MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2

9740 / 01

Paper 1

18 August 2016

JC 2 PRELIMINARY EXAMINATION

Time allowed: **3 hours**

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Index number, Form Class, graphic and/or scientific calculator model/s on the cover page.
Write your Index number and full name on all the work you hand in.
Write in dark blue or black pen on your answer scripts.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in the question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages.

MATHEMATICS DEPARTMENT
JC2 Preliminary Examination 2016

MATHEMATICS 9740
Higher 2
Paper 1

/ 100

Index No: | | | | |

Form Class: ____________

Name: ______________________________

Calculator model: ________________

Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

Question No.	Marks
1	/5
2	/5
3	/5
4	/9
5	/4
6	/8
7	/8
8	/6
9	/9
10	/11
11	/7
12	/12
13	/11

Summary of Areas for Improvement			
Knowledge (K)	Careless Mistakes (C)	Read/Interpret Qn wrongly (R)	Presentation (P)

1 Without the use of a calculator, solve the inequality $\frac{2w-5}{w^2-3}>0$. [2]

Hence solve $\frac{(2|y|-5)\sin x}{y^2-3}\leq 0$, given that $\pi<x\leq\frac{3\pi}{2}$. [3]

2 The equation of the curve C is given by $y=\ln x$. The line ℓ with the equation $y=\frac{x}{e}$ is tangential to the curve C at the point $(e,1)$. The region R is bounded by the curve C, the line ℓ and the x-axis. The solid S is formed by rotating the region R through 2π radians about the x-axis. Find the exact volume of the solid S in terms of π and e. [5]

3 **(i)** Every year Warren Gate's net worth increases by 100% of the previous year. His net worth was estimated to be \$1 000 000 on 31st December 1993. In what year will his fortune first surpass 2.5 billion dollars (1 billion = 10^9). [3]

(ii) On 1st January 2005, Warren Gates deposits \$100 000 in an investment account and receives an interest of \$1000 on 31st December 2005. After that the amount of interest earned at the end of the year is 1.5 times the amount of interest earned in the previous year. Taking year 2005 as the first year, find the amount of savings that Warren Gates has in his account at the end of the 15th year giving your answer to the nearest integer. [2]

4 **(a)** It is given that $g(x)=\frac{1}{\cos(\frac{\pi}{4}+x)\cos(\frac{\pi}{4}-x)}$ where x is sufficiently small for x^3 and higher powers of x to be neglected.

Show that $g(x)\approx 2+ax+bx^2$, where a and b are constants. [3]

Comment on the value of m for this expression $\int_{-m}^{m} g(x)\,dx \approx \int_{-m}^{m}(2+ax+bx^2)\,dx$ to be valid. [1]

(b) Find the first four non-zero terms of the expansion of $\left(1-x^2\right)^{-\frac{1}{2}}$ in ascending powers of x where $|x|<1$. [2]

Hence find the first four non-zero terms of the Maclaurin's series for $\cos^{-1}x$ in ascending powers of x. [3]

[Turn Over

5 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \text{ and } \frac{1}{3^r}\left(\frac{u_{r+1}}{3} - u_r\right) = 2r \text{ for all } r \geq 1.$$

Use the method of differences to prove that $u_{n+1} = 3^n(3n^2 + 3n + 1)$ for all $n \geq 1$. [4]

6 An investor deposits \$$K$ in a bank account. The bank offers an annual interest rate of 5% compounded continuously. No further deposits are made. The amount of money in the account at time t years is denoted by M. Both M and t are taken to be continuous variables. Money is withdrawn at a continuous rate of \$4000 per year. Set up a differential equation and show that for $t > 0$, $\frac{dM}{dt} = aM + b$, where a and b are constants to be determined. [1]

For $t > 0$, find M in terms of t and K. [4]

On a single clearly labelled diagram, show the graph of M against t for

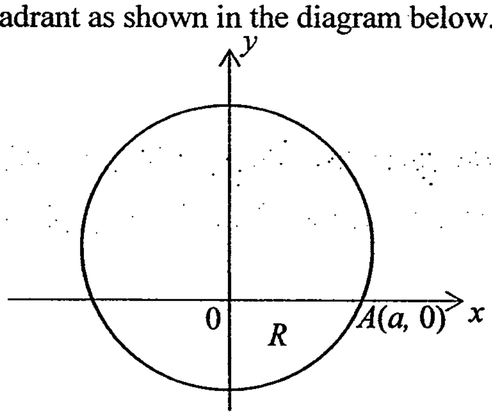
(i) $K > 80\,000$, [1]

(ii) $K < 80\,000$. [1]

Hence state the condition for which the money deposited initially will be completely withdrawn in a finite period of time. [1]

7 **(i)** Use the substitution $x = 5\sin\theta$ to find $\int \sqrt{25 - x^2}\,dx$. [5]

(ii) The circle with equation $x^2 + (y - b)^2 = 25$ where $0 < b < 5$, cuts the positive x-axis at $A(a, 0)$. The region R is bounded by the x and y axes, and the part of the circle lying in the fourth quadrant as shown in the diagram below.



Use your result in **(i)** to find the area of the region R in terms of a. [3]

8 The complex number z is given by $z=k+\mathrm{i}$ where k is a non-zero real number.

(i) Find the possible values of k if $z=k+\mathrm{i}$ satisfies the equation $z^3-\mathrm{i}z^2-2z-4\mathrm{i}=0$. [3]

(ii) For the complex number z found in part **(i)** for which $k>0$, find the smallest integer value of n such that $|z^n|>100$ and z^n is real. [3]

9 Use the method of mathematical induction to prove that

$$\sum_{r=1}^{n}\frac{3r+1}{r(r+1)(r+2)}=\frac{n(7n+9)}{4(n+1)(n+2)},\text{ for all } n\geq 1.$$ [5]

(i) Show that $\dfrac{n(7n+9)}{4(n+1)(n+2)}<\dfrac{7}{4}$. [2]

(ii) Hence show that $\displaystyle\sum_{r=1}^{n}\frac{3r}{(r+1)^3}<\frac{7}{4}$. [2]

10 The curve C has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\dfrac{x^2-4k^2}{x-k}$. It is given that k is a constant and $x\neq k$.

Find the set of possible values that y can take. [3]

For the case $k>1$,

(i) Sketch the graph of C, stating in terms of k, the coordinates of any points of intersection with the axes and equations of any asymptotes. [3]

(ii) Hence find $\int_{-1}^{1}\mathrm{f}(|x|)\,\mathrm{d}x$ in terms of k. [3]

(iii) The graph of curve C is transformed by a scaling of factor 2 parallel to the x-axis, followed by a translation of $2k$ units in the negative x-direction. Find the equation of the new curve. You need not simplify your answer. [2]

11 Referred to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Given that $\mathbf{a}\times\mathbf{b}=4\mathbf{a}\times\mathbf{c}$, where $\mathbf{b}\neq 4\mathbf{c}$ and $\mathbf{a}$ is a non-zero vector,

(i) show that $\mathbf{b}-4\mathbf{c}=\alpha\mathbf{a}$ where α is a scalar. [1]

(ii) Hence evaluate $|\mathbf{b}\times\mathbf{c}|$, given that the area of triangle OAB is $\sqrt{126}$ and $\alpha=\sqrt{3}$. [2]

(iii) Give the geometrical meaning of $|\mathbf{b}\times\mathbf{c}|$. [1]

It is also given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}|=5$, $|\mathbf{c}|=2$ and $\mathbf{b}-4\mathbf{c}=\sqrt{3}\mathbf{a}$.

(iv) By considering $(\mathbf{b}-4\mathbf{c})\cdot(\mathbf{b}-4\mathbf{c})$, find the angle between $\mathbf{b}$ and $\mathbf{c}$. [3]

[Turn Over

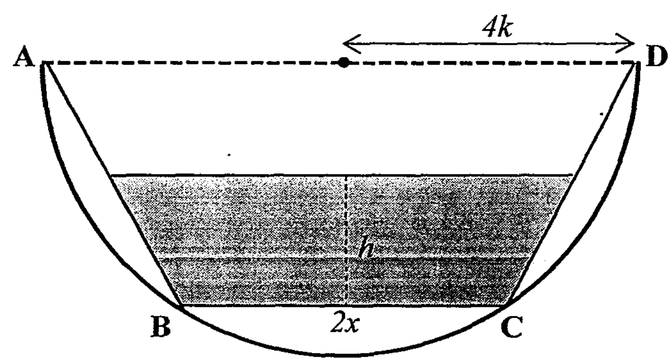
12 (i) Solve the equation $z^5 - i = 0$, giving the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Show the roots z_1, z_2, z_3, z_4 and z_5 on an Argand diagram where $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \leq \pi$. [5]

(ii) Find the exact cartesian equation of the locus of all points z such that $|z - z_2| = |z - z_3|$ and sketch this locus on an Argand diagram. Find the least possible value of $|z - z_1|$. [4]

(iii) Sketch on the same Argand diagram in **(ii)**, the locus $\arg(z - z_1) = \arg(z_4)$. [1]

(iv) Find the complex number z that satisfy the 2 equations $|z - z_2| = |z - z_3|$ and $\arg(z - z_1) = \arg(z_4)$, giving your answer in the form $a + ib$. [2]

13



The diagram shows the cross-section of a container. It is in the shape of a semicircle of fixed radius $4k$ metres with a hole in the shape of a trapezium *ABCD*.

(i) If $BC = 2x$ metres, show that the area S of the trapezium *ABCD* is given by $S = (x + 4k)\sqrt{16k^2 - x^2}$. [2]

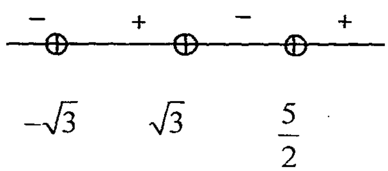
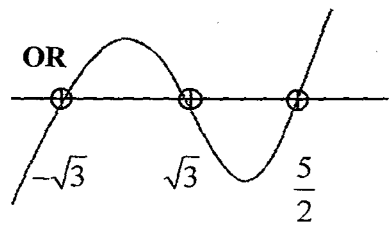
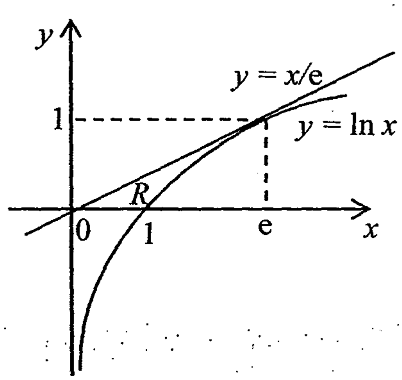
(ii) Use differentiation to show that the area of the trapezium is maximum when $x = 2k$ metres. [4]

It is given that $x = 2k$ metres and the length of the container is given as 3 metres. This container is filled with water at a constant rate of 0.2 m^3/s. At time t seconds the depth of water in the container is h metres as shown.

(iii) Show that the volume V of water in the container is given by $V = 3h\left(4k + \frac{h}{\sqrt{3}}\right)$. [2]

(iv) Find, in terms of k, the rate at which the depth is increasing at the instant when the depth is $k\sqrt{3}$ metres. [3]

- End of Paper -

Qn	Solution	Remarks
1	$\dfrac{2w-5}{w^2-3}>0$  OR  $-\sqrt{3}<w<\sqrt{3}$ or $w>\dfrac{5}{2}$ $\dfrac{(2\lvert y\rvert-5)\sin x}{y^2-3}\le 0$, Since $\sin x<0$ for $\pi<x\le\dfrac{3\pi}{2}$, $\dfrac{2\lvert y\rvert-5}{y^2-3}\ge 0$ From above, $-\sqrt{3}<\lvert y\rvert<\sqrt{3}$ or $\lvert y\rvert\ge\dfrac{5}{2}$ $0\le\lvert y\rvert<\sqrt{3}$ or $\lvert y\rvert\ge\dfrac{5}{2}$ $-\sqrt{3}<y<\sqrt{3}$ or $y\ge\dfrac{5}{2}$ or $y\le-\dfrac{5}{2}$	
2	 **Method 1 - Integration** Volume of solid S $=\pi\int_0^e\left(\dfrac{x}{e}\right)^2dx-\pi\int_1^e(\ln x)^2\,dx$ $=\dfrac{\pi}{e^2}\int_0^e x^2dx-\pi\int_1^e(\ln x)^2\,dx$ $=\dfrac{\pi}{e^2}\left[\dfrac{x^3}{3}\right]_0^e-\pi\int_1^e(\ln x)^2\,dx$ **Method 2 – Volume of Cone** Volume of solid S $=$ Vol of cone $-\pi\int_1^e(\ln x)^2\,dx$ $=\dfrac{1}{3}\pi(1)^2(e)-\pi\int_1^e(\ln x)^2\,dx$ $=\dfrac{1}{3}\pi e-\pi\int_1^e(\ln x)^2\,dx$	

	$=\frac{\pi}{e^2}\left[\frac{e^3}{3}\right]-\pi\int_1^e(\ln x)^2\,dx$ $=\frac{1}{3}\pi e-\pi\int_1^e(\ln x)^2\,dx$ $\int_1^e(\ln x)^2\,dx$ $=\left[x(\ln x)^2\right]_1^e-2\int_1^e \ln x\,dx$ $\quad u_1=(\ln x)^2 \quad \frac{dv_1}{dx}=1$ $=\left[x(\ln x)^2\right]_1^e-2\left(\left[x\ln x\right]_1^e-\int_1^e 1\,dx\right)$ $\quad \frac{du_1}{dx}=2(\ln x)(\frac{1}{x}) \quad v_1=x$ $=\left[x(\ln x)^2\right]_1^e-2\left[x\ln x\right]_1^e+2\int_1^e 1\,dx$ $\quad u_2=\ln x \quad \frac{dv_2}{dx}=1$ $=\left[x(\ln x)^2\right]_1^e-2\left[e\ln e-\ln 1\right]_1^e+2\left[x\right]_1^e$ $\quad \frac{du_2}{dx}=\frac{1}{x} \quad v_2=x$ $=\left[x(\ln x)^2\right]_1^e-2\left[e\ln e-\ln 1\right]+2\left[e-1\right]$ $=e-2e+2e-2$ $=e-2$ Hence, volume of solid S $=\frac{1}{3}\pi e-\pi(e-2)$ $=\frac{1}{3}\pi e-\pi e+2\pi$ $=2\pi-\frac{2}{3}\pi e$ $=\frac{2}{3}\pi(3-e)$	
3 (i)	$2\,500\,000\,000=1000\,000(2)^{n-1}$ $2^{n-1}=2500$ $n-1=\frac{\ln 2500}{\ln 2}=11.2877$ $n=12.2877$ His net worth will first exceed 2.5 billion when $n=13$ The year 1993 + (13 -1)(1) = 2005 or 1993 + 13 – 1 = 2005	
3 (ii)	$100\,000+1000+1000(1.5)+1000(1.5^2)+\ldots\ldots.(15\text{ terms})=$ $100\,000+\frac{1000(1.5^{15}-1)}{1.5-1}=\$\,973\,787.7808=\$\,973\,788$	
4 (a)	$g(x)=\frac{1}{\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)}=\frac{1}{\left(\frac{1}{2}\cos^2 x-\frac{1}{2}\sin^2 x\right)}=$	

	$\frac{1}{\frac{1}{2}(\cos^2 x-\sin^2 x)}=\frac{2}{\cos 2x}\approx\frac{2}{(1-\frac{(2x)^2}{2})}=\frac{2}{1-2x^2}=2(1-2x^2)^{-1}$ $\approx 2(1+2x^2)=2+4x^2$ m must be **sufficiently** small for $g(x)\approx 2+ax+bx^2$	
4 (b)	$(1-x^2)^{-1/2}=1+\frac{\left(-\frac{1}{2}\right)}{1}\left(-x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-x^2\right)^2+$ $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(-x^2\right)^3=1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+.....$ $\cos^{-1}x=\int\frac{-1}{\sqrt{1-x^2}}dx=-\int(1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+....)\,dx$ $\approx-(x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7)+C$ When $x=0$, $\cos^{-1}0=\frac{\pi}{2}=C$ $\cos^{-1}(x)=-(x+\frac{1}{6}x^3+\frac{3}{40}x^5)+\frac{\pi}{2}$	
5	$\frac{1}{3^r}\left(\frac{u_{r+1}}{3}-u_r\right)=2r$ $\sum_{r=1}^{n}\frac{u_{r+1}}{3^{r+1}}-\frac{u_r}{3^r}=2\sum_{r=1}^{n}r$ $\frac{u_2}{3^2}-\frac{u_1}{3^1}$ $+\frac{u_3}{3^3}-\frac{u_2}{3^2}$ $+...$ $+\frac{u_n}{3^n}-\frac{u_{n-1}}{3^{n-1}}$ $+\frac{u_{n+1}}{3^{n+1}}-\frac{u_n}{3^n}=2\left(\frac{n}{2}(1+n)\right)$ $\frac{u_{n+1}}{3^{n+1}}-\frac{1}{3}=n(n+1)$ $u_{n+1}=3^{n+1}\left(n(n+1)+\frac{1}{3}\right)=3^n\left(3n^2+3n+1\right)$	
6	Rate of change = rate of growth – rate of decrease Rate of change = rate of earning interest – rate of withdrawal $\frac{dM}{dt}=0.05M-4000$	

$$\int \frac{1}{0.05M-4000}\,dM = \int 1\,dt$$

$$\frac{1}{0.05}\ln|0.05M-4000| = t + C$$

$$\ln|0.05M-4000| = \frac{t}{20} + \frac{C}{20}$$

$$|0.05M-4000| = e^{\frac{t}{20}+\frac{C}{20}}$$

$0.05M - 4000 = Ae^{\frac{t}{20}}$ where $A = \pm e^{\frac{C}{20}}$

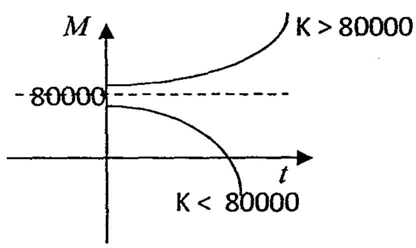
$$M = 80000 + 20Ae^{\frac{t}{20}}$$

When $t = 0$, $M = K$

$K = 80000 + 20A$

$20A = K - 80000$

Hence $M = 80000 + (K-80000)e^{\frac{t}{20}}$



Money is completely withdrawn if $K < 80000$

7(i) Let $x = 5\sin\theta$

$$\frac{dx}{d\theta} = 5\cos\theta$$

$$\int \sqrt{25-x^2}\,dx$$

$$= \int \sqrt{25-(5\sin\theta)^2}\,(5\cos\theta)\,d\theta$$

$$= \int \sqrt{25-25\sin^2\theta}\,(5\cos\theta)\,d\theta$$

$$= \int \sqrt{25(1-\sin^2\theta)}\,(5\cos\theta)\,d\theta$$

$$= \int \sqrt{25(\cos^2\theta)}\,(5\cos\theta)\,d\theta$$

$$= \int (5\cos\theta)^2\,d\theta$$

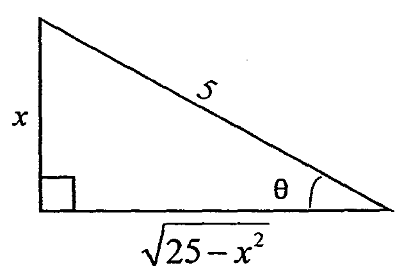
$$= 25\int \cos^2\theta\,d\theta$$

$$= \frac{25}{2}\int 1+\cos 2\theta\,d\theta$$

$$= \frac{25}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$$

$$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + c$$

$$= \frac{25}{2}\left(\sin^{-1}\frac{x}{5} + \left(\frac{x}{5}\right)\left(\frac{\sqrt{25-x^2}}{5}\right)\right) + c$$



Let $x = 5\sin\theta$

$$\sin\theta = \frac{x}{5}$$

$$\theta = \sin^{-1}\frac{x}{5}$$

$$\cos\theta = \frac{1}{5}\sqrt{25-x^2}$$

$$= \frac{25}{2}\sin^{-1}\frac{x}{5} + \frac{1}{2}x\sqrt{25-x^2} + c$$

7(ii)

$$x^2 + (y-b)^2 = 25$$

$$(y-b)^2 = 25 - x^2$$

Since $y < 0$, $y - b < 0$,

$$y - b = -\sqrt{25-x^2}$$

$$y = b - \sqrt{25-x^2}$$

Area of region R

$$= \left|\int_0^a b - \sqrt{25-x^2}\,dx\right|$$

$$= -\int_0^a b - \sqrt{25-x^2}\,dx$$

$$= \int_0^a \sqrt{25-x^2}\ dx - \int_0^a b\ dx$$

$$= \left[\frac{25}{2}\sin^{-1}\frac{x}{5} + \frac{1}{2}x\sqrt{25-x^2}\right]_0^a - [bx]_0^a$$

$$= \frac{25}{2}\sin^{-1}\frac{a}{5} + \frac{1}{2}a\sqrt{25-a^2} - ab$$

$$= \frac{1}{2}a\sqrt{25-a^2} + \frac{25}{2}\sin^{-1}\frac{a}{5} - a\sqrt{25-a^2}$$

$$= \frac{25}{2}\sin^{-1}\frac{a}{5} - \frac{1}{2}a\sqrt{25-a^2}.$$

Substituting $x = a$ and $y = 0$ into the equation

$$x^2 + (y-b)^2 = 25,$$

we have

$$a^2 + (0-b)^2 = 25$$

$$b = \sqrt{25-a^2}$$

8 (i)

$$z = k + i$$

$$z^2 = (k+i)^2 = k^2 + 2(k)(i) + (i)^2 = (k^2-1) + (2k)i$$

$$z^3 = (k+i)^3 = k^3 + 3(k)^2(i) + 3(k)(i)^2 + (i)^3$$

$$= (k^3-3k) + (3k^2-1)i$$

$$z^3 - iz^2 - 2z - 4i = 0$$

$$[(k^3-3k) + (3k^2-1)i] - i[(k^2-1) + (2k)i] - 2[k+i] - 4i = 0$$

$$[(k^3-3k) + 2k - 2k] + i[(3k^2-1) - (k^2-1) - 2 - 4] = 0$$

$$(k^3-3k) + i(2k^2-6) = 0$$

$$k(k^2-3) = 0 \text{ and } 2k^2 - 6 = 0$$

$$(k = 0 \text{ or } k = \pm\sqrt{3}) \text{ and } k = \pm\sqrt{3}$$

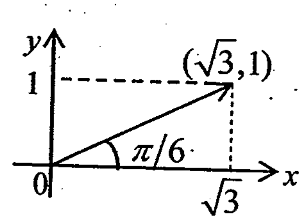
Hence, $k = \pm\sqrt{3}$

8 (ii)

$$z = \sqrt{3} + i \quad (\because k > 0)$$

$$|z| = \sqrt{1+3} = 2$$

$$\arg(z) = \frac{\pi}{6}$$



	Method 1: By Polar Form & Trigonometry $z = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $z^n = 2^n e^{in\pi/6} = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ z^n is real $\Leftrightarrow \sin\frac{n\pi}{6} = 0$ $\Leftrightarrow \frac{n\pi}{6} = k\pi$, where $k \in \mathbf{Z}$ $\Leftrightarrow n = 6k$, where $k \in \mathbf{Z}$ Hence, $n = 0, \pm 6, \pm 12, \pm 18, \ldots$ **Method 2: By Properties of arg(z)** $\arg(z^n) = n\arg(z) = \frac{n\pi}{6}$ z^n is real, the point representing z^n on the Argand diagram is on the x-axis. Thus, $\arg(z^n) = \frac{n\pi}{6} = k\pi$, where $k \in \mathbf{Z}$ $\therefore n = 6k$, where $k \in \mathbf{Z}$ i.e. $n = 0, \pm 6, \pm 12, \pm 18, \ldots$ Given $\lvert z^n \rvert > 100$. $\lvert z^n \rvert = \lvert z \rvert^n = 2^n$ Hence, $2^n > 100$ But n is a multiple of 6. We then have $2^6 = 64 < 100$ $2^{12} = 4096 > 100$ The least value of n is then 12.	
9	Let P_n be the statement $\sum_{r=1}^{n} \frac{3r+1}{r(r+1)(r+2)} = \frac{n(7n+9)}{4(n+1)(n+2)}$ for all integers $n \geq 1$. When $n = 1$, LHS $= \sum_{r=1}^{1} \frac{3r+1}{r(r+1)(r+2)} = \frac{3+1}{1(2)(3)} = \frac{2}{3}$ RHS $= \frac{7+9}{4(2)(3)} = \frac{2}{3} =$ LHS $\therefore P_1$ is true. Assume that P_k is true for **some** positive integer k, $k \geq 1$, i.e. $\sum_{r=1}^{k} \frac{3r+1}{r(r+1)(r+2)} = \frac{k(7k+9)}{4(k+1)(k+2)}$ Need to prove P_{k+1} is true,	

i.e. $\displaystyle\sum_{r=1}^{k+1}\frac{3r+1}{r(r+1)(r+2)}=\frac{(k+1)(7k+16)}{4(k+2)(k+3)}$.

$$\text{LHS of } P_{k+1}=\sum_{r=1}^{k+1}\frac{3r+1}{r(r+1)(r+2)}$$

$$=\sum_{r=1}^{k}\frac{3r+1}{r(r+1)(r+2)}+\frac{3(k+1)+1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(7k+9)}{4(k+1)(k+2)}+\frac{3k+4}{(k+1)(k+2)(k+3)}$$

$$=\frac{7k^3+30k^2+39k+16}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(7k^2+23k+16)(k+1)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)(7k+16)}{4(k+2)(k+3)}=\text{RHS}$$

$\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true.

Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by the Principle of Mathematical Induction, $\displaystyle\sum_{r=1}^{n}\frac{3r+1}{r(r+1)(r+2)}=\frac{n(7n+9)}{4(n+1)(n+2)}$ is true for all integers $n\geq 1$.

9 (i)

$$\frac{n(7n+9)}{4(n+1)(n+2)}=\frac{7}{4}-\frac{6n+7}{2(n+1)(n+2)}$$

Since n is a positive integer, $\dfrac{6n+7}{2(n+1)(n+2)}>0$

$$\therefore\sum_{r=1}^{n}\frac{3r+1}{r(r+1)(r+2)}=\frac{n(7n+9)}{4(n+1)(n+2)}=\frac{7}{4}-\frac{6n+7}{2(n+1)(n+2)}<\frac{7}{4}$$

Alternatively,

$$\sum_{r=1}^{n}\frac{3r+1}{r(r+1)(r+2)}=\sum_{r=1}^{n}\frac{1}{2r}+\frac{2}{r+1}-\frac{5}{2(r+2)}=....=\frac{7}{4}-\frac{1}{2(n+1)}-\frac{5}{2(r+2)}<\frac{7}{4}$$

Since n is a positive integer, $-\dfrac{1}{2(n+1)}-\dfrac{5}{2(n+2)}<0$.

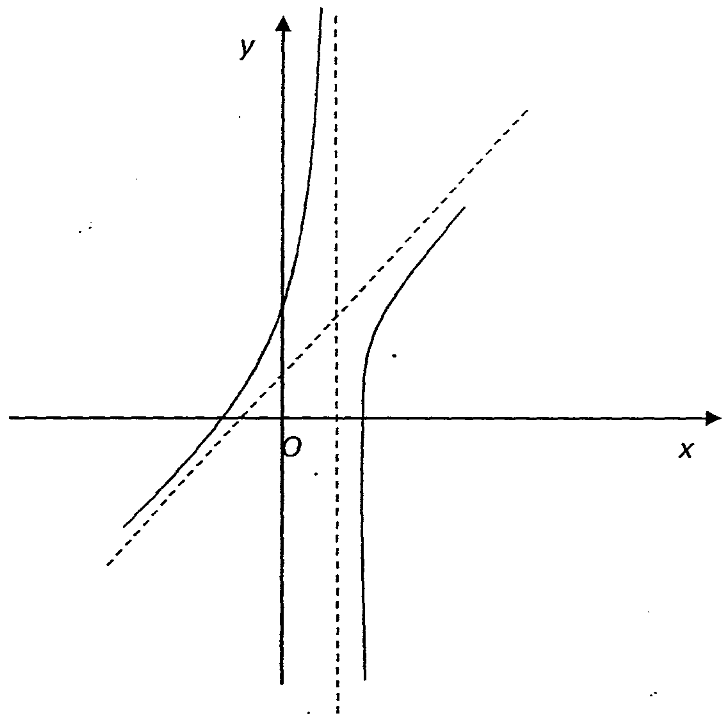
9 (ii)	$(r+1)^3 = r^3 + 3r^2 + 3r + 1$ $r(r+1)(r+2) = r^3 + 3r^2 + 2r$ $\therefore (r+1)^3 > r(r+1)(r+2)$ $\sum_{r=1}^{n} \frac{3r}{(r+1)^3} < \sum_{r=1}^{n} \frac{3r}{r(r+1)(r+2)} < \sum_{r=1}^{n} \frac{3r+1}{r(r+1)(r+2)} < \frac{7}{4}.$	
10	$y = \frac{x^2 - 4k^2}{x-k}$ where k is a constant such that $k \neq 0$ $xy - ky = x^2 - 4k^2$ $x^2 - xy + (ky - 4k^2) = 0$ x is real $\Rightarrow$ discriminant ≥ 0 $y^2 - 4\left(ky - 4k^2\right) \geq 0$ $y^2 - 4ky + 16k^2 \geq 0$ $\left(y - 2k\right)^2 + 12k^2 \geq 0$ This inequality is true for all values of y. Therefore y can take the set of all real numbers. *Alternative Method:* $\frac{dy}{dx} = 0$ $\frac{x^2 - 2xk + 4k^2}{\left(x-k\right)^2} = 0$ $x^2 - 2xk + 4k^2 = 0$ discriminant = $-12k^2 < 0$ no real roots. Hence, no turning pts. $y \in □$.	

10 (i)

$$y=\frac{x^2-4k^2}{x-k}=x+k-\frac{3k^2}{x-k}$$

Asymptotes: $y=x+k$ *and* $x=k$

Points of intercept with axes: $(0,4k)$, $(-2k,0)$, $(2k,0)$



10 (ii)

$$\int_{-1}^{1} \mathrm{f}\left(|x|\right)\,\mathrm{d}x=2\int_0^1 \mathrm{f}\left(x\right)dx=2\int_0^1 x+k-\frac{3k^2}{x-k}\,\mathrm{d}x$$

$$=2\left[\frac{x^2}{2}+kx-3k^2\ln|x-k|\right]_0^1$$

$$=2\left[\frac{1}{2}+k-3k^2\ln|1-k|+3k^2\ln|-k|\right]$$

$$=1+2k+6k^2\ln\frac{k}{k-1}.$$

10 (iii)

$$y=\frac{x^2-4k^2}{x-k}\rightarrow y=\frac{\left(\frac{x}{2}\right)^2-4k^2}{\left(\frac{x}{2}\right)-k}\rightarrow y=\frac{x^2-16k^2}{2x-4k}$$

$$\rightarrow y=\frac{\left(x+2k\right)^2-16k^2}{2\left(x+2k\right)-4k}=\frac{\left(x+2k\right)^2-16k^2}{2x}$$

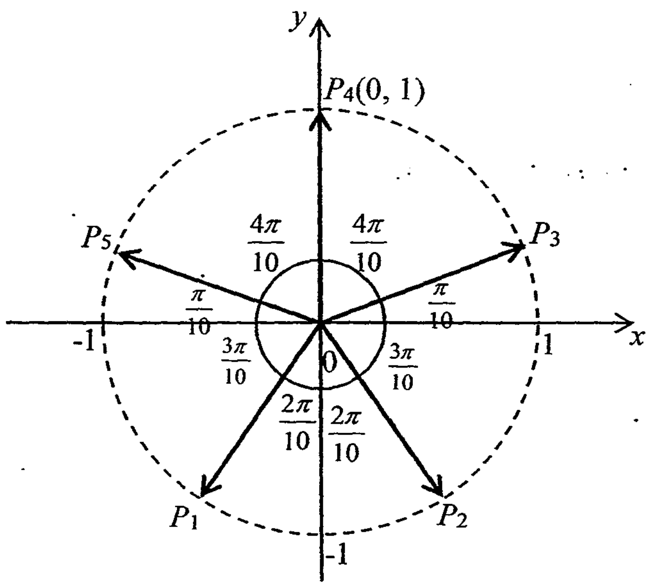
11 **(i)**	$\mathbf{a}\times\mathbf{b}=4\mathbf{a}\times\mathbf{c}$ $(\mathbf{a}\times\mathbf{b})-(4\mathbf{a}\times\mathbf{c})=0$ $(\mathbf{a}\times\mathbf{b})-(\mathbf{a}\times 4\mathbf{c})=0$ $\mathbf{a}\times(\mathbf{b}-4\mathbf{c})=0$ $\boldsymbol{a}$ is parallel to $\boldsymbol{b}-4\boldsymbol{c}$ $\boldsymbol{b}-4\boldsymbol{c}=\alpha\,\boldsymbol{a}$	
(ii)	$\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert=\sqrt{126}$ $\frac{1}{2}\lvert 4\mathbf{a}\times\mathbf{c}\rvert=\sqrt{126}$ $\lvert\mathbf{a}\times\mathbf{c}\rvert=\frac{\sqrt{126}}{2}$ $\left\lvert\left(\frac{\mathbf{b}-4\mathbf{c}}{\sqrt{3}}\right)\times\mathbf{c}\right\rvert=\frac{\sqrt{126}}{2}$ $\lvert(\mathbf{b}\times\mathbf{c})-(4\mathbf{c}\times\mathbf{c})\rvert=\frac{\sqrt{3}\sqrt{126}}{2}$ $\lvert(\mathbf{b}\times\mathbf{c})\rvert=\frac{\sqrt{378}}{2}$ Alternatively, $\lvert\mathbf{b}\times\mathbf{c}\rvert=\left\lvert\mathbf{b}\times\left(\frac{\mathbf{b}-\sqrt{3}\mathbf{a}}{4}\right)\right\rvert=\frac{1}{4}\lvert\mathbf{b}\times\mathbf{b}-\mathbf{b}\times\sqrt{3}\mathbf{a}\rvert$ $=\frac{\sqrt{3}}{4}\lvert\mathbf{b}\times\mathbf{a}\rvert=\frac{\sqrt{3}}{2}.\frac{1}{2}\lvert\mathbf{b}\times\mathbf{a}\rvert=\frac{\sqrt{3}}{2}.\sqrt{126}$ $\therefore\ \lvert(\mathbf{b}\times\mathbf{c})\rvert=\frac{\sqrt{378}}{2}$	
(iii)	Area of parallelogram with adjacent sides *OB* and *OC*.	
(iv)	$(\boldsymbol{b}-4\boldsymbol{c})\,.\,(\boldsymbol{b}-4\boldsymbol{c})=3\lvert\boldsymbol{a}\rvert^2$ $\lvert\boldsymbol{b}\rvert^2-8\,\boldsymbol{b}\,.\,\boldsymbol{c}+16\lvert\boldsymbol{c}\rvert^2=3\lvert\boldsymbol{a}\rvert^2$ $\boldsymbol{b}\,.\,\boldsymbol{c}=-\frac{10}{8}$ $\cos\theta=\frac{\boldsymbol{b}\,.\,\mathbf{c}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{c}\rvert}=\frac{-\frac{10}{8}}{1(2)}$ $\theta=128.7^\circ$	
12 **(i)**	$z^5-i=0$ $z^5=i$ $z^5=e^{i\pi/2}=e^{i(2k\pi+\frac{\pi}{2})}$, where $k\in\mathbf{Z}$ $z=e^{i(\frac{2k\pi}{5}+\frac{\pi}{10})}$ Putting $n=-2,-1,0,1,2$	

$z = e^{-\frac{7\pi}{10}i}, e^{-\frac{3\pi}{10}i}, e^{\frac{\pi}{10}i}, e^{\frac{\pi}{2}i}, e^{\frac{9\pi}{10}i}$

Given $-\pi < \arg(z_1) < \arg(z_2) < \arg(z_3) < \arg(z_4) < \arg(z_5) \le \pi$.

i.e. $z_1 = e^{-\frac{7\pi}{10}i}, z_2 = e^{-\frac{3\pi}{10}i}, z_3 = e^{\frac{\pi}{10}i}, z_4 = e^{\frac{\pi}{2}i}, z_5 = e^{\frac{9\pi}{10}i}$

Let the points P_1, P_2, P_3, P_4 and P_5 on the Argand diagram represents the complex numbers z_1, z_2, z_3, z_4, and z_5.



The locus $|z - z_2| = |z - z_3|$ is a perpendicular bisector of the line segment P_2P_3 where $P_2 \equiv z_2$ and $P_3 \equiv z_3$.

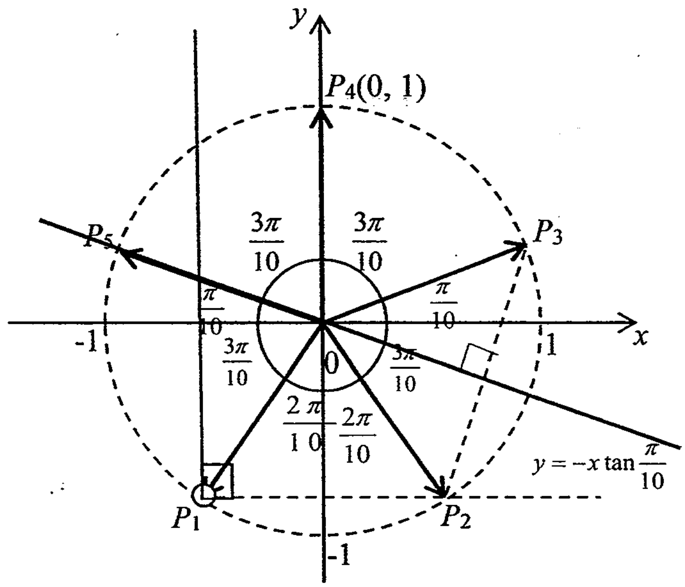
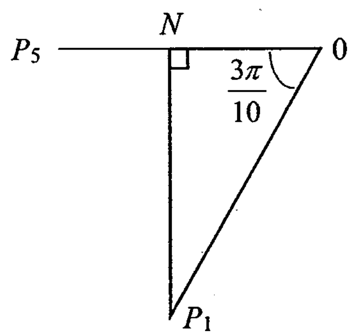
12 (ii) Since OP_2P_3 is an isosceles triangle, the perpendicular cuts through the origin and bisect the angle P_2OP_3.

Thus the perpendicular bisector is inclined at angle $\frac{\pi}{10}$ radian below the positive x-axis.

Hence, the Cartesian equation of the locus $|z - z_2| = |z - z_3|$ is

$y = -x\tan(\frac{\pi}{10})$ OR $y = x\tan(\frac{9\pi}{10})$

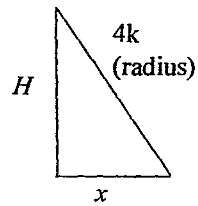
$\arg(z - z_1) = \arg(z_4)$

	 $z \equiv \overrightarrow{OP}$ $z_1 \equiv \overrightarrow{OP_1}$, $z - z_1 \equiv \overrightarrow{OP} - \overrightarrow{OP_1}$ i.e. $z - z_1 \equiv \overrightarrow{P_1P}$ The least value of $\lvert z - z_1 \rvert$ is the shortest distance from P_1 to the perpendicular bisector (line OP_5). $\frac{P_1N}{OP_1} = \sin\frac{4\pi}{10}$ $P_1N = OP_1 \sin\frac{4\pi}{10}$ $= \sin\frac{2\pi}{5} = 0.951$  	
12 (iii)	The locus $\arg(z - z_1) = \arg(z_4)$ is a half-line with its initial point at P_1 and above and excluding P_1, parallel to the y-axis. (Refer to diagram.)	
12 (iii)	The intersection point between the 2 loci has the same x-coordinates as the point P_1, i.e. $x = \cos\frac{-7\pi}{10} = -\cos\frac{3\pi}{10}$. Substituting $x = -\cos\frac{3\pi}{10}$ into the equation of the perpendicular bisector $y = -x\tan(\frac{\pi}{10})$, we have $y = -(-\cos\frac{3\pi}{10})\tan\frac{\pi}{10} = \cos\frac{3\pi}{10}\tan\frac{\pi}{10} = 0.191$ $x = -\cos\frac{3\pi}{10} = -0.588$ Coordinates of the intersection point are $(-0.588, 0.191)$.	

13 (i)

$$S=\frac{1}{2}H(2x+8k)=\frac{1}{2}(2x+8k)\sqrt{(4k)^2-x^2}$$

$$=(x+4k)\sqrt{16k^2-x^2}\ (shown)$$



(ii)

$$\frac{dS}{dx}=\sqrt{16k^2-x^2}-(x+4k)x\left(16k^2-x^2\right)^{-\frac{1}{2}}$$

$$=\frac{-2x^2-4kx+16k^2}{\sqrt{16k^2-x^2}}=0$$

$$x^2+2kx-8k^2=0$$

$$(x-2k)(x+4k)=0$$

$$\Rightarrow x=2k \quad or \quad x=-4k\ (\text{N.A}, x>0)$$

$$\frac{dS}{dx}=\frac{-2x^2-4kx+16k^2}{\sqrt{16k^2-x^2}}=\frac{-2(x^2+2kx-8k^2)}{\sqrt{16k^2-x^2}}$$

$$\frac{d^2S}{dx^2}=-2\left[\frac{\sqrt{16k^2-x^2}\,(2x+2k)-(x^2+2kx-8k^2)\frac{1}{2}\left(16k^2-x^2\right)^{-\frac{1}{2}}(-2x)}{16k^2-x^2}\right]$$

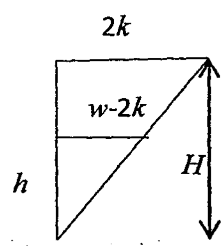
when $x=2k$

$$\frac{d^2S}{dx^2}=-2\frac{\sqrt{12k^2}\,6k}{12k^2}=-\sqrt{12}<0$$

Area of trapezium is maximum when $x=2k$.

(iii)

Using similar triangles:



$$\frac{h}{H}=\frac{w-2k}{2k}$$

$$When\ x=2k, H=\sqrt{12k^2}=2\sqrt{3}k$$

$$w=\frac{2kh}{H}+2k=\frac{h}{\sqrt{3}}+2k$$

$$\therefore V=\frac{3}{2}h(4k+2w)=\frac{3}{2}h\left(4k+2\left(\frac{h}{\sqrt{3}}+2k\right)\right)$$

$$=3h\left(4k+\frac{h}{\sqrt{3}}\right)(shown)$$

(iv)	$\frac{dV}{dh}=\left(4k+\frac{h}{\sqrt{3}}\right)3+3h\left(\frac{1}{\sqrt{3}}\right)$ $=12k+\frac{6h}{\sqrt{3}}$ *When* $h=\sqrt{3}k,\ \frac{dV}{dh}=18k$ $\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt}=\frac{0.2}{18k}=\frac{1}{90k}m/s$	

MATHEMATICS DEPARTMENT

MATHEMATICS
Higher 2

9740 / 02

Paper 2

22 August 2016

JC 2 PRELIMINARY EXAMINATION

Time allowed: **3 hours**

Additional Materials: List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Index number, Form Class, graphic and/or scientific calculator model/s on the cover page.
Write your Index number and full name on all the work you hand in.
Write in dark blue or black pen on your answer scripts.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in the question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 8 printed pages.

[Turn Over

MATHEMATICS DEPARTMENT
JC2 Preliminary Examination 2016

MATHEMATICS 9740
Higher 2
Paper 2

/ 100

Index No: ☐☐☐☐ Form Class: ____________

Name: ______________________________

Calculator model: ________________

Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

Question No.	Marks
1	/3
2	/14
3	/8
4	/15
5	/3
6	/5
7	/10
8	/8
9	/15
10	/9
11	/10

Summary of Areas for Improvement			
Knowledge (K)	Careless Mistakes (C)	Read/Interpret Qn wrongly (R)	Presentation (P)

Section A: Pure Mathematics [40 marks]

1 A triangular region R is drawn on a large sheet of graphing paper marked in 1mm squares. The region R is bounded by the x-axis, the line $y = \frac{1}{20}x$ and the line $y = -\frac{1}{20}x + 40$. The scales on both the axes are such that 1 mm represents 1 unit.

By using the table below or otherwise, find the number of complete 1mm squares which lie inside region R. [3]

Range of x	Number of complete 1 mm squares which lies inside region R
$20 \le x \le 40$	
$40 \le x \le 60$	
$60 \le x \le 80$	
⋮	

2 The parametric equations of curve C are

$$x = a\cos^3 t, \quad y = a\sin^3 t \quad \text{for } 0 \le t \le \pi,$$

where a is a positive constant.

(i) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis and the coordinates of the points at which the tangent is parallel to the y-axis. [5]

(ii) Hence sketch the curve C. [1]

(iii) If the cartesian equation of curve C is $y = \mathrm{f}(x)$, use the curve in part **(ii)** to sketch the graphs of

(a) $y = \dfrac{1}{\mathrm{f}(x)}$, **(b)** $y = \mathrm{f}'(x)$,

stating in each case, the equations of any asymptotes and the coordinates of any points of intersection with the axes. [4]

(iv) The point P on the curve has parameter p. Show that the equation of tangent at P is $x\sin p + y\cos p = a\sin p\cos p$. [2]

(v) The tangent at P is perpendicular to the tangent at another point Q, on the curve. If $p = \dfrac{\pi}{3}$, find the value of the parameter t at point Q. [2]

3 The functions h and g are defined by

$$\mathrm{h}: x \mapsto \mathrm{e}^{|2x+1|}+1,\ x \in \mathbb{R},\ x \le k,$$

$$\mathrm{g}: x \mapsto \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{for } \frac{1}{2} \le x \le 1. \end{cases}$$

(i) Given that the function h has an inverse, state the greatest value of k. Find $\mathrm{h}^{-1}(x)$ and write down the domain of h^{-1}. [4]

(ii) Explain why the composite function gg exist. [1]

(iii) Sketch the graph of $y = \mathrm{gg}(x)$. [2]

(iv) Given that $\mathrm{f}: x \mapsto \mathrm{e}^{|2x+1|}+1,\ x \in \mathbb{R}$, find the range of fg exactly. [1]

4 The equations of three planes p_1, p_2, p_3 are

$$\begin{aligned} 2x+3y-6z&=10, \\ -2x-3y+6z&=a, \\ x+\ y+bz&=5, \end{aligned}$$

respectively, where a, b are constants.

The planes p_1 and p_3 intersect in the line l with cartesian equation $\frac{5-x}{3}=\frac{y}{4}=z$.

(i) Show that $b=-1$. [2]

(ii) The point S lies on p_1 and the point R has coordinates $(-2, 4, 1)$. Given that RS is perpendicular to p_3, find the coordinates of S. [4]

The planes p_1 and p_2 are $\frac{8}{7}$ units apart.

(iii) Given that $a<0$, find the possible values of a. [4]

(iv) The point P with coordinates $(5, 2, c)$ lies on p_1. Find the value of c. [1]

(v) The point F is the foot of the perpendicular from P to the line l. The point Q is the reflection of F in the plane p_2. Find the distance PF and hence find the area of triangle FPQ. [4]

Section B: Statistics [60 marks]

5 Florida fitness club wants to carry out a survey to find out from their members the facilities that the club can improve on. The club has a list of all the 15000 members' names.

(i) Describe how to obtain a systematic sample of 500 members from the list to take part in the survey. [2]

(ii) State one disadvantage of using a systematic sample in this context. [1]

6 A factory manufactures rectangular glass panels. The length and breadth of each panel, in cm, are modelled as having independent normal distributions with means and standard deviations as shown in the table.

Glass Panel	Mean (cm)	Standard Deviation (cm)
Length	300	0.5
Breadth	150	0.2

The probability that the total perimeter of 2 randomly selected glass panels exceeds the mean length of n randomly selected glass panels by more than 1501cm is less than 0.2576. Find the least value of n. [5]

7 The mean number of guests checking into a hotel in an hour is 3.6 and can be modelled by a Poisson distribution.

(i) Find the probability that not more than 4 guests checked into the hotel in a given hour. [1]

(ii) Given that three non-overlapping one-hour blocks are chosen at random, find the probability that one of the blocks has not more than 4 guests checking into the hotel and the remaining two blocks have no guests checking into the hotel. [2]

(iii) Given that each day consists of 24 non-overlapping one-hour blocks. Use a suitable approximation, to find the probability that between 85 and 90 guests checked into the hotel in a particular day. State the parameters of the distribution that you use. [3]

(iv) The probability of at least n one-hour blocks in a day of 24 non-overlapping one-hour blocks has not more than 4 guests checking into the hotel is less than 0.124. Find the least value of n. [3]

(v) Explain why the Poisson distribution may not be a good model for the number of guests checking into the hotel in a year. [1]

8 Tandao Café has an outlet at North Vista and another outlet at South Parc. On weekdays, the waiting time during lunch periods in each outlet follows a normal distribution with mean μ minutes.

(i) Hono has lunch regularly at the North Vista outlet. On 10 randomly selected weekdays, his waiting times per visit were recorded, in minutes, as follows:

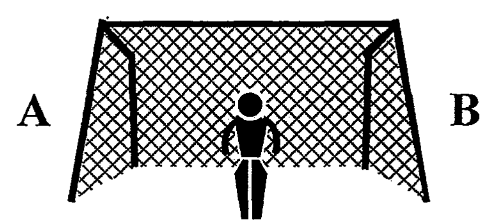
49, 38, 43, 70, 45, 51, 57, 85, 39, 44

Test, at the 10% significance level, whether the mean waiting time is less than one hour. [4]

(ii) Lulu has lunch regularly at the South Parc outlet. On 56 randomly selected weekdays, her waiting times per visit were recorded, in minutes. It was found that the sample mean waiting time is $\bar{t}$ minutes and the sample variance is 69.8 minutes^2. A test is to be carried out at the 5% level of significance to determine whether the average waiting time at the South Parc outlet is not one hour.

Find the range of values of $\bar{t}$ for which the result of the test would be that the null hypothesis is rejected, leaving your answers in 2 decimal places. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [4]

9



(a) The diagram above shows a goalkeeper and a front view of a goal post labelled A on one side and B on the other side. Aaron is a goalkeeper in a football club. Based on his past experiences as a goalkeeper in penalty shoot-outs, the probability that he dives to side B is 0.72. In a particular match, Aaron's team went into a penalty shoot-out. The probability that a penalty kicker kicks the ball to side B is p, where $0 < p < 1$. Assume that Aaron's choice of direction to dive is independent of the penalty kicker's choice of direction to kick the ball.

(i) Show that the probability Aaron dives in the same direction as the ball is kicked is $0.44p + 0.28$. [1]

(ii) If Aaron dives in the same direction as the ball is kicked, the probability that he saves the ball is 0.4. Find, in terms of p, the probability that Aaron fails to save the ball. [3]

Hence find the values between which this probability must lie. [1]

(b) In any football match, the expected number of saves made by Aaron is 4.8, with a standard deviation of 1.2. Find the probability that in 50 matches, the average number of saves per match made by Aaron is less than 5. [2]

(c) Aaron can either be a goalkeeper or a forward in a match, depending on the strategy that the team uses. In the team, there is another goalkeeper, 6 defenders, 5 midfielders and 5 forwards. Altogether, there are 18 members in the team.

(i) Find the number of ways to select 4 defenders, 4 midfielders, 2 forwards and a goalkeeper from the team, given that Aaron is selected. [3]

After a match, the team stands in a straight line to take a photo.

(ii) Find the number of ways such that all the defenders are standing alternately with all the midfielders. [2]

The team is then asked to sit in a circle for a debrief.

(iii) Find the probability that the two goalkeepers sit opposite each other given that a group consisting of 2 particular midfielders and 6 defenders are seated together. [3]

10 **(i)** Sketch a scatter diagram that might be expected when h and s are related approximately as given in each of the models (A) and (B) below. In each model, your diagram should include 6 points, approximately equally spaced with respect to h, and all h- and s-values positive. The letters a, b, c and d represent constants.

(A) $s = a + b\ln h$, where a is negative and b is positive.

(B) $s = c + \dfrac{d}{h}$, where c is positive and d is negative. [2]

A company recently launched a new product in Singapore and wanted to know more about the relationship between the number of promoters, h, and the product's monthly sales, s, in Singapore dollars. They collected data for the past 9 months and the results are given in the table.

h	50	60	70	80	90	100	110	120	130
s	40 000	47 000	52 000	55 000	57 800	60 000	61 500	62 500	63 000

(ii) Draw a scatter diagram for these values, labelling the axes. [1]

(iii) Comment on whether a linear model would be appropriate, referring to both the scatter diagram and the context of the question. [2]

(iv) It is required to estimate the number of promoters needed to achieve a monthly sales of \$75,000. Using an appropriate model in part **(i)** to find the equation of the suitable regression line, correct to 3 decimal places. Use your equation to find the required estimate, correct to the nearest integer. [2]

(v) Comment on the reliability of your estimate. [1]

(vi) Given that 1 US dollar = 1.34 Singapore dollars, re-write your equation from part **(iv)**, correct to 3 decimal places, so that it can be used to estimate the number of promotors when the product's monthly sales is given in US dollars. [1]

11 The manager of a car show room wants to study the number of cars sold by the 2 car salesman under his charge. The number of potential car-buyers that they meet in a particular week and the average probabilities that each salesman is successful in closing a deal with each customer is given in the table below.

Salesman	Number of potential car-buyers	Probability of closing a deal
X	60	0.2
Y	50	0.3

(i) It is assumed that the deals closed are independent of one another. State, in context, another assumption needed for the number of deals closed by a car salesman to be well modelled by a binomial distribution. [1]

(ii) Explain why the assumption that the deals closed are independent of one another may not hold in this context. [1]

Assume now that the assumptions stated in part **(i)** do in fact hold and the deals closed by salesman X is independent of the deals closed by salesman Y.

(iii) Use suitable approximations to find the probability that both salesmen collectively closed a total of more than 20 deals in a particular week. State the parameters of the distributions that you use. [4]

A new salesman joined the company. During his probation week, he met 60 potential car-buyers. The number of car deals he closed during his probation week is denoted by C with the distribution $B(60, p)$.

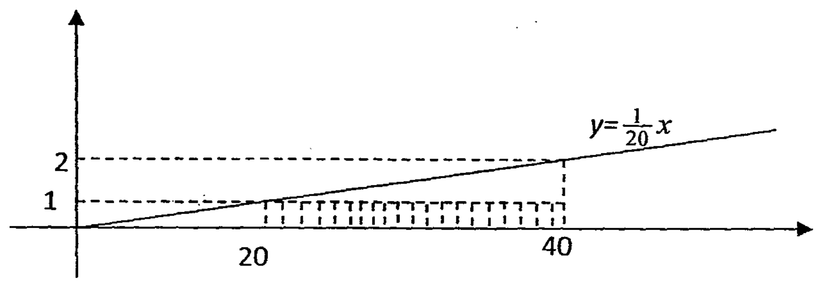
(iv) Given that $P(C = 30) = 0.03014$. Find an equation for p. Hence find the value of p, correct to 1 decimal place, given that $p < 0.5$. [2]

(v) Given that $p = 0.05$, use a suitable approximation, which should be stated, to find the probability that he sold more than 4 cars. [2]

- End of Paper -

Qn	Solutions	Remarks

1



Number of squares for $20 \le x \le 40$ is 20

Column	Range of x	Number of squares
1	$20 \le x \le 40$	20
2	$40 \le x \le 60$	40
3	$60 \le x \le 80$	60
n	$380 \le x \le 400$	

Number of columns $= n = \dfrac{380}{20} = 19$ for $0 \le x \le 400$

OR $380 = 20 + (n\text{-}1)(20) \Rightarrow n = 19$

OR $400 = 40 + (n\text{-}1)(20) \Rightarrow n = 19$

AP sequence : 20, 40, 60, …….. 19 terms

Number of complete squares in region R for $0 \le x \le 400$ is

$$\frac{19}{2}\left[2(20)+18(20)\right] = 3800$$

Region R is symmetrical about the line $x = 400$.

Total number of squares in region $R = 3800 \times 2 = 7600$

2(i)

$x = a\cos^3 t,\quad y = a\sin^3 t \quad for\ \ 0 \le t \le \pi$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

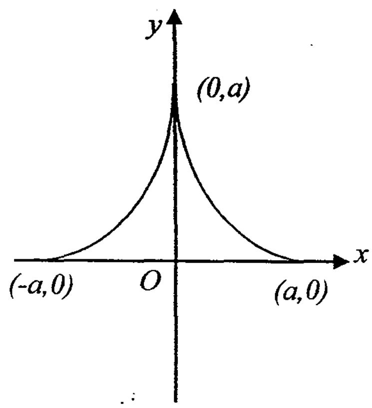
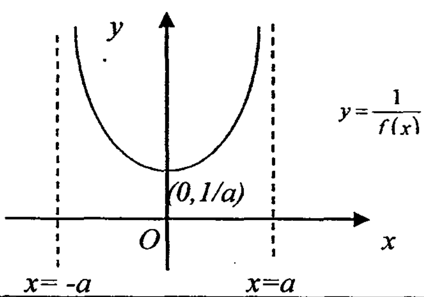
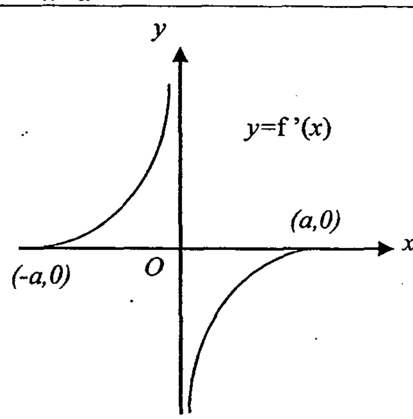
$Let\ \dfrac{dy}{dx} = 0$

$-\tan t = 0 \Rightarrow t = 0, \pi$

Points on curve where tangent is parallel to x-axis are $(a,0)$ and $(-a,0)$

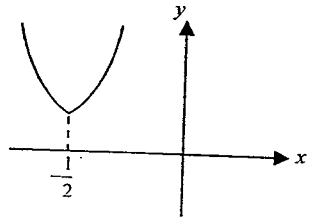
Let $\dfrac{dx}{dy} = 0 \Rightarrow \dfrac{1}{-\tan t} = 0 \Rightarrow t = \dfrac{\pi}{2}$

Point on curve where tangent is parallel to y-axis are $(0,a)$

2 (ii)		
2 (iii) (a)		
2 (iii) (b)		
2 (iv)	Equation of tangent at point *P* with parameter *p:* $y-a\sin^3 p=-\tan p(x-a\cos^3 p)$ $y\cos p-a\sin^3 p\cos p=-\sin p(x-a\cos^3 p)$ $x\sin p+y\cos p=(a\sin p\cos p)(\cos^2 p+\sin^2 p)$ $x\sin p+y\cos p=a\sin p\cos p$ (*shown*)	
2 (v)	Given $p=\frac{\pi}{3}$, gradient of tangent at P is $-\tan\frac{\pi}{3}=-\sqrt{3}$ $\therefore$ gradient of tangent at Q $=\frac{1}{\sqrt{3}}$ Value of parmeter at Q: $-\tan t=\frac{1}{\sqrt{3}}\Rightarrow t=\frac{5\pi}{6}$	

3

(i) Sketch y=f(x)



h(x) has an inverse if $x \le -\frac{1}{2}$. Greatest value of k is $-\frac{1}{2}$

$y-1 = e^{|2x+1|}$

$\ln(y-1) = |2x+1|$

$2x+1 = \pm\ln(y-1)$

$x = -\frac{1}{2} \pm \frac{1}{2}\ln(y-1)$

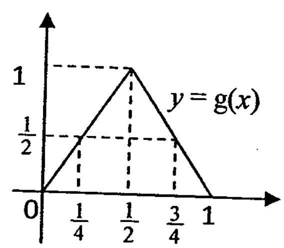
$x = -\frac{1}{2} - \frac{1}{2}\ln(y-1)$ since $x \le -\frac{1}{2}$

$h^{-1}(x) = -\frac{1}{2} - \frac{1}{2}\ln(x-1)$

$h(-\frac{1}{2}) = 2$

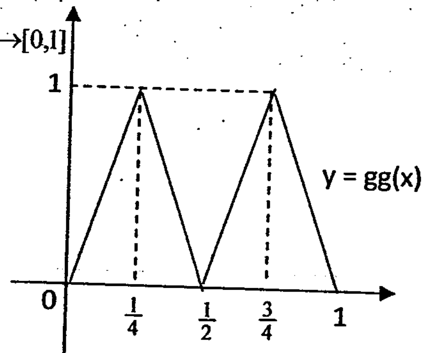
Domain of $h^{-1}(x)$ = Range of h(x) = $[2, \infty)$

(ii)



Composite function exist because
$R_g = [0,1] \subseteq D_g = [0,1]$

(iii) $[0,\frac{1}{4}] \xrightarrow{g} [0,\frac{1}{2}] \xrightarrow{g} [0,1]$



$[\frac{1}{4},\frac{1}{2}] \xrightarrow{g} [\frac{1}{2},1] \xrightarrow{g} [0,1]$

$[\frac{1}{2},\frac{3}{4}] \xrightarrow{g} [\frac{1}{2},1] \xrightarrow{g} [0,1]$

$[\frac{3}{4},1] \xrightarrow{g} [0,\frac{1}{2}] \xrightarrow{g} [0,1]$

	(iv) $[0,1] \xrightarrow{g} [0,1] \xrightarrow{f} [e+1, e^3+1]$ $R_{fg} = [e+1, e^3+1]$	
4 (i)	$\begin{pmatrix}1\\1\\b\end{pmatrix}.\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$ $-3+4+b=0$ $b=-1$ **Alternatively,** $\begin{pmatrix}1\\1\\b\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=k\begin{pmatrix}-3\\4\\1\end{pmatrix}$ $\begin{pmatrix}-6-3b\\6+2b\\1\end{pmatrix}=k\begin{pmatrix}-3\\4\\1\end{pmatrix}$ Equating **k** component, $k=1$. Equating **i** component, $-6-3b=-3k$ $\therefore -3b=-3(1)+6$ $b=-1$ (Shown)	
4 (ii)	Line *RS*: $\mathbf{r}=\begin{pmatrix}-2\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$, $\lambda \in \square$ Substitute this line to p_1, $\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}.\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$ $\lambda=\frac{8}{11}$ $\overrightarrow{OS}=\begin{pmatrix}-2+\lambda\\4+\lambda\\1-\lambda\end{pmatrix}=\begin{pmatrix}-\frac{14}{11}\\\frac{52}{11}\\\frac{3}{11}\end{pmatrix}$	

4 (iii)	$p_1: \quad r.\dfrac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\dfrac{10}{7}$ $p_2: \quad r.\dfrac{\begin{pmatrix}2\\3\\-6\end{pmatrix}}{7}=\dfrac{-a}{7}$ distance between two planes : $\dfrac{8}{7}=\dfrac{10-(-a)}{7}$ or $\dfrac{8}{7}=\dfrac{(-a)-10}{7}$ $a=-2$ or $a=-18$	
4 (iv)	$p_1: \quad \begin{pmatrix}5\\2\\c\end{pmatrix}.\begin{pmatrix}2\\3\\-6\end{pmatrix}=10$ $10+6-6c=10$ $c=1$	
4 (v)	Let point *A* be (5, 0, 0) which is a point on line *l*. $PF=\dfrac{\left\|\overline{AP}\times\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\|}=\dfrac{\left\|\begin{pmatrix}0\\2\\1\end{pmatrix}\times\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\|}=\dfrac{\left\|\begin{pmatrix}-2\\-3\\6\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-3\\4\\1\end{pmatrix}\right\|}$ $=\dfrac{7}{\sqrt{26}}=1.3728$ $QF=2\left(\dfrac{8}{7}\right)=\dfrac{16}{7}$ Area $PFQ=\dfrac{1}{2}(1.3728)\left(\dfrac{16}{7}\right)=1.5689=1.57$	

	Alternative method: $\overrightarrow{OF}=\begin{pmatrix}5-3\lambda\\4\lambda\\\lambda\end{pmatrix}$ $\overrightarrow{FP}=\overrightarrow{OP}-\overrightarrow{OF}=\begin{pmatrix}3\lambda\\2-4\lambda\\1-\lambda\end{pmatrix}$ Since $\overrightarrow{FP}\perp\begin{pmatrix}-3\\4\\1\end{pmatrix}$, $\begin{pmatrix}3\lambda\\2-4\lambda\\1-\lambda\end{pmatrix}.\begin{pmatrix}-3\\4\\1\end{pmatrix}=0$ $-9\lambda+8-16\lambda+1-\lambda=0$ $\lambda=\frac{9}{26}$ $\therefore\left\|\overrightarrow{FP}\right\|=\left\|\begin{pmatrix}(3)\left(\frac{9}{26}\right)\\2-(4)\left(\frac{9}{26}\right)\\1-\left(\frac{9}{26}\right)\end{pmatrix}\right\|=\left\|\begin{pmatrix}\frac{27}{26}\\\frac{8}{13}\\\frac{17}{26}\end{pmatrix}\right\|=1.3728$ $QF=2\left(\frac{8}{7}\right)=\frac{16}{7}$ Area $PFQ=\frac{1}{2}(1.3728)\left(\frac{16}{7}\right)=1.5689=1.57$	
5 (i)	Number the club members in order from 1 to 15000 according to the name list. (Alphabetical order) Since $k=\frac{15000}{500}=30$, select a member randomly from the name list. Thereafter, select every 30th member cycling to the start of the list if the end of list is reached until we form a sample of 500 members.	
5(ii)	There is a bias when the name list of the members of the fitness club have a **periodic** or **cyclic** pattern i.e. there is some pattern in the way the names is arranged and the pattern coincides in some way with the sampling interval of 30.	
6	Let the random variable L be the length of a randomly chosen glass panel. Let the random variable B be the breadth of a randomly	

	chosen glass panel. Let $X = 2L_1 + 2B_1 + 2L_2 + 2B_2$ $\therefore X \sim N(1800, 2.32)$ Let $\bar{L} = \dfrac{L_1 + L_2 + L_3 + ... + L_n}{n}$ $\therefore \bar{L} \sim N\left(300, \dfrac{0.5^2}{n}\right)$ $X - \bar{L} \sim N\left(1500, 2.32 + \dfrac{0.5^2}{n}\right)$ $P(X - \bar{L} > 1501) < 0.2576$ $P\left(Z > \dfrac{1501-1500}{\sqrt{2.32 + \dfrac{0.5^2}{n}}}\right) < 0.2576$ $P\left(Z < \dfrac{1}{\sqrt{2.32 + \dfrac{0.5^2}{n}}}\right) > 0.7424$ $\dfrac{1}{\sqrt{2.32 + \dfrac{0.5^2}{n}}} > 0.6507622042$ $\sqrt{2.32 + \dfrac{0.5^2}{n}} < 1.536659618$ $2.32 + \dfrac{0.5^2}{n} < 2.361322781$ $\dfrac{0.5^2}{n} < 0.0413227807$ $n > 6.049931679$ Least $n = 7$.	
7 (i)	Let the random variable X be the number of guests checking into the hotel in a given hour. $P(X \leq 4) = 0.7064384499 = 0.706$ (3s.f)	

7 (ii)	Required probability $= \left[\mathrm{P}(X \le 4)\right] \times \left[\mathrm{P}(X=0)\right]^2 \times \frac{3!}{2!}$ $= 0.0015822508$ $= 0.00158$ (3s.f)	
7 (iii)	Let the random variable Y be the number of guests checking into the hotel in a day. $Y \sim P_O(86.4)$ Since $\lambda > 10$, $Y \sim N(86.4, 86.4)$ approx. $\mathrm{P}(85 < X < 90) \overset{c.c}{\approx} \mathrm{P}(85.5 < X < 89.5) = 0.169$ (3s.f)	
7 (iv)	Let the random variable W be the number of one-hour blocks in a day, which has not more than 4 guests checking into the hotel. $W \sim B(24, 0.7064384499)$ $\mathrm{P}(W \ge n) < 0.124$ $\mathrm{P}(W \le n-1) > 0.876$ Using G.C, $\mathrm{P}(W \le 19) - 0.876 > -2.797 \times 10^{-4}$ $\mathrm{P}(W \le 20) - 0.876 > 0.7534$ $\therefore n-1 = 20$ Least $n = 21$.	
7 (v)	This is because the **mean** number of guests checking into the hotel **per hour** is **unlikely to be constant throughout the year.** The number of guests checking into the hotel is likely to vary across different months in a year due to **seasonal fluctuations** caused by factors such as the **holiday seasons**, etc. Hence a Poisson distribution may not be a good model.	
8 (i)	Let random variable X be a randomly chosen Hono's lunch waiting time at North Vista outlet. To test $H_0: \mu = 60$ Against $H_1: \mu < 60$ at 10% level of significance. Under H_0, $T = \frac{\overline{X} - \mu_0}{s/\sqrt{n}} \sim t(n-1)$, i.e. $T = \frac{\overline{X} - 60}{s/\sqrt{10}} \sim t(9)$	

	Value of test statistic: $t = \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}}$ $= \dfrac{52.1 - 60}{14.9328 / \sqrt{10}} = -1.6729597$ $p\text{-value} = = 0.064332 < 0.1$ $\therefore$ Reject H_0 and conclude that there is sufficient evidence at 10% level of significance that the average waiting time during lunch periods at North Vista is less than one hour.	
8 (ii)	Unbiased estimate of population variance is $s^2 = \left(\dfrac{n}{n-1}\right)(\text{sample variance}) = \left(\dfrac{56}{55}\right)(69.8) = 71.06909$ To test $H_0: \ \mu = 60$ Against $H_1: \ \mu \neq 60$ at 5% level of significance. Under H_0, $Z = \dfrac{\overline{T} - 60}{s / \sqrt{56}} \sim N(0,1)$ *approx.*, Value of test statistic: $z = \dfrac{\overline{t} - 60}{\sqrt{71.06909} / \sqrt{56}} = \dfrac{\overline{t} - 60}{1.12654}$ Since the null hypothesis is rejected, $\dfrac{\overline{t} - 60}{1.12654} < -1.959964$ or $\dfrac{\overline{t} - 60}{1.12654} > 1.959964$ $\overline{t} < 57.792$ or $\overline{t} > 62.208$	
9 (a) (i)	$p(0.72) + (1-p)(0.28) = 0.44p + 0.28$	
(ii)	$(0.44p + 0.28)(0.6) + 1 - (0.44p + 0.28)$ $= 0.888 - 0.176p$ Since $0 < p < 1$, $0 > -0.176p > -0.176$ $0.888 > 0.888 - 0.176p > 0.888 - 0.176$ $0.888 > 0.888 - 0.176p > 0.712$ $\therefore 0.712 < 0.888 - 0.176p < 0.888$	

(b) Since n is large, by Central Limit Theorem,

$$\overline{X}=\frac{X_1+X_2+\ldots+X_{50}}{50}\sim N\left(4.8,\ \frac{1.44}{50}\right)\text{ approx.}$$

Required probability $=\mathrm{P}\left(\overline{X}<5\right)=0.8807035=0.881$ (3s.f)

(ci) If Aaron is the goalkeeper: ${}^{6}C_{4}\,{}^{5}C_{4}\,{}^{5}C_{2}=750$

If Aaron is the forward: ${}^{6}C_{4}\,{}^{5}C_{4}\,{}^{5}C_{1}=375$

Number of ways required = 1125

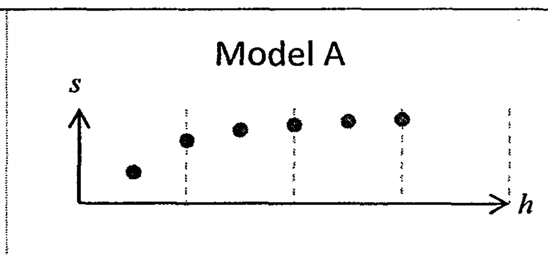
(cii) Number of ways required = $5!6!8!=3483648000$

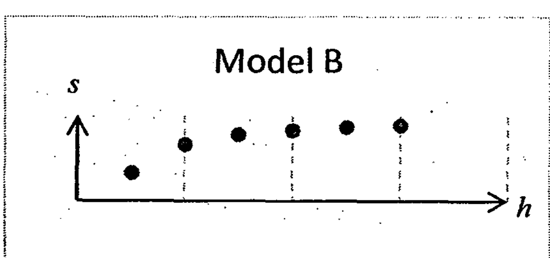
(ciii) Required probability

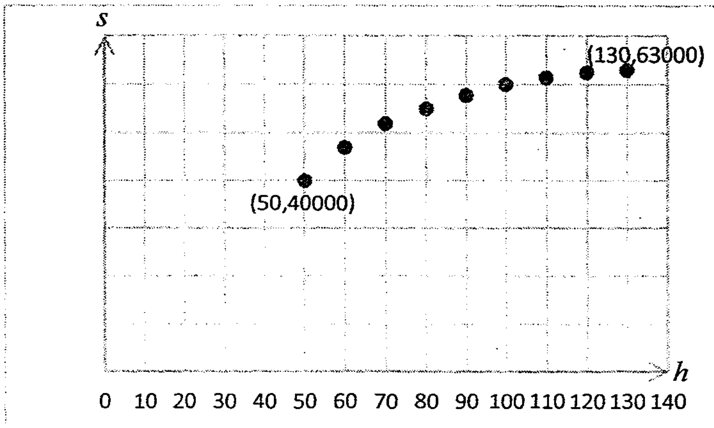
$$=\frac{\mathrm{P}(\text{goalkeepers are opposite \& defenders with 2 particular midfielders are together})}{\mathrm{P}(\text{defenders with 2 particular midfielders are together})}$$

$$=\frac{8!2!8!}{(11-1)!8!}=\frac{2!}{10\times 9}=\frac{1}{45}$$

10 (i)





10 (ii)		
10 (iii)	A Linear model <u>will not</u> be approriate. This is because the scatter diagram indicates that as h increases, s is <u>increasing at a decreasing rate</u> which is not a linear realtionship. Furthermore, a linear model will mean that the product's monthly sales in Singapore will increase indefinitely with the increase of the number of promoters. This is not realistic in the context of the question as the product's monthly sales will likely slow down and perhaps decrease due to market stauration.	
10 (iv)	Correlation coefficent of s on $\ln(h) = 0.981$ Correlation coefficent of s on $1/h = -0.998$ Since correlation coefficent of s on $1/h$ is stronger, hence use least square regression line of s on $1/h$. Least square regression line of s on $1/h$: $s = 78531.62777 - \frac{1896285.284}{h}$ $s = 78531.628 - \frac{1896285.284}{h}$ (3 d.p) when $s = 75000$, $75000 = 78531.62777 - \frac{1896285.284}{h}$ $h = 536.9437001 \approx 537$ (nearest integer).	

10 (v)	The estimate is not reliable as s=75000 does not lie within $40000 \le s \le 63000$. Hence, we are extrapolating.	
10 (vi)	$s = 78531.62777 - \dfrac{1896285.284}{h}$ $\dfrac{s}{1.34} = \dfrac{78531.62777}{1.34} - \dfrac{1896285.284}{1.34h}$ $\dfrac{s}{1.34} = 58605.692365671 - \dfrac{1415138.2716417}{h}$ $\Rightarrow\ u = 58605.692 - \dfrac{1415138.272}{h}$ (3 dp) where u = monthly sales in US dollars (i.e. $u = \dfrac{s}{1.34}$)	
11 (i)	The probability that each salesman is successful in closing a deal is assumed to be constant.	
11 (ii)	<u>Assumption: Deals closed are independent of one another</u> Deals closed may not be independent of one another as customers may collaborate to buy cars as a group for better bargaining power.	
11 (iii)	$X \sim \mathrm{B}(60, 0.2)$ Since $np = 12 > 5$ and $nq = 48 > 5$, $X \sim \mathrm{N}(12, 9.6)$ *approx.* $Y \sim \mathrm{B}(50, 0.3)$ Since $np = 15 > 5$ and $nq = 35 > 5$, $Y \sim \mathrm{N}(15, 10.5)$ *approx.* $X + Y \sim \mathrm{N}(27, 20.1)$ *approx.* $\mathrm{P}(X+Y > 20) \overset{c.c}{\approx} \mathrm{P}(X+Y > 20.5) = 0.926$ (3s.f)	

11 (iv)	$P(C=30)=0.03014$ ${}^{60}C_{30}\times p^{30}\times(1-p)^{30}=0.03014$ $p^2-p+0.2399991029=0$ $\therefore p=0.6$ or 0.3999955 Since $p<0.5$, $p=0.4$ (1 d.p)	
11 (v)	$C\sim B(60,0.05)$ Since $n=60>50$ such that $np=3<5$, $C\sim P_O(3)$ *approx.* $P(C>4)=0.185$ (3s.f)	