2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2

9740/01

Paper 1

Wednesday **14 September 2016** **3 hours**

Additional materials: Answer paper
List of Formula (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT class on all the work you hand in, including the Cover Page which is found on Page 2.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF15).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, place the completed cover page on top of your answer scripts and fasten all your work securely together with the string provided.

This question paper consists of 6 printed pages.

2016 JC2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS

9740
14 September 2016
3 hours

Name: ____________ **CT:** 1 5 _ _ _

COVER PAGE

1. Write your name, CT group and calculator model(s) in the spaces provided.

2. Arrange your answers in numerical order.

3. Detach this cover page and place it on top of your answer paper and fasten them securely together with the string provided.

For Examiner's Use			
Question No.	Marks Obtained	Total Marks	Remarks
1		4	
2		5	
3		8	
4		9	
5		9	
6		9	
7		10	
8		10	
9		11	
10		12	
11		13	
TOTAL		100	

Graphing Calculator Model: ____________

Scientific Calculator Model: ____________

1. A sequence follows the recurrence relation

$$U_{n+1} - U_n = 2\cos\frac{(2n+1)x}{2}\sin\frac{x}{2}\ ,\ U_1 = \sin x \text{ for } n = 1, 2, 3, \ldots .$$

Prove by mathematical induction that $U_n = \sin(nx)$ for all positive integer n. [4]

2. Solve the inequality $\dfrac{2}{4(x+1)^2+1} > 1$. [2]

Hence find $\displaystyle\int_{-1}^{\frac{\sqrt{3}-2}{2}} \left|1 - \frac{2}{4(x+1)^2+1}\right| \mathrm{d}x$, leaving your answer in exact form. [3]

3. Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$ with $\mathbf{a}$ not parallel to $\mathbf{b}$. The point P is on AB produced with $AP : AB = 3:1$ and the position vector of point Q is $2\mathbf{a}$.

(a) Find the position vector of the point of intersection of lines OB and PQ, giving your answer in terms of $\mathbf{b}$. [4]

(b) It is given that $\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and the point $C(0,\ 3,\ 4)$ does not lie on the plane OAB. Find the foot of the perpendicular from C to the plane OAB. [4]

4. Prove that $\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \sqrt{n^2+2n} - \sqrt{n^2-1}$. [2]

Hence find $\displaystyle\sum_{n=1}^{N} \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$. [3]

(a) Deduce the value of $\displaystyle\sum_{n=2}^{N} \frac{2n-1}{\sqrt{n^2-2n}+\sqrt{n^2-1}}$. [3]

(b) Show that $\displaystyle\sum_{n=1}^{N} \frac{2n+1}{2n-1} > \sqrt{N^2+2N}$. [1]

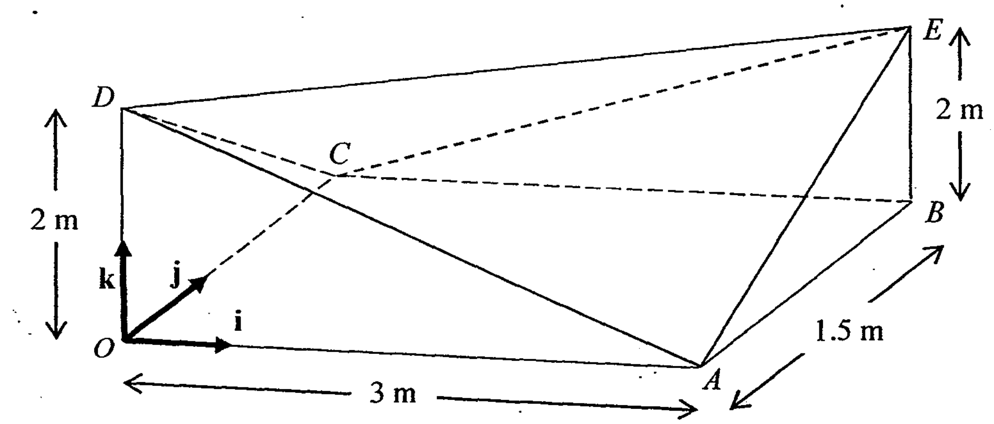
5. Sketch on a single Argand diagram, the loci defined by $-\frac{\pi}{4} < \arg(z+1+2\mathrm{i}) \le \frac{\pi}{4}$ and $|(2+\mathrm{i})w+5| \le \sqrt{5}$. [4]

(i) Find the minimum value of $\arg(w)$. [2]

(ii) Find the minimum value of $|z-w|$. [2]

(iii) Given that $\arg(z-w) < \theta$, $-\pi < \theta \le \pi$, state the minimum value of θ. [1]

6. A group of boys want to set up a camping tent. They lay down a rectangular tarp *OABC* on the horizontal ground with $OA = 3$ m and $AB = 1.5$ m and secure the points *D* and *E* vertically above *O* and *B* respectively, such that $OD = BE = 2$ m.



Assume that the tent takes the shape as shown above with 6 triangular surfaces and a rectangular base. The point *O* is taken as the origin and the unit vectors **i**, **j** and **k** are taken to be in the direction of $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively.

(i) Show that the line *DE* can be expressed as $\mathbf{r} = 2\mathbf{k} + \lambda(2\mathbf{i}+\mathbf{j})$, $\lambda \in \square$. [2]

(ii) Find the Cartesian equation of the plane *ADE*. [3]

(iii) Determine the acute angle between the planes *ADE* and *OABC*. Hence, or otherwise, find the acute angle between the planes *ADE* and *CDE*. [4]

7. The curve *C* has equation $y = \dfrac{x-2}{kx^2+x-2}$, where $k > 1$.

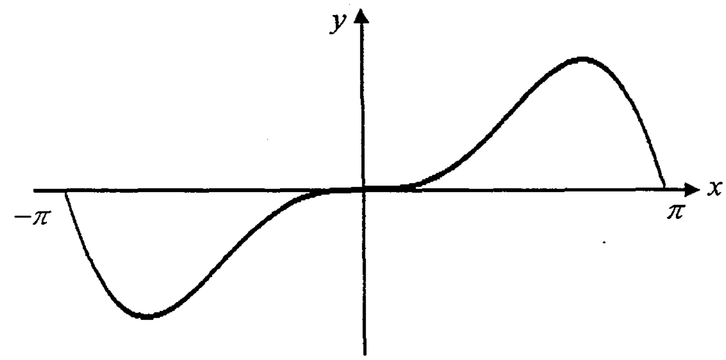
(i) Find the equation of the tangent at the point *A* where *C* cuts the *y*-axis. [2]

(ii) Sketch *C*, giving the equations of asymptotes, the coordinates of turning points and axial intercepts in terms of k, if any. [4]

(iii) Find the equation of the normal at the point *B* where *C* cuts the *x*-axis. Leave your answer in terms of k. [2]

(iv) Hence show that the value of the area bounded by the tangent at *A*, the normal at *B* and both the *x*- and *y*-axes is more than $\frac{15}{8}$ square units. [2]

8. The curve C (as shown in the diagram below) has equation $y = x^2 \sin x,\ -\pi \le x \le \pi$.



(i) Calculate the exact area of the region R enclosed by C and the x-axis. [4]

(ii) Sketch the curve with equation $(y+1)^2 - 4(x+2)^2 = 1$, showing clearly the coordinates of the turning points and the equation(s) of any asymptote(s). [2]

(iii) Hence find the volume of the solid generated when the region bounded by the 2 curves is rotated through 4 right angles about the x-axis. [4]

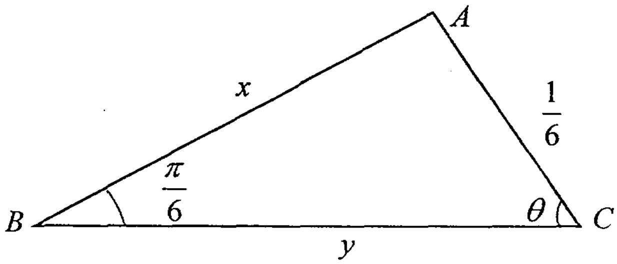
9. A manufacturer produces cylindrical containers using sheet metal of negligible thickness. The cylindrical container has an open top, and a base and curved sides made up of the sheet metal.

(a) (i) It is given that the volume of the cylindrical container is fixed at k cm^3. Show that when the amount of sheet metal used for the cylindrical container is a minimum, the ratio of its height to its radius is $1:1$. [5]

(ii) A product designer proposed a new design where the height of the cylindrical container is always 2.5 times that of its radius. Given that the radius of a cylindrical container produced using the new design equals the radius of the container produced in part **(a)(i)** with minimum sheet metal. Find the ratio of the amount of sheet metal used in this new design to the minimum amount of sheet metal used in part **(a)(i)**. [2]

(b) To reduce cost, plastic with negligible thickness, instead of sheet metal is used to manufacture the new design cylindrical containers in part **(a)(ii)** using *injection blow moulding* technology. In the injection blow moulding process, it is assumed that the cylindrical containers increase in size proportionately with the height to radius ratio remaining constant at 5:2 throughout the process. If the volume of the cylindrical container increases at a rate of 80 cm^3 per second, find the rate of change of the surface area of the cylindrical container when its height is 50 cm. [4]

10.



In the triangle ABC, $AB = x$, $BC = y$, $AC = \frac{1}{6}$, angle $ABC = \frac{\pi}{6}$ radians and angle $ACB = \theta$ radians (see diagram).

(a) **(i)** Show that $\dfrac{x}{y} = \dfrac{2\sin\theta}{\cos\theta + \sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is sufficiently small, express $\dfrac{x}{y}$ as a cubic polynomial in θ. [3]

(b) **(i)** Show that $\theta = \sin^{-1}(3x)$. [1]

(ii) Find the Maclaurin series for θ, up to and including the term in x^3. [5]

11. The functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{1}{2}\mathrm{e}^{1-x^2},\ x \in \mathbb{R},\ x \le 1 \text{ and}$$

$$\mathrm{g} : x \mapsto \sqrt{1 - \ln x},\ x \in \mathbb{R},\ 0 < x \le \mathrm{e}.$$

(i) Show that gf exists, and find the range of gf. [4]

(ii) Justify, with a reason, whether f^{-1} exists. [2]

(iii) The domain of f is restricted to $(-\infty, b]$ such that b is the largest value for which the inverse function f^{-1} exists. State the value of b and define f^{-1} clearly. [4]

(iv) The graph of $y = \mathrm{h}(x)$ is obtained by transforming the graph of $y = \mathrm{g}(x)$ in the following 2 steps.

Step 1: Scale parallel to the x-axis by a factor of 2.

Step 2: Reflect in the x-axis.

Define h in a similar form. [3]

End of Paper

2016 Prelim Paper 1 Solutions

Qn	Solution
1	* Let P_n be statement $U_n = \sin(nx)$ for all $n \in \mathbb{Z}^+$. When $n=1$, LHS $= U_1 = \sin x$, RHS $= \sin x$ $\therefore P_1$ is true. * Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $U_k = \sin(kx)$. Want to prove that P_{k+1} is true, i.e. $U_{k+1} = \sin(k+1)x$. LHS $= U_{k+1}$ $= U_k + 2\cos\frac{(2k+1)x}{2}\sin\frac{x}{2}$ $= \sin(kx) + 2\cos\left(\frac{2k+1}{2}\right)x\sin\left(\frac{1}{2}\right)x$ $= \sin(kx) + \sin(k+1)x - \sin(kx)$ $= \sin(k+1)x = \text{RHS}$ *Since P_1 is true, P_k is true implies P_{k+1} is true, by MI P_n is true for all $n \in \mathbb{Z}^+$.
2	$\frac{2}{4(x+1)^2+1} > 1$ $\frac{-(2x+1)(2x+3)}{4(x+1)^2+1} > 0$ Since $4(x+1)^2+1>0$ for all x, $(2x+1)(2x+3)<0$ $\therefore -\frac{3}{2} < x < -\frac{1}{2}$ $\int_{-1}^{\frac{\sqrt{3}}{2}-1} \left\| 1 - \frac{2}{4(x+1)^2+1} \right\| dx$ $= \int_{-1}^{-\frac{1}{2}} \left(-1 + \frac{2}{4(x+1)^2+1}\right) dx + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1} \left(1 - \frac{2}{4(x+1)^2+1}\right) dx$ $= \left[-x + \tan^{-1}(2x+2)\right]_{-1}^{-\frac{1}{2}} + \left[x - \tan^{-1}(2x+2)\right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}$ $= \left[\frac{1}{2} + \tan^{-1}1 - 1\right] + \left[\frac{\sqrt{3}}{2} - 1 - \tan^{-1}\sqrt{3} + \frac{1}{2} + \tan^{-1}1\right]$ $= \frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1$

3 (a)

$\overrightarrow{OP} = \underset{\sim}{a} + 3\overrightarrow{AB} = \underset{\sim}{a} + 3\left(\underset{\sim}{b} - \underset{\sim}{a}\right) = 3\underset{\sim}{b} - 2\underset{\sim}{a}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\underset{\sim}{a} - \left(3\underset{\sim}{b} - 2\underset{\sim}{a}\right) = 4\underset{\sim}{a} - 3\underset{\sim}{b}$

$l_{PQ} : \underset{\sim}{r} = 2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right),\ \lambda \in \square$

$l_{OB} : \underset{\sim}{r} = \mu\underset{\sim}{b},\ \mu \in \square$

At point of intersection, $2\underset{\sim}{a} + \lambda\left(4\underset{\sim}{a} - 3\underset{\sim}{b}\right) = \mu\underset{\sim}{b}$

Comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\lambda = -\frac{1}{2},\ \mu = \frac{3}{2}$

$\therefore$ position vector of the point of intersection $= \frac{3}{2}\underset{\sim}{b}$

(b)

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Let F be the foot of perpendicular.

<u>Method 1</u>

$$l_{FC} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\ s \in \square \quad , \ \Pi_{OAB} : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$$

$$-6 + 4 + s\left(1 + 4 + 1\right) = 0$$

$$s = \frac{1}{3}$$

$$\therefore \overrightarrow{OF} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$$

$$\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$$

Method 2

$$\overrightarrow{FC} = \left(\overrightarrow{OC} \cdot \hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$$

$$= \left(\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right|}\right) \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right|}$$

$$= \frac{-6+4}{\sqrt{6}} \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{6}}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$$

$$\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$$

4

Method 1

$$\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} \times \frac{\sqrt{n^2+2n}+\sqrt{n^2-1}}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$$

$$= \frac{(2n+1)\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)}{\left(n^2+2n\right)-\left(n^2-1\right)}$$

$$= \sqrt{n^2+2n}-\sqrt{n^2-1}$$

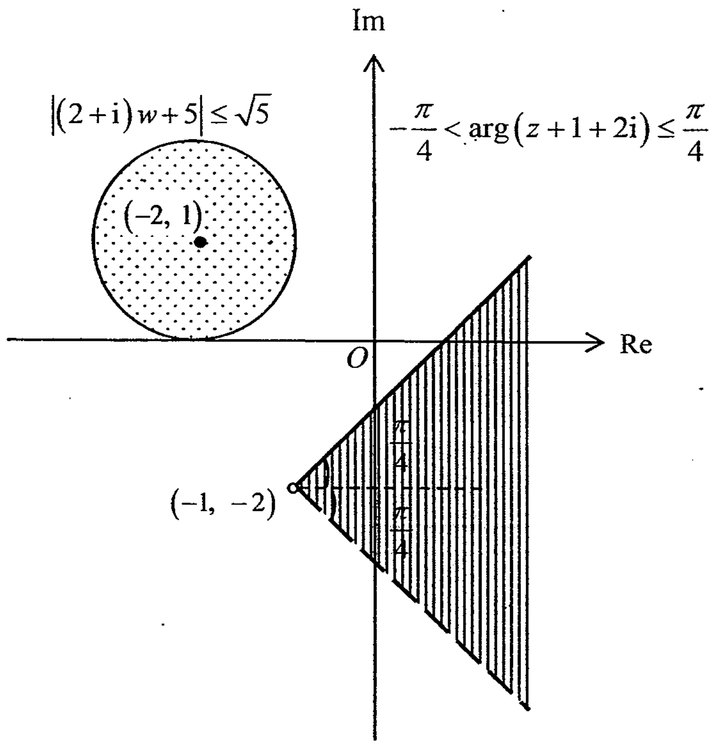
	<u>Method 2</u> $\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-1}\right)$ $=\left(n^2+2n-\left(n^2-1\right)\right)$ $=2n+1$ $\therefore \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{n^2+2n}-\sqrt{n^2-1}$
	$\sum_{n=1}^{N}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sum_{n=1}^{N}\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)$ $=\begin{bmatrix} \sqrt{3}-\sqrt{0} \\ +\sqrt{8}-\sqrt{3} \\ \vdots \\ +\sqrt{N^2+2N}-\sqrt{N^2-1} \end{bmatrix}$ $=\sqrt{N^2+2N}$
(a)	Replace n by $n+1$, $\sum_{n=2}^{N}\dfrac{2n-1}{\sqrt{n^2-1}+\sqrt{n(n-2)}}$ $=\sum_{n=1}^{N-1}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sqrt{(N-1)^2+2(N-1)}$ $=\sqrt{N^2-1}$
(b)	Notice that $\sqrt{n^2+2n}>n$ and $\left(\sqrt{n^2-1}\right)^2-(n-1)^2=2n-2\geq 0$. $\Rightarrow \sqrt{n^2-1}\geq n-1$ $\Rightarrow \sqrt{n^2+2n}+\sqrt{n^2-1}>2n-1$ $\Rightarrow \dfrac{1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}<\dfrac{1}{2n-1}$ $\therefore \sum_{n=1}^{N}\dfrac{2n+1}{2n-1}>\sum_{n=1}^{N}\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{N^2+2N}$

5

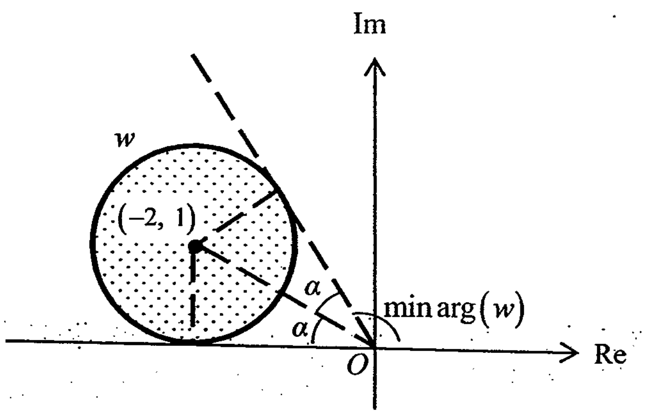
$\left|(2+\mathrm{i})w+5\right| \le \sqrt{5}$

$\left|2+\mathrm{i}\right|\left|w+\dfrac{5}{2+\mathrm{i}}\right| \le \sqrt{5}$

$\left|w+2-\mathrm{i}\right| \le 1 \Rightarrow$ circle centre $(-2,\ 1)$, radius 1



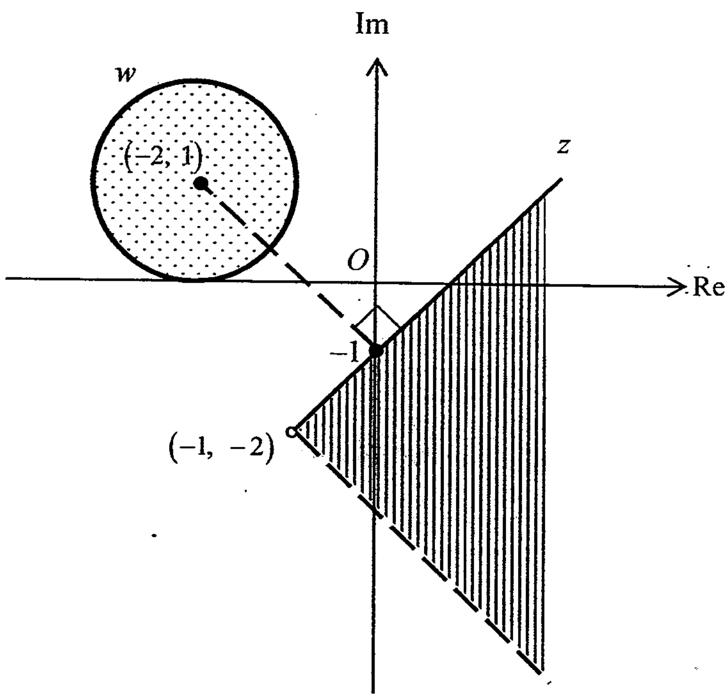
(i)



$\alpha = \sin^{-1}\dfrac{1}{\sqrt{(-2)^2+1^2}} = \sin^{-1}\dfrac{1}{\sqrt{5}}$

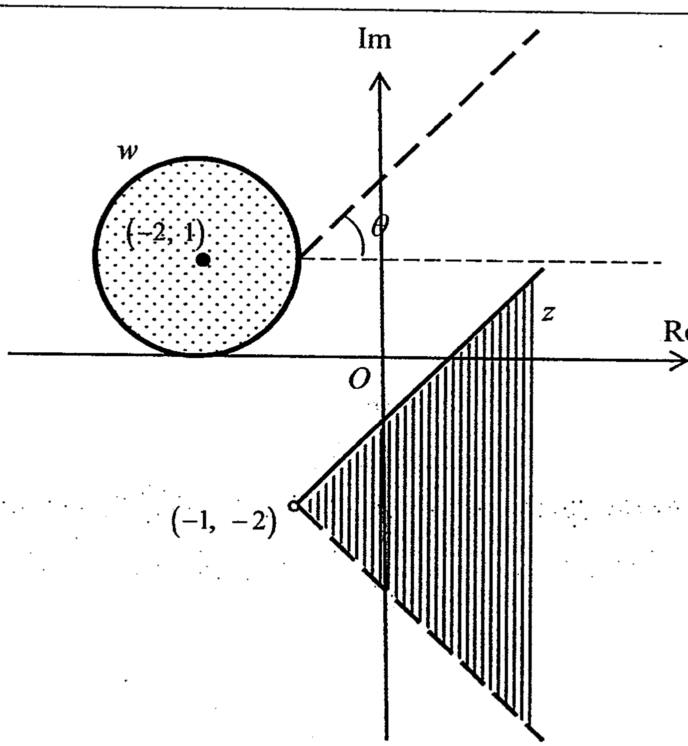
$\min\arg(w) = \pi - 2\alpha = 2.2143 = 2.21(3\text{sf})$

(ii)



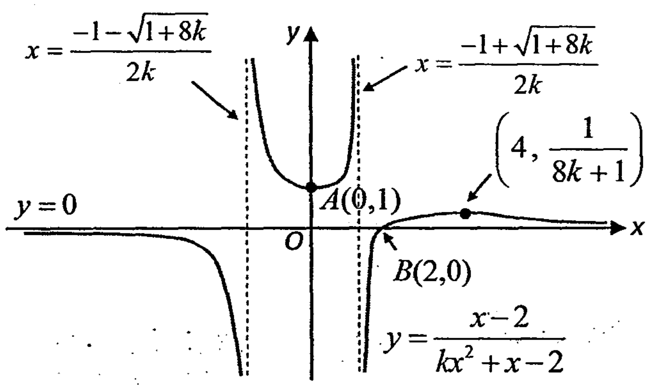
$\min|z-w| = 2\sqrt{2}-1$

(iii)

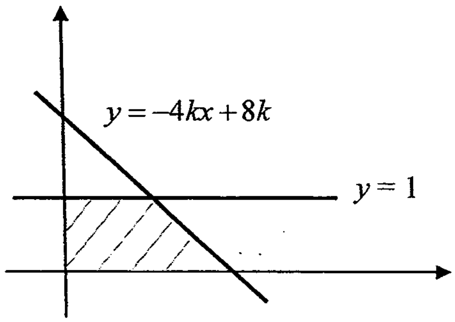


$\theta = \dfrac{\pi}{4}$

6 (i)	$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix},\ \overrightarrow{OE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}$ $\overrightarrow{DE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}3\\1.5\\0\end{pmatrix}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}$ $l_{DE}:\underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	$\overrightarrow{AD}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\2\end{pmatrix}$ $\overrightarrow{DE}\times\overrightarrow{AD}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-3\\0\\2\end{pmatrix}=1.5\begin{pmatrix}2\\-4\\3\end{pmatrix}\Rightarrow\underset{\sim}{n}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ $\Pi_{ADE}:\underset{\sim}{r}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=6$ $\therefore 2x-4y+3z=6$
(iii)	$\underset{\sim}{n}_{OABC}=\begin{pmatrix}0\\0\\1\end{pmatrix},\ \underset{\sim}{n}_{ADE}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ angle between planes $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}$ $=\cos^{-1}\dfrac{3}{\sqrt{4+16+9}}$ $=\cos^{-1}\dfrac{3}{\sqrt{29}}$ $=56.1^\circ\text{(1dp)}$

	$\text{Angle} = 180° - 2\cos^{-1}\dfrac{3}{\sqrt{29}}$ $= 67.7°$ (1 d.p.)
7 (i)	$\dfrac{dy}{dx} = \dfrac{(kx^2+x-2)-(x-2)(2kx+1)}{(kx^2+x-2)^2} = \dfrac{-kx^2+4kx}{(kx^2+x-2)^2}$ When $x=0$, $\dfrac{dy}{dx} = \dfrac{0}{(-2)^2} = 0$ and $y = \dfrac{-2}{-2} = 1$ Hence required equation of tangent is $y = 1$.
(ii)	For axial intercepts, when $y=0$, $x=2$. when $x=0$, $y=1$. For vertical asymptotes, $kx^2+x-2=0$ $\therefore x = \dfrac{-1\pm\sqrt{1+8k}}{2k}$ For turning points, $\dfrac{dy}{dx}=0$ $-kx^2+4kx=0$ $-kx(x-4)=0$ $\therefore x=0$ or $x=4$ 
(iii)	At $x=2$, $\dfrac{dy}{dx} = \dfrac{-4k+8k}{(4k)^2} = \dfrac{4k}{16k^2} = \dfrac{1}{4k}$ $\therefore$ gradient of normal $= -4k$ Hence required equation of normal is $y-0=-4k(x-2)$ $y=-4kx+8k$

(iv)



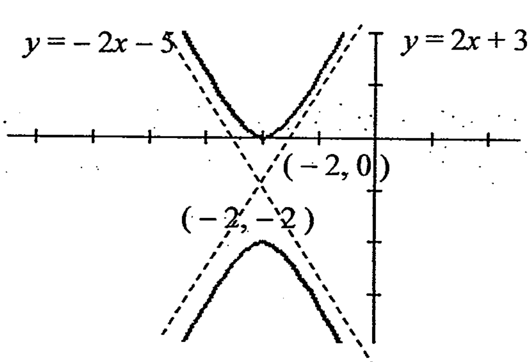
When $y=1$, $1=-4kx+8k \Rightarrow x=\dfrac{8k-1}{4k}$

$$\therefore \text{ required area } =\frac{1}{2}\left(\frac{8k-1}{4k}+2\right)(1)$$
$$=\frac{16k-1}{8k}$$
$$=2-\frac{1}{8k}$$
$$>2-\frac{1}{8} \qquad (\text{since } k>1)$$
$$>\frac{15}{8}$$

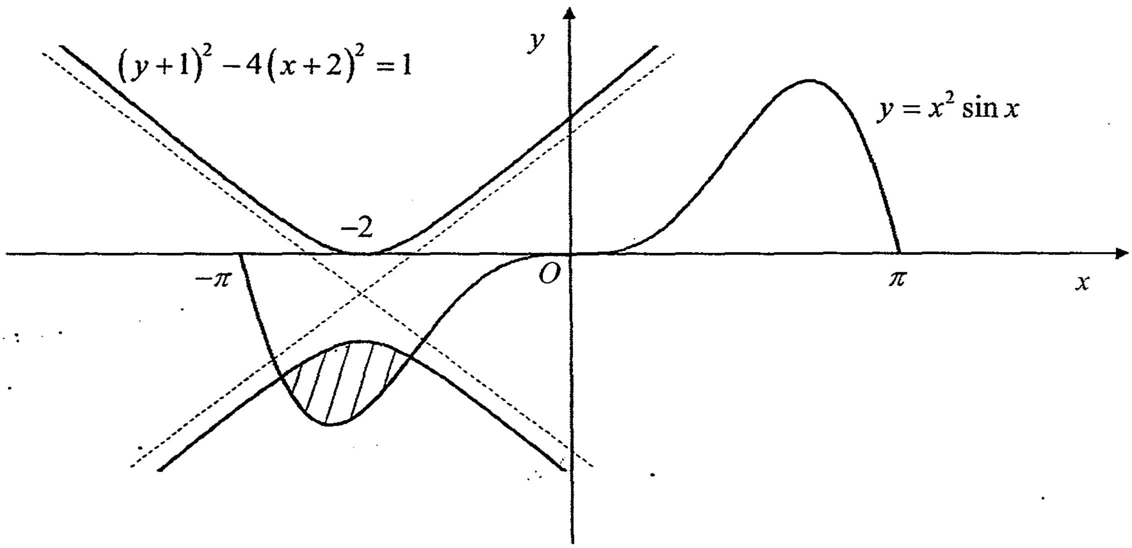
8 (i)

$$\text{Area}=2\int_0^{\pi} x^2 \sin x\,dx$$
$$=2\left[\left[-x^2\cos x\right]_0^{\pi}+\int_0^{\pi}2x\cos x\,dx\right]$$
$$=2\left[\pi^2+2(\left[x\sin x\right]_0^{\pi}-\int_0^{\pi}\sin x\,dx)\right]$$
$$=2\left[\pi^2+2\left[\cos x\right]_0^{\pi}\right]$$
$$=2(\pi^2-4)\text{ units}^2$$

(ii)



(iii)



Coordinates of the points of intersections of the 2 curves are $(-1.5374, -2.3623)$ and $(-2.7626, -2.8238)$.

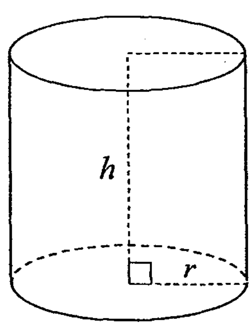
Volume of solid generated

$$= \pi\int_{-2.7626}^{-1.5374} (x^2 \sin x)^2 \, dx - \pi\int_{-2.7626}^{-1.5374} (-1-\sqrt{1+4(x+2)^2})^2 dx = 26.8 \text{ units}^3$$

9 (i)

Let A cm² be the surface area of the cylindrical container.

Let r cm and h cm be the radius and height of the cylindrical container respectively.



$$\text{Volume} = \pi r^2 h = k$$

$$\therefore \ h = \frac{k}{\pi r^2}$$

$$A = 2\pi r h + \pi r^2$$

$$= 2\pi r\left(\frac{k}{\pi r^2}\right) + \pi r^2$$

$$= \frac{2k}{r} + \pi r^2$$

Can also express r in terms of h and find A in terms of h, then let $\frac{dA}{dh} = 0$ to obtain h and subsequently r.

Hence $\frac{dA}{dr} = -\frac{2k}{r^2} + 2\pi r$

When $\frac{dA}{dr} = 0$,

	$-\frac{2k}{r^2}+2\pi r=0$ $r^3=\frac{k}{\pi}$ $r=\sqrt[3]{\frac{k}{\pi}}$ $\therefore\ h=\frac{k}{\pi r^2}=\frac{k}{\pi\left[\left(\frac{k}{\pi}\right)^{\frac{1}{3}}\right]^2}=\sqrt[3]{\frac{k}{\pi}}$ Hence $h:r=\sqrt[3]{\frac{k}{\pi}}:\sqrt[3]{\frac{k}{\pi}}=1:1$ (shown) $\frac{d^2A}{dr^2}=\frac{4k}{r^3}+2\pi>0$ since $p>0$ and $k>0$ Hence A is a minimum when $r=\sqrt[3]{\frac{k}{\pi}}$
(ii)	From (i), $h:r=1:1$ Hence $A=2\pi rh+\pi r^2=2\pi r(r)+\pi r^2=3\pi r^2$ For new design, $h:r=5:2$ Hence new $A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\therefore$ required ratio is $6\pi r^2:3\pi r^2=2:1$
(b)	<u>Method 1</u> Let V cm³ be the volume of the cylindrical container. $V=\pi r^2h=\pi r^2(\frac{5}{2}r)=\frac{5}{2}\pi r^3$ $A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$ $\frac{dV}{dr}=\frac{15}{2}\pi r^2$ ← Can also find $\frac{dV}{dh}$ and $\frac{dA}{dh}$ and use $\frac{dA}{dt}=\frac{dA}{dh}\times\frac{dh}{dV}\times\frac{dV}{dt}$ $\frac{dA}{dr}=12\pi r$ $\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dV}\times\frac{dV}{dt}$ $=12\pi r\times\frac{2}{15\pi r^2}\times 80$ $=\frac{128}{r}$ When $h=50$, $r=\frac{2}{5}(50)=20$

Hence $\frac{dA}{dt}=\frac{128}{20}=6.4\ \text{cm}^2/\text{s}$

<u>Method 2</u>

$A=6\pi r^2 \quad \therefore\ r=\sqrt{\frac{A}{6\pi}} \quad$ (reject $r=-\sqrt{\frac{A}{6\pi}}$ since $r\geq 0$)

Hence $V=\pi r^2 h=\pi r^2\left(\frac{5}{2}r\right)=\frac{5}{2}\pi r^3$

$$=\frac{5}{2}\pi\left(\sqrt{\frac{A}{6\pi}}\right)^3$$

$$=\frac{5A^{\frac{3}{2}}}{2(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$$

$$\frac{dV}{dA}=\frac{15A^{\frac{1}{2}}}{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$$

When $h=50$, $r=\frac{2}{5}(50)=20$

$\therefore\ A=6\pi(20)^2=2400\pi$

Hence $\frac{dA}{dt}=\frac{dA}{dV}\times\frac{dV}{dt}$

$$=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15A^{\frac{1}{2}}}\times 80$$

$$=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15(2400\pi)^{\frac{1}{2}}}\times 80$$

$$=6.4\quad \text{cm}^2/\text{s}$$

10 (a) (i)

$$\frac{\sin\theta}{x}=\frac{\sin\left(\pi-\frac{\pi}{6}-\theta\right)}{y}$$

$$\frac{x}{y}=\frac{\sin\theta}{\sin\left(\frac{5\pi}{6}-\theta\right)}$$

$$\frac{x}{y}=\frac{\sin\theta}{\sin\frac{5\pi}{6}\cos\theta-\sin\theta\cos\frac{5\pi}{6}}$$

$$\frac{x}{y}=\frac{\sin\theta}{\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}\ \text{(shown)}$$

(a) (ii)	$\frac{x}{y}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$ $\frac{x}{y}=\frac{2\left(\theta-\frac{\theta^3}{3!}+...\right)}{1+\sqrt{3}\theta-\frac{\theta^2}{2}+...}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\right)^{-1}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+(-1)\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2\right)$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2\right)$ $\frac{x}{y}\approx 2\theta-2\sqrt{3}\theta^2+\frac{20}{3}\theta^3$
(b) (i)	Using sine rule, $\frac{\sin\theta}{x}=\frac{\sin\frac{\pi}{6}}{\frac{1}{6}}=3 \quad \therefore \theta=\sin^{-1}3x$
(b) (ii)	<u>Method 1</u> $\sin\theta=3x$ $\cos\theta\frac{d\theta}{dx}=3$ ---- (1) $\cos\theta\frac{d^2\theta}{dx^2}-\sin\theta\left(\frac{d\theta}{dx}\right)^2=0$ --- (2) $\cos\theta\frac{d^3\theta}{dx^3}-\sin\theta\frac{d\theta}{dx}\frac{d^2\theta}{dx^2}-2\sin\theta\frac{d\theta}{dx}\frac{d^2\theta}{dx^2}-\cos\theta\left(\frac{d\theta}{dx}\right)^3$ $=0$ ---(3) When $x=0$, $\theta=0$, $\frac{d\theta}{dx}=3$, $\frac{d^2\theta}{dx^2}=0$, $\frac{d^3\theta}{dx^3}=27$ $\theta=3x+\frac{27}{3!}x^3+...=3x+\frac{9}{2}x^3+...$

Method 2

$\theta=\sin^{-1}(3x)$

$$\frac{d\theta}{dx}=\frac{3}{\sqrt{1-9x^2}}=3\left(1-9x^2\right)^{-\frac{1}{2}}$$

$$\frac{d^2\theta}{dx^2}=3\left(-\frac{1}{2}\right)\left(1-9x^2\right)^{-\frac{3}{2}}(-18x)=27x\left(1-9x^2\right)^{-\frac{3}{2}}$$

$$\frac{d^3\theta}{dx^3}=-\frac{81}{2}x(-18x)\left(1-9x^2\right)^{-\frac{5}{2}}+27\left(1-9x^2\right)^{-\frac{3}{2}}$$

$$=729x^2\left(1-9x^2\right)^{-\frac{5}{2}}+27\left(1-9x^2\right)^{-\frac{3}{2}}$$

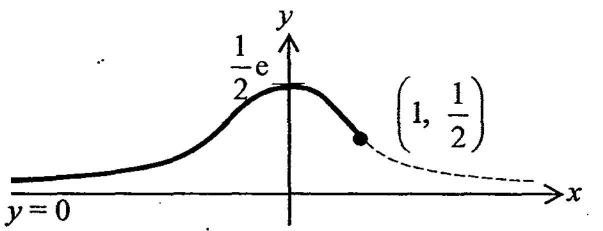
When $x=0$,

$$\theta=0,\ \frac{d\theta}{dx}=3,\ \frac{d^2\theta}{dx^2}=0,\ \frac{d^3\theta}{dx^3}=27$$

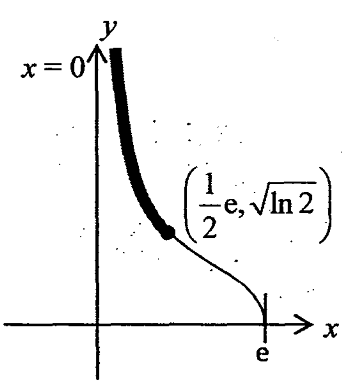
$$\theta=3x+\frac{27}{3!}x^3+\ldots=3x+\frac{9}{2}x^3+\ldots$$

11

(i)

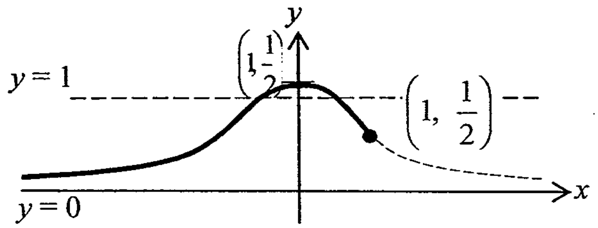


Since $R_f=\left(0,\frac{1}{2}e\right]\subseteq D_g=(0,e]$, $R_f\subseteq D_g$ and gf exists.



$R_{gf}=\left[\sqrt{\ln 2},\infty\right)$

(ii)



Since a horizontal line $y = 1$ cuts the graph of $y = \mathrm{f}(x)$ twice, f is not a one-to-one function and f^{-1} does not exist.

(iii)

$b = 0$

Let $y = \mathrm{f}(x)$

$$y = \frac{1}{2}\mathrm{e}^{1-x^2}$$

$$\ln(2y) = 1 - x^2$$

$$x = \pm\sqrt{1 - \ln(2y)}$$

Since $x \le 0$, $x = -\sqrt{1 - \ln(2y)}$

$$\mathrm{f}^{-1} : x \mapsto -\sqrt{1 - \ln(2x)},\ x \in □,\ 0 < x \le \frac{1}{2}\mathrm{e}$$

(iv)

$$y = \sqrt{1 - \ln x} \xrightarrow{\text{Step 1}} y = \sqrt{1 - \ln\left(\frac{x}{2}\right)}$$

$$y = \sqrt{1 - \ln\left(\frac{x}{2}\right)} \xrightarrow{\text{Step 2}} y = -\sqrt{1 - \ln\left(\frac{x}{2}\right)}$$

$$0 < x \le \mathrm{e} \rightarrow 0 < \frac{x}{2} \le \mathrm{e}$$

$$\therefore 0 < x \le 2\mathrm{e}$$

$$\mathrm{h} : x \mapsto -\sqrt{1 - \ln\left(\frac{x}{2}\right)},\ x \in □,\ 0 < x \le 2\mathrm{e}$$

2016 JC2 PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2

9740/02

Paper 2

Tuesday **20 September 2016** **3 hours**

Additional materials: Answer paper
List of Formula (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT class on all the work you hand in, including the Cover Page.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF15).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, place the completed cover page on top of your answer scripts and fasten all your work securely together with the string provided.

This question paper consists of 6 printed pages.

Section A: Pure Mathematics [40 marks]

1. On the first day of last month, the temperature of a machine in a manufacturing plant was found to be 90 °C and was deemed as too hot by the supervisor. On each day waste heat was produced by the machine as a by-product and caused the temperature of the machine to increase by 2 °C. In an attempt to make it cooler, the supervisor decided to adjust the thermostat and decreased the temperature by 3% at the end of each day.
   (i) Show that the temperature of the machine was 87.8 °C at the end of the 3rd day. [2]
   (ii) At the end of which day would the temperature of the machine first dropped below 70°C? [3]
   (iii) Will the temperature continue to drop indefinitely? Justify your answer. If not, what is the long term temperature of the machine? [3]

2. **(a)** Given that $z_1 = -\frac{\mathrm{i}}{2}$ is a root of the equation $2z^3+(\mathrm{i}-8)z^2+az+13\mathrm{i}=0$, find the complex number a and solve the equation, giving your answer in Cartesian form $x+\mathrm{i}y$. [4]
   Hence, find in Cartesian form the roots of the equation
   $$2w^3+(1+8\mathrm{i})w^2-aw-13=0.$$ [2]
   (b) Solve the equation $z^6+729=0$, expressing your answers in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [2]
   Given that z_1 is a root of the above equation and $0<\arg z_1<\frac{\pi}{2}$.
   If $\frac{z_1^{\ n}}{z_1^{\ *}}$ is a positive real number, find the smallest positive integer n. [2]

3. **(i)** Use the substitution $x = 3\sin\theta+1,\ 0<\theta<\frac{\pi}{2}$ to find $\int\frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x$. [4]
   (ii) A curve has parametric equations
   $$x=\frac{1}{\sqrt{9-(t-1)^2}},\quad y=t^2,\quad 0<t<4.$$
   (a) Sketch the curve, indicating the end point and the equation of the asymptote. [3]
   (b) Using the result in part (i), find the exact area of the region bounded by the curve, the lines $y=1$, $y=\frac{25}{4}$ and $x=\frac{8}{7}$. [4]

4. In harvesting of renewable natural resources, it is desirable that policies are formulated to allow maximal harvest of the natural resources, and yet not deplete the resources below a sustainable level. A simple harvesting model devised for the rate of change of the population of wild salmon in a particular region in the Pacific Ocean is given by

$$\frac{dP}{dt} = P(4-P)-h,$$

where P is the population of wild salmon in millions at time t years and h is the constant harvest rate in millions.

(i) Sketch a graph of $\frac{dP}{dt}$ against P, expressing the turning point in terms of h. [2]

(ii) The *Maximum Sustainable Yield (MSY)* is the largest harvest rate h that allows for a sustainable harvest of wild salmon without long-term depletion. State the *MSY* for the wild salmon. [1]

(iii) It is given that the population of wild salmon in that region was 3.2 million in 2015 and the constant harvest rate is 3 million. Find an expression for P at any time t. [5]
Hence find the population of the wild salmon in that region in 2016. [2]

(iv) State one assumption you made in your calculation. [1]

Section B: Statistics [60 marks]

5. In the last election, there were speculations from unofficial sources before the counting of votes is completed. For the current election, to prevent unnecessary speculations, the election office of Sunny Island will be conducting a sample count in each electoral division after voting is done. Each electoral division has a different number of registered voters and a sample of 400 votes will to be sampled from each electoral division.

(i) Identify and describe an appropriate method to obtain the sample. [3]

(ii) State an advantage and a disadvantage of the sampling method used in part (i). [2]

6. The random variable X has a binomial distribution $B(n, p)$, where $0 < p < 1$, and n is an integer. Show that $\frac{P(X=r)}{P(X=r-1)} = \left(\frac{n-r+1}{r}\right)\left(\frac{p}{1-p}\right)$. [3]

Hence find a condition relating n and p such that X has two values for its mode, and determine these two values, giving your answer in terms of n and p. [3]

7. A group of ten people consists of four single women, two single men and 2 couples. The ten people are arranged randomly in a circle.

(i) Find the probability that the four single women are all separated. [2]

(ii) Find the probability that either the four single women are next to one another or the two single men are next to each other but not both. [3]

One of the ten people left the group and the remaining nine decided to sit at a round table with ten identical chairs equally spaced around the table. The chairs are decorated such that every alternate chair is tied with an identical chair sash. Given that the nine people have no preference to which seat to take, find the number of possible seating arrangements. [2]

8. In an examination, the score, X, for paper 1 of a student is found to follow a normal distribution with mean 62 and standard deviation σ, and the score, Y, for paper 2 of a student is found to follow a normal distribution with mean 71 and standard deviation 8. The final score of a student for the examination is the average score of the 2 papers and it is assumed that X and Y are independent random variables.

(i) Find the probability that for two randomly selected students A and B taking the examination for paper 2, A has at most 2 marks less than the marks of B. [2]

(ii) Given that 15% of the students have at least a final score of 75, find σ. [4]

(iii) Using the value of σ found in part **(ii)**, find the probability that a randomly selected student performs better in her Paper 1 than in her Paper 2.

(iv) Comment on the validity of the answer obtained in part **(ii)** and **(iii)**. [3]

9. To reduce the number of speeding incidents on the road, traffic police in Country S set up traffic cameras at 3 busy traffic Junctions A, B and C to monitor the speeds of vehicles passing through these junctions. The average number of speeding vehicles caught by the camera at Junctions A, B and C are 2 in every 3 hours, 5 in every 4 hours and λ in every hour respectively. It is assumed that the number of speeding vehicles caught by the cameras at the three junctions followed Poisson distributions.

(a) Find the probability that there are at least 2 speeding vehicles caught at Junctions A and B in an hour. [2]

(b) Given that there are 2 speeding vehicles caught at the three junctions in an hour, find the probability that at least one speeding vehicle caught is at Junction C. Leave your answer in terms of λ. [3]

(c) Given that the traffic cameras are in operation 24 hours in a day, using a suitable approximation, find the probability that there will be more speeding vehicles caught at Junction A than at Junction B in a day. State an assumption for the calculation to be valid. [5]

10. A nutritionist claims that the mean number of calories in an energy bar is 350 cal. The nutritionist collected and measured the number of calories of a random sample of 15 energy bars. The mean and variance of the sample was 347.2 cal and 20.74 cal^2 respectively.

(i) The nutritionist wishes to carry out a hypothesis test on his claim. Explain why t-test instead of z-test is to be used. State an assumption for the test to be valid. [2]

(ii) Test at 5% level of significance, whether the mean number of calories in an energy bar is 350 cal, defining any symbols that you use. [4]

(iii) Suppose the nutritionist uses a different test in part **(ii)**. Without further calculation, explain and state whether the conclusion will be different. [2]

The manufacturer of the energy bar refines the manufacturing process and the new energy bars follow a normal distribution with mean μ cal and variance 20.74 cal^2. The manufacturer then provides the nutritionist with another sample of 15 energy bars.

(iv) Find the range of mean number of calories, $\bar{x}$, of the second sample of 15 energy bar so that the null hypothesis in part **(ii)** is not rejected at 5% level of significance. Leave your answer correct to one decimal place. [3]

11. A group of scientists is interested to find out the correlation between the number of species and the size of the natural habitat. The scientists sampled non-overlapping lands of different areas (x) in square kilometres, and noted the corresponding number of species (y) found. The results are shown in the table below.

Area (x)	300	400	500	600	700	800	900	1000	1100	1200
Number of species (y)	12	15	18	21	k	25	26	27	27	28

(i) Given that the equation of the regression line is $y = 0.01758x + 9.018$, show that the value of k to the nearest whole number is 23. [3]

Take k to be 23.

(ii) Draw the scatter diagram for the given data, labelling the axes clearly. [2]

(iii) Calculate the product moment correlation coefficient r. With reference to both the scatter diagram and r, explain why a linear model is not appropriate. [2]

(iv) The following models are suggested for the data.

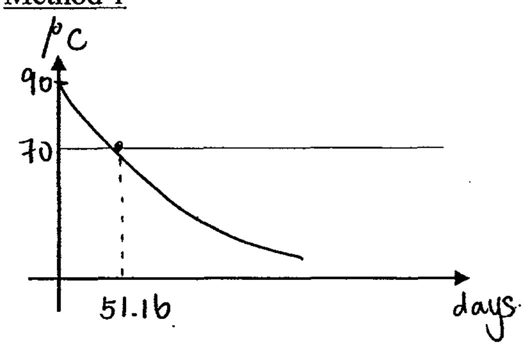
(A) $y = a + bx^2$, (B) $y = ax^b$, where $a > 0$ and $0 < b < 1$.

Use a graphical approach to determine which model is more appropriate. [2]

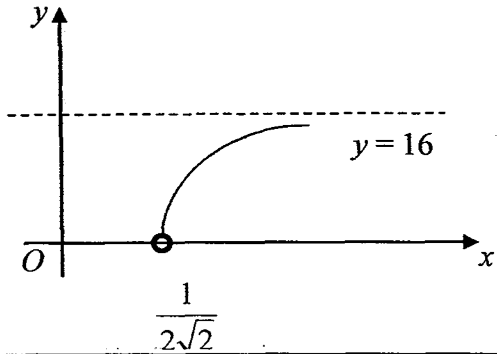
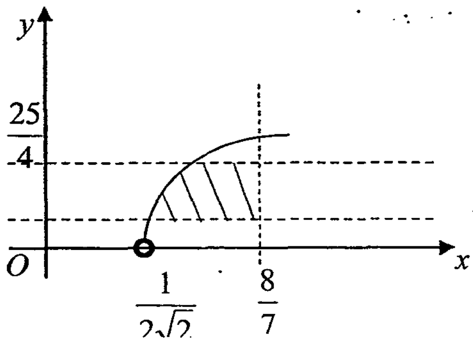
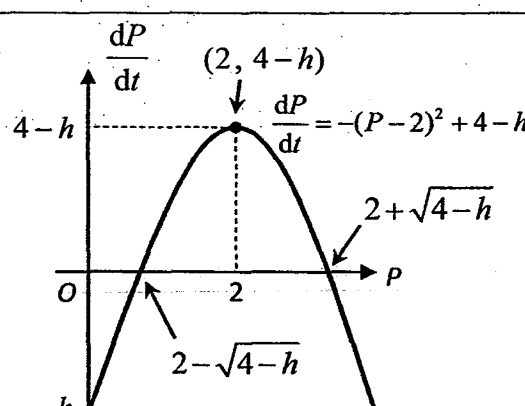
(v) Use the more appropriate model to estimate the area of the natural habitat when the number of species found is 24. Comment on the reliability of your estimation. [3]

End of Paper

2016 Prelim Paper 2 Solutions

Qn	Solution
1 (i)	1^{st} day: $0.97(92)$ 2^{nd} day: $0.97(0.97(92)+2)$ 3^{rd} day: $0.97(0.97(0.97(92)+2)+2)$ $=0.97^3(92)+0.97^2(2)+0.97(2)$ $=87.788$ $=87.8\ °\text{C}$ (3 s.f.)
(ii)	n^{th} day: $=0.97^n(92)+0.97^{n-1}(2)+...+0.97(2)$ $=0.97^n(92)+2\left(\dfrac{0.97(1-0.97^{n-1})}{1-0.97}\right)$ <u>Method 1</u>  <u>Method 2</u> Using GC, When $n=51$, temperature $=70.025°\text{C}$ When $n=52$, temperature $=69.865°\text{C}$ $\therefore$ 52 days
(iii)	No, temperature will not drop infinitely since $r=0.97<1$. As $n\to\infty$, temperature approaches $0+\dfrac{2\times0.97}{1-0.97}=64.7\ °\text{C}$ in the long run.
2 (a)	$2\left(-\dfrac{i}{2}\right)^3+(i-8)\left(-\dfrac{i}{2}\right)^2+a\left(-\dfrac{i}{2}\right)+13i=0$ $\dfrac{1}{4}i+2-\dfrac{1}{4}i-\dfrac{ai}{2}+13i=0$ $a=26-4i$ $2z^3+(i-8)z^2+(26-4i)z+13i=(2z+i)(z^2+bz+13)$ Comparing the coefficient of z^2, $i-8=2b+i$ $\therefore b=-4$ $(2z+i)(z^2-4z+13)=0$ $\therefore z=-\dfrac{i}{2}$ or $z=2\pm3i$.

	Replace z with iw, $2(\mathrm{i}w)^3+(\mathrm{i}-8)(\mathrm{i}w)^2+a(\mathrm{i}w)+13\mathrm{i}=0$ $-2\mathrm{i}w^3+(8-\mathrm{i})w^2+a\mathrm{i}w+13\mathrm{i}=0$ Dividing throughout by $-\mathrm{i}$, $2w^3+(1+8\mathrm{i})w^2-aw-13=0$. $\mathrm{i}w=-\frac{\mathrm{i}}{2}$ or $\mathrm{i}w=2\pm3\mathrm{i}$. $\therefore w=-\frac{1}{2}$ or $w=\pm3-2\mathrm{i}$
(b)	$z^6=-729=729\mathrm{e}^{\mathrm{i}\pi}=729\mathrm{e}^{(\pi+2k\pi)\mathrm{i}}$ $z=3\mathrm{e}^{\left(\frac{\pi}{6}+\frac{2k\pi}{6}\right)\mathrm{i}},k=0,\pm1,\pm2,-3$ OR $z=3\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},3\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}},3\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ $z_1=3\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ $\arg\left(\frac{{z_1}^n}{z_1^*}\right)=n\arg z_1+\arg z_1=(n+1)\frac{\pi}{6}$ Positive real number $(n+1)\frac{\pi}{6}=2k\pi$ $\therefore(n+1)\frac{\pi}{6}=2\pi$ Minimum $n=11$.
3 (i)	$x=3\sin\theta+1$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\cos\theta$ $\int\frac{x}{\sqrt{9-(x-1)^2}}\,\mathrm{d}x$, $=\int\frac{3\sin\theta+1}{\sqrt{9-(3\sin\theta)^2}}(3\cos\theta)\,\mathrm{d}\theta$ $=\int(3\sin\theta+1)\,\mathrm{d}\theta$ $=-3\cos\theta+\theta+C$ $=-\sqrt{9-(x-1)^2}+\sin^{-1}\frac{x-1}{3}+C$

(ii) (a)	
(ii) (b)	 $$\text{Area} = -\int_1^{\frac{25}{4}} x\,\mathrm{d}y + \left(\frac{25}{4} - 1\right) \times \frac{8}{7}$$ $$= -\int_1^{\frac{5}{2}} \frac{2t}{\sqrt{9-(t-1)^2}}\,\mathrm{d}t + 6$$ $$= -2\left[-\sqrt{9-(t-1)^2} + \sin^{-1}\frac{t-1}{3}\right]_1^{\frac{5}{2}} + 6$$ $$= -2\left[\left(-\sqrt{\frac{27}{4}} + \sin^{-1}\frac{1}{2}\right) - \left(-\sqrt{9} + \sin^{-1} 0\right)\right] + 6$$ $$= 3\sqrt{3} - \frac{\pi}{3}$$
4(i)	$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(4-P) - h$$ $$= -P^2 + 4P - h$$ $$= -(P^2 - 4P) - h$$ $$= -(P-2)^2 + 4 - h$$ 

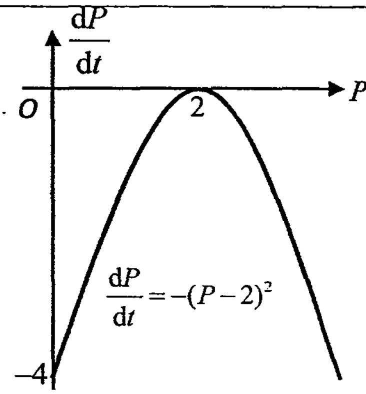
(ii)

From graph in (i),
when $h=4$,

$$\frac{dP}{dt}=-(P-2)^2=0 \text{ at } P=2.$$

$\therefore$ largest $h=4$ where the population P remains constant at $P=2$.

Hence $MSY=4$ million



(iii)

$$\frac{dP}{dt}=P(4-P)-3=-\left[(P^2-4P)+3\right]=-\left[(P-2)^2-1\right]$$

<u>Method 1</u>

$$\int\frac{1}{(P-2)^2-1}\,dP=-\int 1\,dt$$

$$\frac{1}{2}\ln\left|\frac{(P-2)-1}{(P-2)+1}\right|=-t+C$$

$$\frac{P-3}{P-1}=\pm e^{-2t+2C}=Ae^{-2t} \quad \text{where } A=\pm e^{2C}$$

In 2015, let $t=0$, $P=3.2$; hence $A=\frac{1}{11}$

$$\therefore\ P-3=\frac{1}{11}e^{-2t}(P-1)$$

$$11P-33=Pe^{-2t}-e^{-2t}$$

Hence $P=\dfrac{33-e^{-2t}}{11-e^{-2t}}=\dfrac{33e^{2t}-1}{11e^{2t}-1}$

<u>Method 2</u>

$$\frac{dP}{dt}=1-(P-2)^2$$

$$\int\frac{1}{1-(P-2)^2}\,dP=\int 1\,dt$$

$$\frac{1}{2}\ln\left|\frac{1+(P-2)}{1-(P-2)}\right|=t+C$$

$$\frac{P-1}{3-P}=\pm e^{2t+2C}=Ae^{2t} \quad \text{where } A=\pm e^{2C}$$

In 2015, let $t=0$, $P=3.2$; hence $A=-11$

$$P-1=-11e^{2t}(3-P)$$

$$P-1=-33e^{2t}+11Pe^{2t}$$

$$P=\frac{1-33e^{2t}}{1-11e^{2t}} \text{ or } P=\frac{33e^{2t}-1}{11e^{2t}-1}$$

	Method 3 $-\int \frac{1}{P^2-4P+3}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{(P-3)(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ $-\int \frac{1}{2(P-3)}\,\mathrm{d}P + \int \frac{1}{2(P-1)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ (using partial fractions) $\frac{1}{2}\ln\left\|\frac{P-1}{P-3}\right\| = t + C$ $\frac{P-1}{P-3} = \pm e^{2t+2C} = Ae^{2t}$ In 2015, let $t=0$, $P=3.2$; hence $A=11$ $P-1 = 11e^{2t}(P-3)$ $P-1 = 11Pe^{2t} - 33e^{2t}$ $P = \frac{1-33e^{2t}}{1-11e^{2t}}$ or $P = \frac{33e^{2t}-1}{11e^{2t}-1}$ In 2016, $t=1$ Hence $P = \frac{33e^2-1}{11e^2-1} = 3.02$ $\therefore$ the population of wild salmon is 3.02 million in 2016.
(iv)	There are no external factors such as marine pollution or climate change that drastically affect the population of wild salmon in that region.

5 (i)	Simple Random Sampling: Using a random number generator to generate 400 numbers and use select the voting slips corresponding to these 400 numbers Systematic Sampling: Consider N registered voter in the electoral division such that the sampling interval $\frac{N}{400}$ is an integer. Using a random number generator, select a number from 1 to k and take every kth number thereafter until a sample of 400 is obtained. Choose the voting slips corresponding to the numbers. Stratified Sampling: Use each polling station as the stratum. The number of votes in each stratum is calculated by $\frac{\text{no. of voters polling at the station}}{\text{total number of voters in the electoral division}} \times 400$. The votes is then obtained from each stratum using simple random sampling. Quota Sampling: Consider N registered voter in the electoral division such that the sampling interval $\frac{N}{400}$ is an integer. Using a random number generator, select a number from 1 to k and take every kth number thereafter until a sample of 400 is obtained. Choose the voting slips corresponding to the numbers.

	Use each polling station as the stratum, decide on the number of votes to be obtained from each stratum. For example if there are 4 polling stations, obtain 100 votes from each polling station. The voting slips are then selected based on the officer's discretion.
(ii)	Simple random sampling: Advantage: The sample obtained is free from bias The sampling procedures are easy to follow Disadvantage The sample obtained might not be a good representation of the electoral division Systematic Sampling: Advantage: It is easy execute because only the first number needs to be chosen The electoral division will be evenly sampled as the voting slips is chosen at regular intervals Disadvantage: If there is a periodic trend like every *k*th voters are of the same gender, systematic sampling may produce a biased sample Stratified Sampling: Advantage: Stratified sampling will provide a sample of voter that is representative of electoral division The results in each polling station can be analysed separately. Easy to conduct as the sampling frame (registered voters) is known. Disadvantage: It is time consuming to carry out stratified sampling Quota Sampling: Advantage: The sample of voters can be obtained quickly Disadvantage: The sample of voters obtained may not be a good representative of the electoral division
6	$\mathrm{P}(X=r-1)=\mathrm{P}(X=r)$ $\binom{n}{r-1}p^{r-1}(1-p)^{n-r+1}=\binom{n}{r}p^{r}(1-p)^{n-r}$ $\frac{n!}{(r-1)!(n-r+1)!}p^{r-1}(1-p)^{n-r+1}=\frac{n!}{(r)!(n-r)!}p^{r}(1-p)^{n-r}$ $\frac{r!(n-r)!}{(r-1)!(n-r+1)!}=\frac{p^{r}(1-p)^{n-r}}{p^{r-1}(1-p)^{n-r+1}}$ $\frac{r}{n-r+1}=\frac{p}{1-p}$ $\frac{\mathrm{P}(X=r)}{\mathrm{P}(X=r-1)}=\left(\frac{n-r+1}{r}\right)\frac{p}{1-p}$ (shown)

	$r(1-p)=p(n-r+1)$ $r-pr=np-pr+p$ $r=(n+1)p$ X will have two modes when $(n+1)p$ is a positive integer. $r=(n+1)p$, $r=(n+1)p-1$
7 (i)	No of ways that the single women are all separated $= {}^6C_4 \times 4! \times (6-1)! = 43200$ Probability $= \dfrac{43200}{9!} = \dfrac{5}{42} = 0.119$
(ii)	Probability that the single women are next to one another $= \text{P}(S) = \dfrac{(7-1)! \times 4!}{9!} = \dfrac{1}{21}$ Probability that the single men are next to each other $= \text{P}(B) = \dfrac{(9-1)! \times 2!}{9!} = \dfrac{2}{9}$ Probability that the single women are next to one another and the single men are next to each other $= \text{P}(S \cap B) = \dfrac{(6-1)! \times 2! \times 4!}{9!} = \dfrac{1}{63}$ Therefore probability $= \text{P}(S)+\text{P}(B)-2\text{P}(S \cap B) = \dfrac{1}{21}+\dfrac{2}{9}-\dfrac{2}{63}=\dfrac{5}{21}=0.238$
	No of ways $= 9! \times 2! = 725760$
8 (i)	$A, B \sim \text{N}(71, 8^2)$ $A-B \sim \text{N}(0,128)$ $\text{P}(0 \le A-B \le 2) = 0.0702$ (3.s.f)
(ii)	$X \sim \text{N}(62, \sigma^2)$, $Y \sim \text{N}(71, 8^2)$ Let $M = \dfrac{X+Y}{2} \sim \text{N}(66.5, \dfrac{\sigma^2+8^2}{4})$ $\text{P}(M \ge 75) = 0.15$ $\text{P}(M \le 75) = 0.85$ $\text{P}(Z \le \dfrac{75-66.5}{\sqrt{\dfrac{\sigma^2+64}{4}}}) = 0.85$ $\dfrac{8.5}{\sqrt{\dfrac{\sigma^2+64}{4}}} = 1.03643338$ $\sigma = 14.319$ $\sigma = 14.3$
(iii)	$X-Y \sim \text{N}(-9, 269.0389)$ $\text{P}(X>Y) = \text{P}(X-Y>0)$ $= 0.292$ (3 s.f.)
(iv)	Not valid because X and Y are not be independent for the same student.

9(a)	Let X be the total number of speeding incidents caught at Junctions A and B in an hour. $X \sim \text{Po}(\frac{23}{12})$ $\text{P}(X \geq 2) = 1 - \text{P}(X \leq 1) = 0.571$
(b)	$A+B+C \sim \text{Po}\left(\frac{23}{12}+\lambda\right)$ Required Probability $= \text{P}(C \geq 1 \mid A+B+C=2)$ $= \frac{\text{P}(A+B=1)\text{P}(C=1)+\text{P}(A+B=0)\text{P}(C=2)}{\text{P}(A+B+C=2)}$ $= \frac{\left(\frac{23}{12}e^{-\frac{23}{12}}\right)(\lambda e^{-\lambda})+\left(e^{-\frac{23}{12}}\right)\left(e^{-\lambda}\frac{\lambda^2}{2}\right)}{\left(e^{-\frac{23}{12}-\lambda}\right)\frac{\left(\frac{23}{12}+\lambda\right)^2}{2}}$ $= \frac{\frac{e^{-\frac{23}{12}-\lambda}}{2}\left(\frac{23}{6}\lambda+\lambda^2\right)}{\frac{e^{-\frac{23}{12}-\lambda}}{2}\left(\frac{23}{12}+\lambda\right)^2}$ $= \frac{\frac{\lambda}{6}(23+6\lambda)}{\frac{1}{144}(23+12\lambda)^2}$ $= \frac{24\lambda(23+6\lambda)}{(23+12\lambda)^2}$
(c)	Let A be the number of speeding incidents caught at Junctions A, and B be the number of speeding incidents caught at Junction B in a day $A \sim \text{Po}(16)$, $B \sim \text{Po}(30)$ Since both 16 and 30 are greater than 10, $A \sim \text{N}(16,16)$ and $B \sim \text{N}(30,30)$ approximately $\Rightarrow A-B \sim \text{N}(-14,46)$ $\text{P}(A-B>0) \xrightarrow{c.c} \text{P}(A-B>0.5) = 0.0163$ The occurrence of speeding incidents caught at Junction A and Junction B are independent of each other.
10 (i)	Since n is small and population variance unknown, the nutritionist should use t-test. It is assumed that the calories count of the energy bar follows a normal distribution.

(ii)

$s^2 = \frac{15}{14}(20.74) = 22.221$

Let H_0 be the null hypothesis, H_1 be the alternative hypothesis. Let μ be the population mean number of calories in an energy bar and $\overline{X}$ be the sample mean.

$H_0 : \mu = 350$

$H_1 : \mu \neq 350$

Under H_0, Test statistic, $T = \dfrac{\overline{X} - 350}{\sqrt{\dfrac{22.221}{15}}} \sim t_{14}$

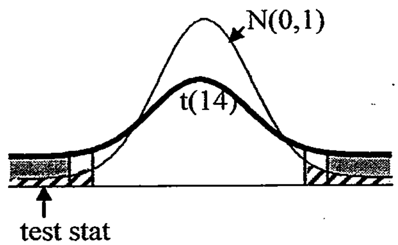
p value $= 0.0373 < 0.05$

Since the p – value < 0.05, reject H_0. There is sufficient evidence at 5% level of significance to conclude that the mean number of calories in an energy bar is not 350.

(iii)

Since H_0 is rejected for t-test, $p_t < 0.05$ and since $p_z < p_t < 0.05$. Therefore H_0 will be rejected under z test. So the conclusion will not be different.

Alternative method



Ho is rejected under t test

$\Rightarrow$ test statistic is inside critical region for t-test

$\Rightarrow$ test statistic is inside critical region for Z-test

$\Rightarrow$ Ho is rejected under Z test

The conclusion would be the same.

(iv)

Since H_0 is not rejected,

$\left|\dfrac{\bar{x} - 350}{\sqrt{1.3827}}\right| < 1.95996$

$\Rightarrow -1.95996 < \dfrac{\bar{x} - 350}{\sqrt{1.3827}} < 1.95996$

$-2.3047 < \bar{x} - 350 < 2.3047$

$347.7 < \bar{x} < 352.3$

11(i)

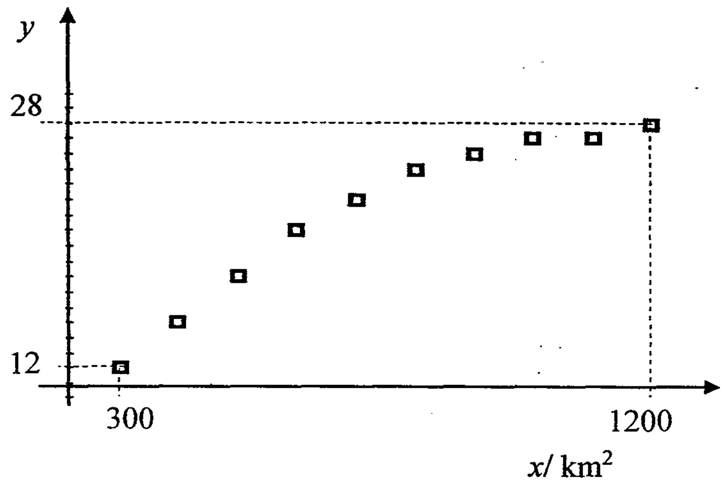
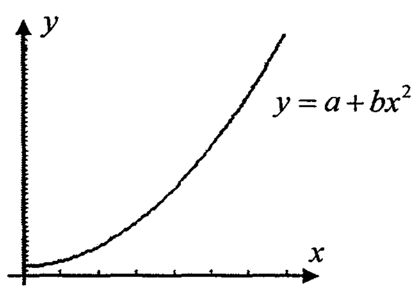
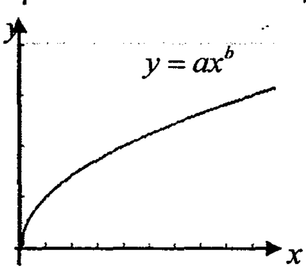
$\bar{x} = 750$

$\bar{y} = 0.01758(750) + 9.018$

$= 22.203$

$22.203(10) = k + 199$

$\therefore k = 23.03 \approx 23$

(ii)	
(iii)	$r = 0.958$. Though r is close to 1, the shape of the scatter plot is curvilinear therefore a linear model is not appropriate
(iv)	 Model B is a more appropriate model as graph concave downwards like the scatter diagram $\ln y = -0.96684 + 0.61722 \ln x$ $\ln y = -0.967 + 0.617 \ln x$ $\ln(24) = -0.96684 + 0.61722 \ln x$ $0.61722 \ln x = \ln(24) + 0.96684$ $x = 825$ Since $r = 0.982$ is close to 1 and $y = 24$ is within the data range, the prediction is appropriate