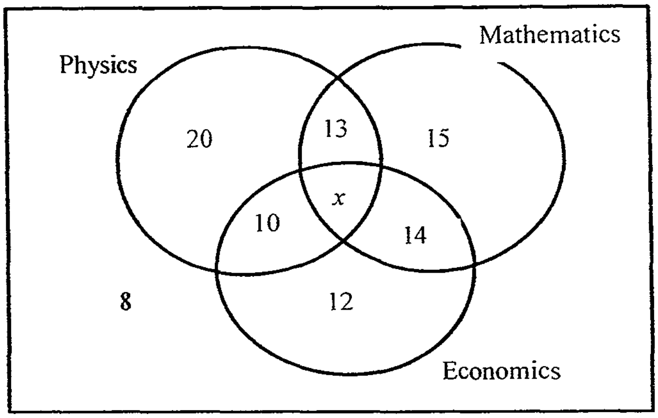
For the rest of the question, the data point in **(ii)** is removed from the data set.

(iii) Calculate the revised product moment correlation coefficient and comment on the effect on the value of the product moment correlation coefficient after removing the data point in **(ii)**. [2]

(iv) Calculate the equation of the regression line of y on x. [1]

(v) A new trainee obtained a score of 127 in performance assessment 1. Calculate an estimate of his score in performance assessment 2 and comment on the reliability of the estimate. [2]

12



A group of students were asked about the subjects they took at the A level examination. The numbers who have taken Physics, Mathematics and Economics are shown in the Venn diagram. The number of students who have taken all three subjects is x. One student is chosen at random.

S is the event that the student has taken Physics.
M is the event that the student has taken Mathematics.
E is the event that the student has taken Economics.

(i) Write down expressions for $P(M)$ and $P(E)$ in terms of x. Given that M and E are independent, show that $x = 8$. [4]

Using this value of x, find

(ii) $P(S' \cap M)$, [1]

(iii) $P(S \cap E \mid M)$. [1]

Two students from the whole group are chosen at random.

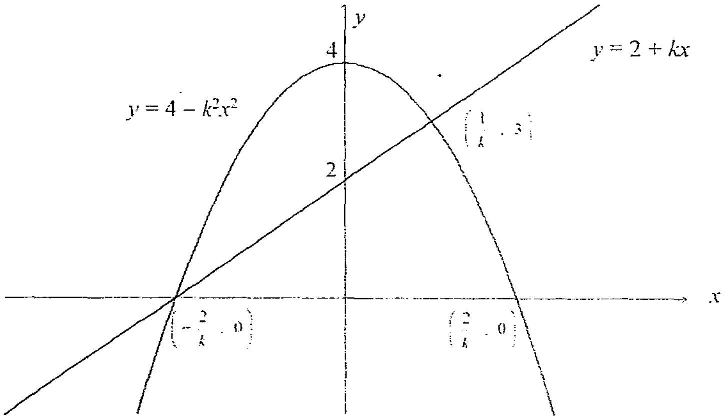
(iv) Find the probability that only one of them has taken all 3 subjects. [2]

~ ~ ***End of paper*** ~~

2016 JC2 H1 Prelim Exam

Solutions

Qn	Solution
1	$y = px + 4$ --- (1) $x^2 - xy = 2$ --- (2) Substitute (1) into (2) $x^2 - x(px + 4) = 2$ $(1 - p)x^2 - 4x - 2 = 0$ No real roots $\Rightarrow b^2 - 4\,ac < 0$ i.e. $(-4)^2 - 4(1 - p)(-2) < 0$ $16 + 8 - 8p < 0$ $24 < 8p$ $p > 3$ Set of values = $\{p \in \square : p > 3\}$
2	$u = 2^x$ $2^{2x} - 2^{1+x} = 8 \Rightarrow u^2 - 2u - 8 = 0$ $\Rightarrow (u + 2)(u - 4) = 0$ $\Rightarrow u = -2$ or $u = 4$ $\Rightarrow 2^x = -2$ (rejected as $2^x > 0$) $\therefore\ 2^x = 4 = 2^2 \Rightarrow x = 2$
3 (i)	**(a)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(3\ln\left(1 + 3x^2\right)\right) = 3\left(\frac{1}{1+3x^2}\right)(6x)$ $= \frac{18x}{1+3x^2}$ **(b)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{1-2x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}(1-2x)^{-\frac{1}{2}}$ $= -\frac{1}{2}(1-2x)^{-\frac{3}{2}}(-2)$ $= (1-2x)^{-\frac{3}{2}}$
3 (ii)	$\int_1^4 \left(\frac{\sqrt{x} - 2}{x}\right)\mathrm{d}x = \int_1^4 \left(x^{-\frac{1}{2}} - \frac{2}{x}\right)\mathrm{d}x$ $= \left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - 2\ln\lvert x\rvert\right]_1^4$ $= \left[2\sqrt{4} - 2\ln\lvert 4\rvert\right] - \left[2\sqrt{1} - 2\ln\lvert 1\rvert\right]$ $= 2 - 2\ln 4$ (or $= 2 - 4\ln 2$)

Qn	Solution
4 (i)	$\frac{dy}{dx} = e^{1-ax}$ At $P, x = 1 - \ln 4, y = 0$, $\frac{dy}{dx} = 4$ $\therefore 4 = e^{1-a(1-\ln 4)}$ $\ln 4 = 1 - a\,(1 - \ln 4)$ $a(1 - \ln 4) = 1 - \ln 4$ $\therefore a = 1$ $\frac{dy}{dx} = e^{1-x} \Rightarrow y = \int e^{1-x} dx = \frac{e^{1-x}}{-1} + C$ When $x = 1 - \ln 4, y = 0$ $\Rightarrow 0 = -e^{1-(1-\ln 4)} + C$ $\Rightarrow C = e^{\ln 4} = 4$ $\therefore$ equation of the curve is $y = 4 - e^{1-x}$
4 (ii)	Gradient of tangent at $P = m = 4$ $\therefore$ gradient of normal at $P = -\frac{1}{4}$ $\therefore$ equation of the normal at P is $y - 0 = -\frac{1}{4}(x - (1 - \ln 4))$ $\Rightarrow y = -\frac{1}{4}x + \frac{1}{4}(1 - \ln 4)$
5 (i)	C: $y = 4 - k^2x^2$ …… (1) L: $y = 2 + kx$ …… (2) Solving (1) & (2), $4 - k^2x^2 = 2 + kx$ $k^2x^2 + kx - 2 = 0$ …… (3) $(kx + 2)(kx - 1) = 0$ $\therefore x = -\frac{2}{k}$ or $\frac{1}{k}$
(ii)	

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>(iii)</td><td>

Required area $= \text{area of }\Delta + \int_{\frac{1}{k}}^{\frac{2}{k}} (4-k^2x^2)\mathrm{d}x$

$$= \frac{1}{2}\left(\frac{3}{k}\right)(3) + \left[4x - k^2\frac{x^3}{3}\right]_{\frac{1}{k}}^{\frac{2}{k}}$$

$$= \frac{9}{2k} + \left[\left(\frac{8}{k} - \frac{k^2}{3}\left(\frac{2}{k}\right)^3\right) - \left(\frac{4}{k} - \frac{k^2}{3}\left(\frac{1}{k}\right)^3\right)\right]$$

$$= \frac{37}{6k}$$

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<tr><td>6(i)</td><td>

Volume $= 300 = \pi r^2 h$

$$\Rightarrow h = \frac{300}{\pi r^2}$$

Surface area, S = area of top & bottom + curved surface

$$= \pi r^2(2) + 2\pi r h$$

$$= \pi r^2(2) + 2\pi r\left(\frac{300}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{600}{r}$$

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<tr><td>6 (ii)</td><td>

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 2\pi(2r) + 600(-r^{-2})$$

For minimum, let $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$

$$\Rightarrow 4\pi r - \frac{600}{r^2} = 0$$

$$\Rightarrow r^3 = \frac{600}{4\pi} = \frac{150}{\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{150}{\pi}} = 3.62783$$

<table>
<tr><td>r</td><td>$\left(\sqrt[3]{\frac{150}{\pi}}\right)^-$</td><td>$\sqrt[3]{\frac{150}{\pi}}$</td><td>$\left(\sqrt[3]{\frac{150}{\pi}}\right)^+$</td></tr>
<tr><td>$\frac{\mathrm{d}S}{\mathrm{d}r}$</td><td>-ve</td><td>0</td><td>+ve</td></tr>
<tr><td>slope</td><td>\</td><td>—</td><td>/</td></tr>
</table>

$\therefore$ S is minimum when $r = \sqrt[3]{\frac{150}{\pi}}$ =3.62783

Minimum S $= 2\pi(3.62783)^2 + \frac{600}{3.62783}$

$= 248.0821$

$= 248.08$ cm^2 (to 2 dec places)

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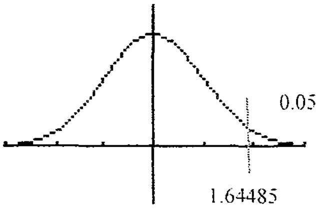
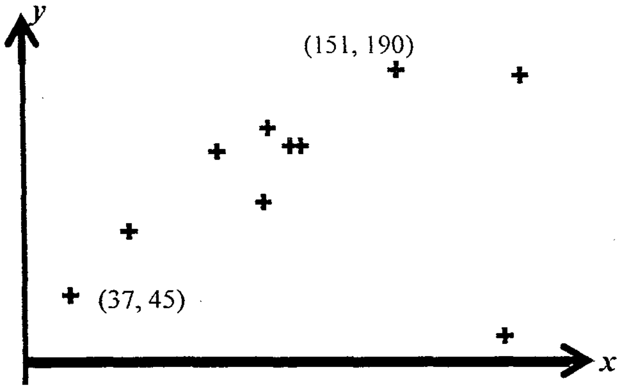
Qn	Solution
7(i)	Systematic sampling.
(ii)	- There may not be a proportionate representation of the employees. - certain category of the employees may not be represented - the sample may consist of only one or two categories of the employees
(iii)	Stratified sampling method is more appropriate. 15 general workers, 5 foremen and 2 management personnel are randomly selected to form the sample.
8(i)	Let D and M be the mass, in kg, of a durian and a mangosteen respectively. Then $D \sim N(1.7, 0.2^2)$, $M \sim N(\mu, 0.05^2)$ $P(M < 0.3) > 0.95$ $P\left(Z < \frac{0.3-\mu}{0.05}\right) > 0.95$ $\Rightarrow \frac{0.3-\mu}{0.05} > 1.64485$ $\Rightarrow \mu < 0.3 - 1.64485(0.05) = 0.21776$ i.e. $\mu < 0.218$ (to 3 s.f)
(ii)	$\mu = 0.2$ i.e. $M \sim N(0.2 , 0.05^2)$ $D_1+D_2+M_1+\ldots+M_4 \sim N(1.7+1.7+0.2+0.2+0.2+0.2,$ $0.2^2+0.2^2+0.05^2+0.05^2+0.05^2+0.05^2)$ i.e. $D_1+D_2+M_1+\ldots+M_4 \sim N(4.2, 0.09)$ $P(D_1+D_2+M_1+\ldots+M_4 > 4.4) = 0.25249 = 0.252$
(iii)	$M_1+\ldots+M_4 - \frac{1}{2}D \sim N(-0.05, 0.02)$ $P(\frac{1}{2}D - 0.2 < M_1+\ldots+M_4 < \frac{1}{2}D + 0.2)$ $= P(-0.2 < M_1+\ldots+M_4 - \frac{1}{2}D < 0.2)$ $=0.8170279 = 0.817$
(iv)	Let A and B be the selling price, in \$, of a durian and a mangosteen respectively. Then $A = 8D \sim N(13.6, 2.56)$, $B = 3M \sim N(0.6, 0.0225)$ $A_1+A_2+A_3+B_1+\ldots+B_{10} \sim N(46.8, 7.905)$ $P(A_1+A_2+A_3+B_1+\ldots+B_{10} < 45) = 0.2610184 = 0.261$
9(i) (a) (b)	Let X be the number of rainy days in a week (out of 7). Then $X \sim B(7, 0.3)$ $P(X = 3) = 0.2268945 = 0.227$ $P(X > 3) = 1 - P(X \leq 3)$ $= 1 - 0.873964 = 0.126036 = 0.126$
(ii)	Let Y be the number of weeks with more dry days in a year (52 weeks) Then $Y \sim B(52, 0.873964)$ $P(Y \geq 40) = 1 - P(Y \leq 39)$ $= 0.989124771 = 0.989$
(iii)	Let $\overline{X}$ be the average number of rainy days per week in a period of 52 weeks.

Qn	Solution
	Then by CLT, $\overline{X} \sim \mathrm{N}(2.1, \frac{1.47}{52})$ approximately $P(\overline{X} > 2) = 0.7239986698 = 0.724$
(iv)	Let W be the number of rainy days in 52 weeks (364 days) Then $W \sim \mathrm{B}(364, 0.3)$ Since n=364 is large and $np = 109.2 > 5$, $nq = 254.8 > 5$ Then $W \sim \mathrm{N}(109.2, 76.44)$ approximately. $\mathrm{P}(100 < W < 120) \xrightarrow{c.c} \mathrm{P}(100.5 < W < 119.5)$ $= 0.7207705 = 0.721$
10 (i)	Unbiased estimate for $\mu = \frac{75}{60} + 50 = 51.25$ Unbiased estimate for $\sigma^2 = \frac{1}{59}\left(2016 - \frac{75^2}{60}\right) = 32.5805$ $= 32.6$
(ii)	$\mathrm{H_0}: \mu = 50$ $\mathrm{H_1}: \mu > 50$ Under $\mathrm{H_0}$, $Z = \frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$ approx. by CLT where $\mu = 50$, $\bar{x} = 51.25$, $s = \sqrt{32.5805}$, $n = 60$ Level of significance: 5% From GC, p-value = 0.0449. Since p-value < 5%, we reject $\mathrm{H_0}$ and conclude that at the 5% level, there is significant evidence to support the resident's claim.
(iii)	"5% level of significance" means that there is a probability of 0.05 to support the resident's claim when the population mean speed is actually 50 km/h.
(iv)	$\mathrm{H_0}: \mu = 50$ $\mathrm{H_1}: \mu > 50$ Under $\mathrm{H_0}$, $Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ approx. by CLT  where $\mu = 50$, $\sigma = 14$, $n = 100$ Level of significance: 5% Reject $\mathrm{H_0} \Rightarrow z > 1.64485$ $\frac{c - 50}{\left(\frac{14}{\sqrt{100}}\right)} > 1.64485$ $c > 50 + 1.64485\left(\frac{14}{\sqrt{100}}\right)$ $c > 52.30279$ $c > 52.3$

Qn	**Solution**
11 (i)	 product moment correlation coefficient = 0.289
(ii)	Trainee 9 with scores 186 and 17 for Performance Assessment 1 and 2 respectively
(iii)	The revised product moment correlation coefficient between x and $y = 0.903$. By removing the incorrect data point, the correlation coefficient is very much closer to 1 compared to the value obtained in (i).
(iv)	The equations of the regression lines of y on x is $y = 0.899x + 34.9$
(v)	When $x = 127, y \approx 149.113 \approx 149$ The estimate is reliable as it is obtained by interpolation (or $x = 127$ is within the range of data given) and $r = 0.903$ is close to 1.
12(i)	$\mathrm{P}(M) = \frac{42+x}{92+x}$, $\mathrm{P}(E) = \frac{36+x}{92+x}$ $\mathrm{P}(M \cap E) = \frac{14+x}{92+x}$ Since M and E are independent, $\mathrm{P}(M \cap E) = \mathrm{P}(M)\mathrm{P}(E)$ $\frac{14+x}{92+x} = \left(\frac{42+x}{92+x}\right) \cdot \left(\frac{36+x}{92+x}\right)$ $(14+x)(92+x) = (42+x)(36+x)$ $1288 + 106x + x^2 = 1512 + 78x + x^2$ $28x = 224$ $\therefore x = 8$ OR using $\mathrm{P}(M \mid E) = \mathrm{P}(M) \Rightarrow \frac{14+x}{36+6} = \frac{42+x}{92+x}$ OR using $\mathrm{P}(E \mid M) = \mathrm{P}(E) \Rightarrow \frac{14+x}{42+x} = \frac{36+x}{92+x}$
(ii)	$P(S' \cap M) = \frac{29}{100} = 0.29$
(iii)	$P(S \cap E \mid M) = \frac{P(S \cap E \cap M)}{P(M)} = \frac{8}{50} = 0.16$
(iv)	P(only one has studied all 3 subjects) $= \left(\frac{8}{100}\right)\left(\frac{92}{99}\right) \times 2 = 0.148687 \approx 0.149$