JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CIVICS GROUP

INDEX NUMBER

Mathematics **9740/01**

Paper 1 **22 August 2016**

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark-blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

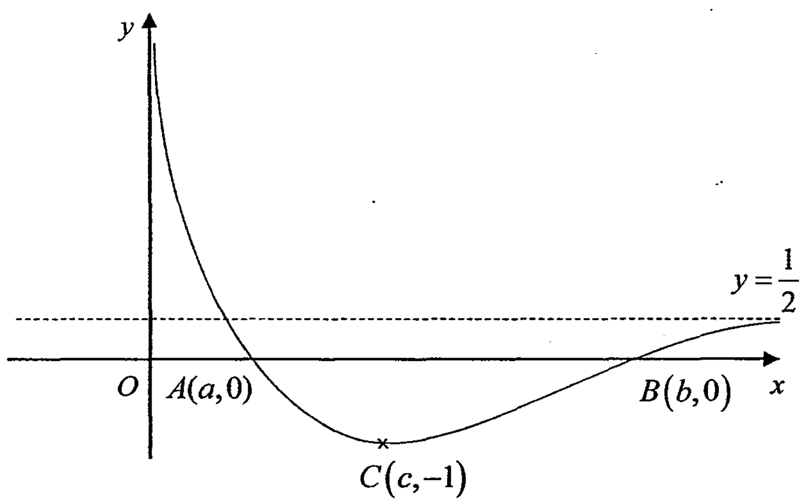
[Turn over

1 A theme park sells tickets at different prices according to the age of the customer. The age categories are senior citizen (ages 60 and above), adult (ages 13 to 59) and child (ages 4 to 12). Four tour groups visited the theme park on the same day. The numbers in each category for three of the groups, together with the total cost of the tickets for each of these groups, are given in the following table.

Group	Senior Citizen	Adult	Child	Total cost
A	2	19	9	$1982
B	0	10	3	$908
C	1	7	4	$778

Find the total cost of the tickets for Tour Group *D*, which consists of four senior citizens, five adults and one child. [4]

2



The diagram shows the curve $y = \mathrm{f}(x)$. The curve passes through the point $A(a,0)$ and the point $B(b,0)$, has a turning point at $C(c,-1)$ and asymptotes $y = \frac{1}{2}$ and $x = 0$.

Sketch, on separate diagrams, the graphs of

(a) $y = 3 - |\mathrm{f}(x)|$, [3]

(b) $y = \dfrac{2}{\mathrm{f}(x)}$. [3]

Label the graph in each case clearly and indicate the equations of the asymptotes and the coordinates of the points corresponding to *A*, *B* and *C*.

3 In the triangle ABC, $AB = 1$, $BC = 4$ and angle $ABC = \theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx \left(9+4\theta^2\right)^{\frac{1}{2}} \approx a+b\theta^2 ,$$

for constants a and b to be determined. [5]

4 [It is given that the volume of a pyramid is $\frac{1}{3}\times(\text{base area})\times(\text{height})$.]

A right pyramid of vertical height h m has a square base with side of length $2x$ m and volume $\frac{8}{3}$ m^3.

(i) Express h in terms of x. [1]

(ii) Show that the surface area S m^2 of the pyramid is given by

$$S = 4x^2\left[1+\sqrt{\left(1+\frac{4}{x^6}\right)}\right].$$ [3]

(iii) Use differentiation to find the value of x, correct to 2 decimal places, that gives a stationary value of S. [3]

5 Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point C on OA is such $OC : OA = 1 : 3$. The line l passes through the points A and B. It is given that angle $BOA = 60^\circ$ and $|\mathbf{a}| = 3|\mathbf{b}|$.

(i) By considering $(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$, or otherwise, express $|\mathbf{b}-\mathbf{a}|$ in the form $k|\mathbf{b}|$, where k is a constant to be found in exact form. [3]

(ii) Find, in terms of $|\mathbf{b}|$, the shortest distance from C to l. [5]

6 A curve has parametric equations

$$x=\cos^2\theta, \quad y=\sin 2\theta, \text{ for } -\frac{\pi}{2}<\theta\le\frac{\pi}{2}.$$

(i) Sketch the curve. [2]

The region enclosed by the curve is denoted by R. The part of R above the x-axis is rotated through 2π radians about the x-axis.

(ii) Show that the volume of the solid formed is given by

$$\pi\int_a^b \sin^3 2\theta \, \mathrm{d}\theta,$$

for limits a and b to be determined. [3]

Use the substitution $u=\cos 2\theta$ to find this volume, leaving your answer in exact form. [4]

7 The equation of a curve C is given by

$$3y^3-8y^2+10y=4-5x.$$

(i) Find the equation of the tangent at the point where $x=\frac{4}{5}$. [5]

(ii) Find the Maclaurin series for y, up to and including the term in x^2. [4]

(iii) State the equation of the tangent to the curve C at the point where $x=0$. [1]

8 **(a)** The complex number w is given by $(\sqrt{3})+k\mathrm{i}$, where $k<0$.

Given that w^5 is real, find the possible values of k in the form $k=(\sqrt{3})\tan(n\pi)$, where n is a constant to be determined. [4]

(b) **(i)** If $z=\cos\theta+\mathrm{i}\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$, show that

$$1-z^2=2\sin\theta(\sin\theta-\mathrm{i}\cos\theta).$$ [2]

(ii) Hence find $|1-z^2|$ and $\arg(1-z^2)$ in terms of θ. [3]

9 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{1}{x^2 - x - 6} + 2, \quad x \in \mathbb{R},\ x \neq -2,\ x \neq 3.$$

(i) Explain why the function f^{-1} does not exist. [2]

(ii) Find, algebraically, the set of values of x for which f is decreasing. [3]

In the rest of the question, the domain of f is further restricted to $x \leq \frac{1}{2}$.

The function g is defined by

$$\mathrm{g}: x \mapsto 2 - x, \quad x \in \mathbb{R}.$$

(iii) Find an expression for $\mathrm{gf}(x)$ and hence, or otherwise, find $(\mathrm{gf})^{-1}\left(\frac{1}{4}\right)$. [3]

10 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = \frac{1}{2}$ and

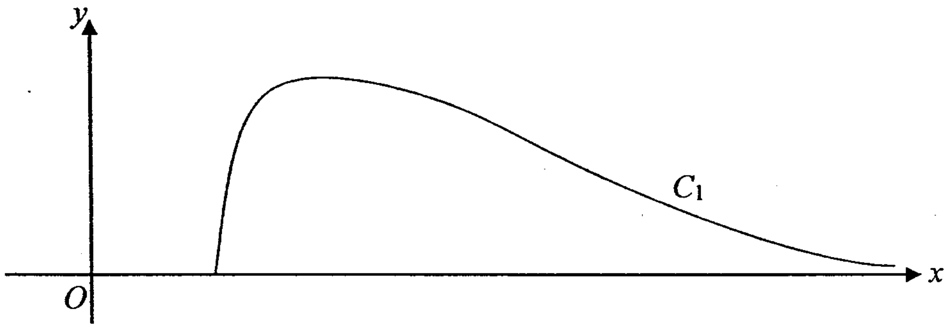
$$u_{n+1} = u_n - \frac{n^2 + n - 1}{(n+2)!}, \quad \text{for all } n \geq 1.$$

(i) Use the method of mathematical induction to prove that $u_n = \frac{n}{(n+1)!}$. [5]

(ii) Hence find $\sum_{n=1}^{N} \frac{n^2 + n - 1}{(n+2)!}$. [3]

(iii) Explain why $\sum_{n=1}^{\infty} \frac{n^2 + n - 1}{(n+2)!}$ is a convergent series, and state the value of the sum to infinity. [2]

11



The diagram above shows the curve C_1 with equation $y = \dfrac{\ln x}{x^2}$, where $x \geq 1$.

(i) Show that the exact coordinates of the turning point on C_1 are $\left(\sqrt{e}, \dfrac{1}{2e}\right)$. [3]

(ii) The curve C_2 has equation $\left(x - \sqrt{e}\right)^2 + (2ey)^2 = 1$, where $y \geq 0$. Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersection with the axes. [3]

(iii) Write down an integral that gives the area of the smaller region bounded by the two curves, C_1 and C_2, and the x-axis. Evaluate this integral numerically. [4]

12 **(a)** **(i)** Solve the equation

$$z^6 - 2i = 0,$$

giving the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

(ii) Show the roots on an Argand diagram. [2]

(iii) The points A, B, C, D, E and F represent the roots z_1, z_2, z_3, z_4, z_5 and z_6 respectively in the Argand diagram. Find the perimeter of the polygon $ABCDEF$, leaving your answer to 3 decimal places. [2]

(b) The complex number w satisfies the relations

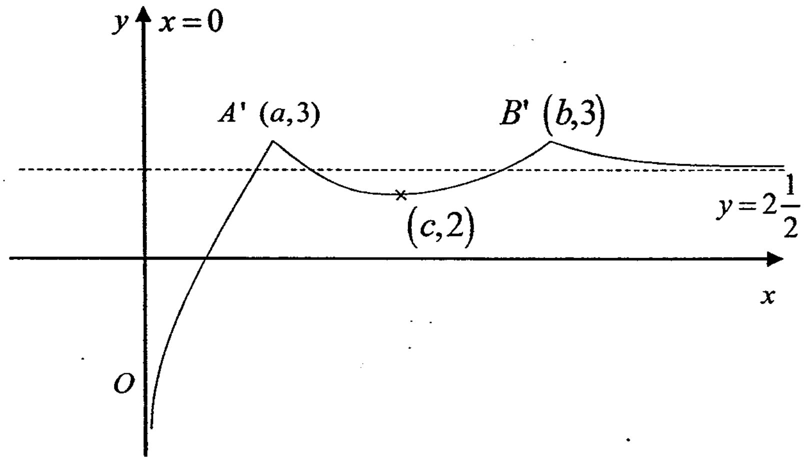
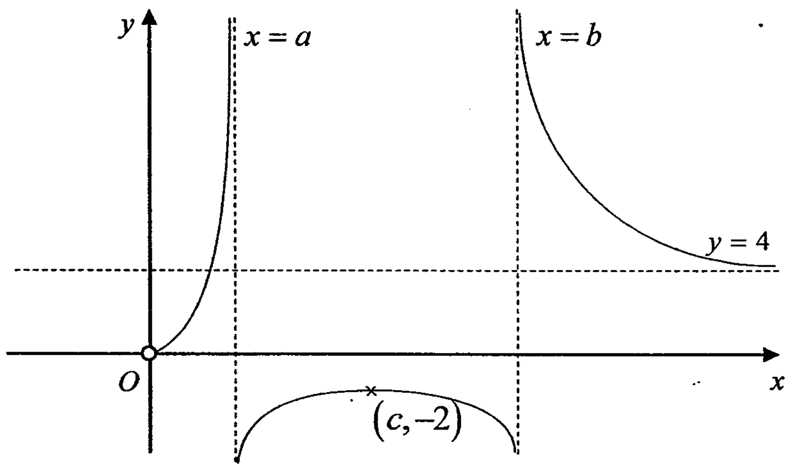
$$|w + 5 - 12i| \leq 13 \text{ and } 0 \leq \arg(w + 18 - 12i) < \frac{\pi}{4}.$$

(i) On an Argand diagram, sketch the region in which the points representing w can lie. [4]

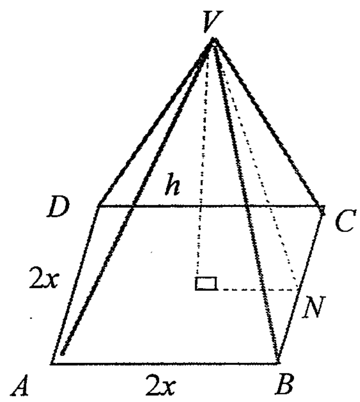
(ii) State the maximum and minimum possible values of $|w + 10|$. [2]

H2 Mathematics
JC2 Preliminary Examinations Paper 1
Solutions

1	**Solution**
	Let \$$x$, \$$y$ and \$$z$ be the cost of a ticket for a senior citizen, adult and child respectively. $2x+19y+9z=1982$ $10y+3z=908$ $x+7y+4z=778$ Using GC, $x=36$ $y=74$ $z=56$ Thus, the cost of a ticket for a senior citizen is \$36, for an adult is \$74 and for a child is \$56. $4(36)+5(74)+1(56)=570$ Therefore, the total cost for Group $D=\$570$

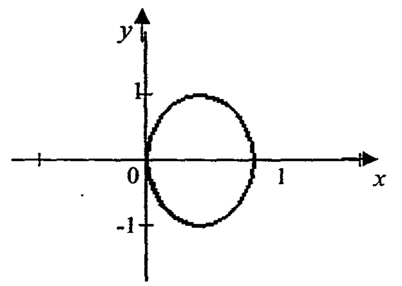
2	Solution
(a)	
(b)	

3	**Solution**
	Using cosine rule, $AC^2 = 1^2 + 4^2 - 2(1)(4)\cos\theta$ $= 1 + 16 - 8\cos\theta$ $= 17 - 8\cos\theta$ $\approx 17 - 8\left(1 - \frac{\theta^2}{2}\right)$ $= 9 + 4\theta^2$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}} \quad (\because AC > 0)$ $AC \approx \left(9 + 4\theta^2\right)^{\frac{1}{2}}$ $= 9^{\frac{1}{2}}\left(1 + \frac{4}{9}\theta^2\right)^{\frac{1}{2}}$ $= 3\left(1 + \frac{1}{2}\left(\frac{4}{9}\theta^2\right) + \ldots\right)$ $\approx 3\left(1 + \frac{2}{9}\theta^2\right)$ $= 3 + \frac{2}{3}\theta^2$ Therefore, $a = 3$ and $b = \frac{2}{3}$

4	Solution
(i)	 Volume of the pyramid $= \frac{8}{3}$ $\Rightarrow \frac{1}{3}(2x)^2 h = \frac{8}{3}$ $\Rightarrow h = \frac{2}{x^2}$
(ii)	In triangle VBC, height $= VN = \sqrt{h^2 + x^2}$ Area of the triangle $VBC = \frac{1}{2}(2x)\sqrt{h^2 + x^2}$ $= x\sqrt{\frac{4}{x^4} + x^2} = x\sqrt{x^2\left(\frac{4}{x^6} + 1\right)}$ $= x^2\sqrt{1 + 4x^{-6}}$ Hence total surface area of the pyramid, $S =$ base area $+ 4\times$ area of triangle VBC $= (2x)^2 + 4\left(x^2\sqrt{1+4x^{-6}}\right)$ $S = 4x^2\left[1 + \sqrt{\left(1 + \frac{4}{x^6}\right)}\right]$ (shown)
(iii)	$S = 4x^2\left[1 + \sqrt{(1+4x^{-6})}\right]$ $\frac{dS}{dx} = 4x^2\left(\frac{1}{2}\left(1+4x^{-6}\right)^{-\frac{1}{2}}\left(-24x^{-7}\right)\right) + (8x)\left[1+\sqrt{(1+4x^{-6})}\right]$ $= -48x^{-5}\left(1+4x^{-6}\right)^{-\frac{1}{2}} + 8x\left[1+\sqrt{(1+4x^{-6})}\right]$ $= -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}} - 1 - \sqrt{(1+4x^{-6})}\right]$ At the stationary value of S, $\frac{dS}{dx} = 0$.

	$\therefore \quad -8x\left[6x^{-6}\left(1+4x^{-6}\right)^{-\frac{1}{2}}-1-\sqrt{(1+4x^{-6})}\right]=0$ By G.C., $x=0.89090=0.89$ (to 2dp)

5	Solution
(i)	$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \lvert\mathbf{b}\rvert^2 + \lvert\mathbf{a}\rvert^2 - 2\mathbf{a}.\mathbf{b}$ $= \lvert\mathbf{b}\rvert^2 + 9\lvert\mathbf{b}\rvert^2 - 2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60°$ $\lvert\mathbf{b}-\mathbf{a}\rvert^2 = 10\lvert\mathbf{b}\rvert^2 - 2(3\lvert\mathbf{b}\rvert)\lvert\mathbf{b}\rvert\frac{1}{2}$ $\lvert\mathbf{b}-\mathbf{a}\rvert = \sqrt{7}\lvert\mathbf{b}\rvert$ Therefore, $k = \sqrt{7}$.
(ii)	$\mathbf{c} = \frac{1}{3}\mathbf{a}$ $\overrightarrow{CA} = \frac{2}{3}\mathbf{a}$ Shortest distance of C to l = $\frac{\left\lvert\frac{2}{3}\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{\left\lvert\frac{2}{3}\mathbf{a}\times\mathbf{b} - \frac{2}{3}\mathbf{a}\times\mathbf{a}\right\rvert}{\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{2\lvert\mathbf{a}\times\mathbf{b}\rvert}{3\lvert\mathbf{b}-\mathbf{a}\rvert} \quad \because \mathbf{a}\times\mathbf{a} = \mathbf{0}$ $= \frac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 60°}{3\lvert\mathbf{b}-\mathbf{a}\rvert}$ $= \frac{6\lvert\mathbf{b}\rvert^2\frac{\sqrt{3}}{2}}{3\sqrt{7}\lvert\mathbf{b}\rvert}$ $= \frac{\sqrt{3}\lvert\mathbf{b}\rvert}{\sqrt{7}}$ $= \sqrt{\frac{3}{7}}\lvert\mathbf{b}\rvert$

6	Solution
(i)	$x=\cos^2\theta,\quad y=\sin 2\theta,$ for $-\frac{\pi}{2}<\theta\le\frac{\pi}{2}.$ 
(ii)	$x=\cos^2\theta$ $dx=-2\cos\theta\sin\theta\,d\theta$ When $y=0,\quad \sin 2\theta=0$ $2\theta=0,\pi$ $\theta=0,\frac{\pi}{2}$ When $\theta=0,\ x=1$; When $\theta=\frac{\pi}{2},\ x=0$ Volume of the solid formed $=\pi\int_0^1 y^2\,dx$ $=\pi\int_{\frac{\pi}{2}}^{0}(\sin 2\theta)^2(-2\cos\theta\sin\theta\,d\theta)$ $=\pi\int_{\frac{\pi}{2}}^{0}(\sin 2\theta)^2(-\sin 2\theta\,d\theta)$ $=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\,d\theta$ (shown) $\therefore\ a=0,\ b=\frac{\pi}{2}.$ Let $u=\cos 2\theta$. $\therefore du=-2\sin 2\theta\ d\theta$ When $\theta=0,\ u=1$; When $\theta=\frac{\pi}{2},\ u=-1$ Volume of the solid formed $=\pi\int_0^{\frac{\pi}{2}}\sin^3 2\theta\,d\theta$

	$=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\sin^2 2\theta(2\sin 2\theta)\mathrm{d}\theta$ $=\frac{1}{2}\pi\int_0^{\frac{\pi}{2}}\left(1-\cos^2 2\theta\right)\left(2\sin 2\theta\,\mathrm{d}\theta\right)$ $=\frac{1}{2}\pi\int_1^{-1}\left(1-u^2\right)(-\mathrm{d}u)$ $=-\frac{1}{2}\pi\left[u-\frac{u^3}{3}\right]_1^{-1}$ $=-\frac{1}{2}\pi\left[\left(-1+\frac{1}{3}\right)-\left(1-\frac{1}{3}\right)\right]$ $=-\frac{1}{2}\pi\left(-\frac{4}{3}\right)$ $=\frac{2}{3}\pi$ units^3

7	**Solution**
(i)	$3y^3-8y^2+10y=4-5x$ Differentiate wrt x; $\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-5}{9y^2-16y+10}$ When $x=\frac{4}{5}$, $3y^3-8y^2+10y=0$. $\therefore y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}$. Eqn of tangent : $y-0=-\frac{1}{2}\left(x-\frac{4}{5}\right)$, ie $y=-\frac{1}{2}x+\frac{2}{5}$
(ii)	$\left(9y^2-16y+10\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-5$ Differentiate wrt x, $\left(9y^2-16y+10\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(18y-16\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. When $x=0$, $y=\frac{2}{3},\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{3}{2},\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{27}{10}$. $\therefore\quad y=\frac{2}{3}-\frac{3}{2}x+\frac{27}{20}x^2+\ldots$
(iii)	$y=\frac{2}{3}-\frac{3}{2}x$

8	Solution
(a)	$\arg\left(w^5\right)=5\arg\left(w\right)=0,\pm\pi,\pm2\pi...$ $\arg\left(w\right)=0,\frac{\pi}{5},-\frac{\pi}{5},\frac{2\pi}{5},\frac{-2\pi}{5},...$ Since $k<0$, $\arg\left(w\right)=-\frac{\pi}{5}$ or $-\frac{2\pi}{5}$. $\frac{k}{\sqrt{3}}=\tan\left(-\frac{\pi}{5}\right)$ or $\frac{k}{\sqrt{3}}=\tan\left(-\frac{2\pi}{5}\right)$ $k=\sqrt{3}\tan\left(-\frac{\pi}{5}\right)$ or $k=\sqrt{3}\tan\left(-\frac{2\pi}{5}\right)$ $n=-\frac{1}{5}$ or $-\frac{2}{5}$
(bi)	**Method 1** $1-z^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-\left(\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta+(\mathrm{i}\sin\theta)^2\right)$ $=1-\left(1-\sin^2\theta+2\mathrm{i}\sin\theta\cos\theta-\sin^2\theta\right)$ $=1-1+2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)$ **Method 2** $1-z^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-(\cos2\theta+\mathrm{i}\sin2\theta)$ $=1-\cos2\theta-\mathrm{i}\sin2\theta$ $=1-\left(1-2\sin^2\theta\right)-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)$
(bii)	**Method 1** $\left\|1-z^2\right\|=\left\|2\sin\theta(\sin\theta-\mathrm{i}\cos\theta)\right\|$ $=2\sin\theta\sqrt{\sin^2\theta+\cos^2\theta}$ $=2\sin\theta$ Given that $0\le\theta\le\frac{\pi}{2}$

$$\arg\left(1-z^2\right)=\arg\left[2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)\right]$$
$$=\arg\left(2\sin\theta\right)+\arg\left(\sin\theta-\mathrm{i}\cos\theta\right)$$
$$=0-\tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$
$$=-\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)$$
$$=-\left(\frac{\pi}{2}-\theta\right)$$
$$=\theta-\frac{\pi}{2}$$

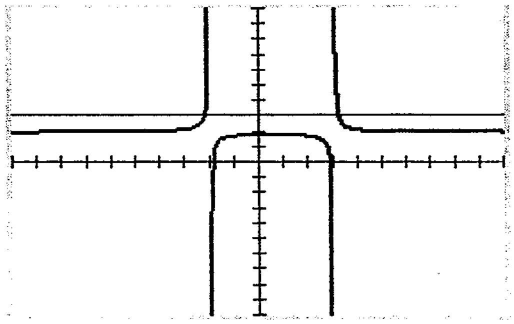
Method 2

$$1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$$
$$=2\sin\theta\left(-\mathrm{i}\right)\left(\cos\theta+\mathrm{i}\sin\theta\right)$$
$$=\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}$$
$$\left|1-z^2\right|=\left|\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right|$$
$$=2\sin\theta$$

$$\arg\left(1-z^2\right)=\arg\left(\left(-2\mathrm{i}\sin\theta\right)\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$=\arg\left(-2\mathrm{i}\sin\theta\right)+\arg\left(\mathrm{e}^{\mathrm{i}\theta}\right)$$
$$=-\frac{\pi}{2}+\theta$$

Method 3

$$1-z^2=2\sin\theta\left(\sin\theta-\mathrm{i}\cos\theta\right)$$
$$=2\sin\theta\left(\cos\left(\frac{\pi}{2}-\theta\right)-\mathrm{i}\sin\left(\frac{\pi}{2}-\theta\right)\right)$$
$$=2\sin\theta\left(\cos\left(\theta-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(\theta-\frac{\pi}{2}\right)\right)$$
$$\left|1-z^2\right|=2\sin\theta$$
$$\arg\left(1-z^2\right)=\theta-\frac{\pi}{2}$$

9	**Solution**
(i)	 From graph, the horizontal line $y = 3$ cuts the graph at two points. Hence f is not a **one-one** function, hence f^{-1} does not exist.
(ii)	$\mathrm{f}(x) = \dfrac{1}{x^2 - x - 6} + 2$ $\mathrm{f}'(x) = -\left(x^2 - x - 6\right)^{-2}(2x - 1)$ $= \dfrac{1 - 2x}{\left(x^2 - x - 6\right)^2}$ For the function to be decreasing, $\mathrm{f}'(x) \le 0$. $1 - 2x \le 0$ $1 \le 2x$ $x \ge 0.5$ $\{x \in \square: x \ge 0.5, x \ne 3\}$
(iii)	$\mathrm{gf}(x) = 2 - \dfrac{1}{x^2 - x - 6} - 2$ $= -\dfrac{1}{x^2 - x - 6}, \quad x \le \dfrac{1}{2}$ $(\mathrm{gf})^{-1}\left(\dfrac{1}{4}\right) = x$ $\mathrm{gf}(x) = \dfrac{1}{4}$ $-\dfrac{1}{x^2 - x - 6} = \dfrac{1}{4}$ $x^2 - x - 6 = -4$ $x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ (rejected) or $x = -1$ $x = -1$

10 **Solution**

(i)

Let P_n be the statement $u_n = \frac{n}{(n+1)!}$ for $n \in \mathbb{Z}^+$.

P_1 is true since $u_1 = \frac{1}{2!} = \frac{1}{2}$.

Assume that P_k is true for some $k \in \mathbb{Z}^+$,

i.e. $u_k = \frac{k}{(k+1)!}$

Consider P_{k+1}:

i.e. $u_{k+1} = \frac{k+1}{(k+2)!}$

$$
\begin{aligned}
u_{k+1} &= \frac{k}{(k+1)!} - \frac{k^2+k-1}{(k+2)!} \\
&= \frac{k(k+2)-(k^2+k-1)}{(k+2)!} \\
&= \frac{k^2+2k-k^2-k+1}{(k+2)!} \\
&= \frac{k+1}{(k+2)!}
\end{aligned}
$$

Thus, P_k is true $\Rightarrow P_{k+1}$ is true.

Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.

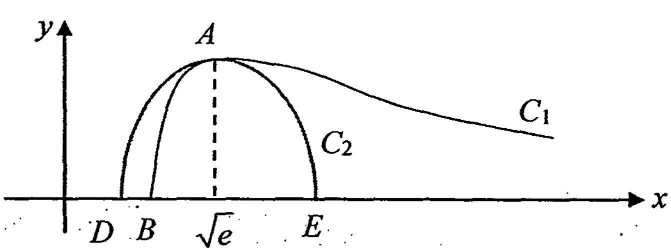
(ii)

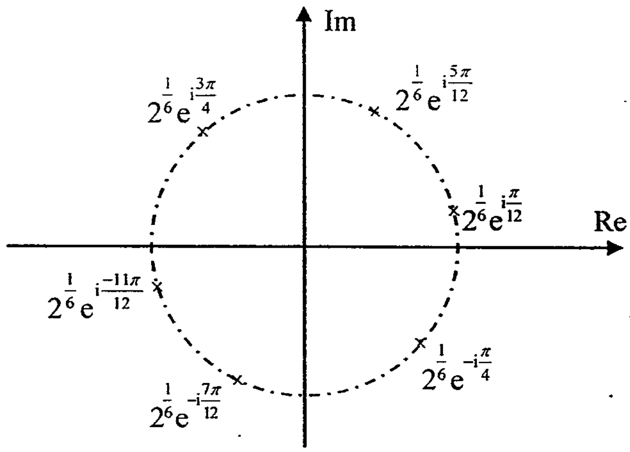
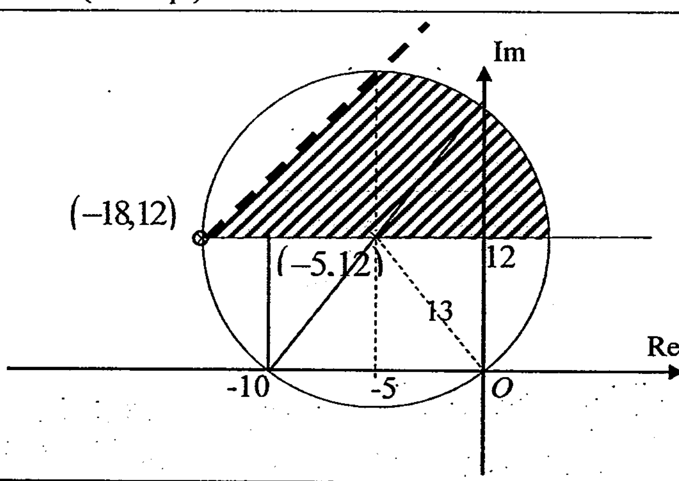
$$\sum_{n=1}^{N} \frac{n^2+n-1}{(n+2)!} = \sum_{n=1}^{N} (u_n - u_{n+1})$$

$$
\begin{aligned}
[\, & u_1 - u_2 \\
+ & u_2 - u_3 \\
+ & u_3 - u_4 \\
+ & \ldots \\
+ & u_{N-1} - u_N \\
+ & u_N - u_{N+1} \,]
\end{aligned}
$$

$$= u_1 - u_{N+1} = \frac{1}{2} - \frac{N+1}{(N+2)!}$$

(iii)	As $N \to \infty$, $\dfrac{N+1}{(N+2)!} \to 0$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n-1}{(n+2)!} \to \frac{1}{2}$ which is a constant, hence it is a convergent series.

11	**Solution**
(i)	$y=\dfrac{\ln x}{x^2},\ x\geq 1$ $\dfrac{dy}{dx}=\dfrac{x^2\cdot\dfrac{1}{x}-\ln x\cdot 2x}{x^4}$ $=\dfrac{x-2x\ln x}{x^4}$ When $\dfrac{dy}{dx}=0$ and since $x\neq 0$, $1-2\ln x=0$ $2\ln x=1$ $\ln x=\dfrac{1}{2}$ $x=e^{\frac{1}{2}}=\sqrt{e}$ When $x=\sqrt{e}$, $y=\dfrac{\ln\sqrt{e}}{\left(\sqrt{e}\right)^2}$ $=\dfrac{1}{2e}$ Hence the coordinates of A is $\left(\sqrt{e},\dfrac{1}{2e}\right)$.
(ii)	$B(1,0)$, $D\left(\sqrt{e}-1,0\right)$ and $E\left(\sqrt{e}+1,0\right)$ 
(iii)	Area $=\int_{\sqrt{e}-1}^{\sqrt{e}}\dfrac{\sqrt{1-\left(x-\sqrt{e}\right)^2}}{2e}\,dx-\int_1^{\sqrt{e}}\dfrac{\ln x}{x^2}\,dx$ $=0.14446942-0.09020401$ $=0.05426541$ $=0.0543$ (correct to 3 s.f.)

12	**Solution**
(ai)	$z^6 - 2\mathrm{i} = 0$ $z^6 = 2\mathrm{i}$ $z^6 = 2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+2k\pi\right)}$, $k = 0, \pm1, \pm2, -3$ $z = 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-11\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-7\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{-\pi}{4}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}, 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$
(aii)	
(aiii)	Since *ABCDEF* is a regular hexagon, the triangles *OAB*, *OBC* … are equilateral triangles. Perimeter of the polygon $= 6 \times 2^{\frac{1}{6}}$ $= 6.735$ (to 3 d.p.)
(bi)	
(bii)	Minimum $\lvert w+10 \rvert = 12$ Maximum $\lvert w+10 \rvert = 26$ (diameter of circle)

H

JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 2

CANDIDATE NAME

CIVICS GROUP | INDEX NUMBER

Mathematics **9740/02**

Paper 2 **24 August 2016**

3 hours

Additional materials: Answer Paper
Cover Page
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Without using a calculator, solve the inequality

$$\frac{3}{4x+3} \le \frac{x}{x+1}.$$ [5]

Hence, or otherwise, solve the inequality

$$\frac{3}{4e^x+3} > \frac{e^x}{e^x+1}.$$ [2]

2 Analysts estimate that when a viral video is posted online, the video attracts comments in such a way that at the end of every hour, the number of comments added for the video is thrice the number of comments at the start of that hour.

In a particular instance, a viral video was posted online and there was one comment immediately after the video was posted. Using the above model proposed by analysts, there will be 3 additional comments by the end of the first hour, 12 additional comments by the end of the second hour, and so on.

(i) Find the number of complete hours for the total number of comments posted online to exceed 200 000. [3]

When the number of these comments posted online reaches 200 000 exactly, Software X is immediately activated to remove the comments. Software X works in such a way that it removes x comments at the start of each day. Once Software X is activated, it is also known that the number of comments at the end of the day is 2% more than the number of comments at the start of the day.

(ii) Show that the number of comments at the end of day n is

$$1.02^n(200\,000) - 51x\left(1.02^n - 1\right),$$

where day 1 is the day that the number of comments is exactly 200 000. [3]

(iii) Hence find the range of values of x such that all comments are removed by the end of day 30. Leave your answer to the nearest integer. [2]

Software Y is able to remove comments at the following rate.

- Day 1: 15 000 comments removed
- Subsequent Day: 90% of the number of comments removed on the preceding day

Without using Software X, explain whether Software Y alone is able to remove all 200 000 comments eventually. [2]

3 A team of naturalists is studying the change in population of wild boars on an island. It is suggested that the population of wild boars, x hundred, at time t years, can be modelled by the differential equation

$$\frac{dx}{dt} = \frac{1}{10}x(5-x).$$

(i) Find an expression for x in terms of t, given that $x=1$ when $t=0$. [7]

(ii) Find the exact time taken for the population of wild boars to reach 200. [2]

(iii) Explain in simple terms what will eventually happen to the population of wild boars on the island using this model. [1]

4 The line l has equation $\frac{x-1}{-2} = y = \frac{z+7}{4}$, and the plane p has equation $x-z=2$.

(i) Find the acute angle between l and p. [3]

(ii) Find the coordinates of the point at which l intersects p. [3]

(iii) The perpendicular to p from the point with coordinates $(1,0,-7)$ meets p at the point N. Find the position vector of N. [4]

(iv) Find a vector equation of the line which is a reflection of l in p. [3]

Section B: Statistics [60 marks]

5 A company wants to find out the transportation habits of their employees. On one particular workday, the interviewer selects a sample of employees to interview from those walking into the company building by

- standing at the entrance of building and choosing at random one of the first 10 employees who walks into the building,
- then choosing every 10th employee after the first employee is chosen.

(i) What is this type of sampling method called? [1]

(ii) State, in this context, a disadvantage of the sampling method stated in part (i). [1]

(iii) Explain briefly how the interviewer could select a sample of 30 employees using quota sampling. [2]

6 Historical data shows that the number of goals scored per match at European Football Championships has a mean of 1.93 and a variance of 1.4. A large random sample of n matches is taken. Find the least value of n such that the probability that the average number of goals scored per match exceeds 2 goals is less than 0.24. [5]

7 A class of twenty four pupils consists of 11 girls and 13 boys. To form the class committee, four of the pupils are chosen at random as "Chairperson", "Vice Chairperson", "Treasurer" and "Secretary".

(i) Find the probability that the committee will consist of at least one girl and at least one boy. [3]

(ii) Find the probability that the "Treasurer" and "Secretary" are both girls. [3]

8 Under normal continuous use, the average battery life of a PI-99 calculator is claimed to be k hours. A random sample of 13 calculators were obtained, and the battery life, x hours, of each calculator was measured. The results are summarised by

$$\sum x = 573.39 \quad \text{and} \quad \sum (x - \bar{x})^2 = 42.22.$$

(i) Find unbiased estimates of the population mean and variance. [2]

A test is to be carried out at the 5% level of significance to determine if the claim made is valid.

(ii) State a necessary assumption to carry out the test. [1]

(iii) State the appropriate hypotheses for the test, defining any symbols that you use. [2]

(iv) Find the set of values of k for which the result of the test would be that the null hypothesis is not rejected. Leave all numerical answers in 2 decimal places. [3]

9 A roller-coaster ride has two separate safety systems to detect faults on the track and on the roller-coaster train itself. Over a long period of time, it is found that the average number of faults detected per day by the systems are 0.25 for the track and 0.15 for the train. Assume that the faults detected on the track are independent of those detected on the train.

(i) State, in this context, a condition that must be met for a Poisson distribution to be a suitable model for the number of faults occurring on a randomly chosen day. [1]

(ii) Find the probability that a total of at most 4 faults is detected by the two systems in a period of 10 days. [2]

(iii) Find the smallest number of days for which the probability that no fault is detected by the two systems is less than 0.05. [2]

(iv) Find the probability that, in a randomly chosen period of 10 days, there are at least 3 faults detected on the track, given that there are a total of at most 4 faults detected by the two systems. [3]

10 Alex and Ben play with each other a set of ten games at table tennis and for each game, the probability that Ben loses is 0.7.

(i) State, in this context, an assumption needed to use a binomial distribution to model the number of games that Ben loses. [1]

Assume that the assumption made in part **(i)** holds.

(ii) Find the probability that Ben loses more than half of the games. [2]

In order to improve his skills at table tennis, Ben attends an intensive training programme. After completing the training, Ben decides to play another set of n games with Alex. Assume that the number of games Ben loses, out of these n games, has the distribution $\mathrm{B}(n, 0.3)$.

(iii) Find the greatest value of n such that the probability that Ben loses more than 8 games is at most 0.01. [3]

(iv) Given that $n = 50$, use a suitable approximation to find the probability that the number of games Ben loses is between 10 and 20 inclusive. State the parameters of the distribution that you use. [3]

11 Research is being carried out into how the concentration of a drug in the bloodstream varies with time, measured from when the drug is given. Observations at successive times give the data shown in the following table.

Time (t minutes)	20	40	70	100	130	190	250
Concentration (m micrograms per litre)	85	62	51	33	29	14	6

(i) Draw a scatter diagram of these values, labelling the axes. Explain how you know from your diagram that the relationship between m and t should not be modelled by an equation of the form $m = a + bt$. [2]

It is thought that the concentration of the drug in the bloodstream at different times can be modelled by one of the formulae

$$m = ct^2 + d \quad \text{or} \quad m = e\ln t + f$$

where c, d, e and f are constants.

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) t^2 and m,

(b) $\ln t$ and m. [2]

(iii) Explain which of $m = ct^2 + d$ or $m = e\ln t + f$ is the better model and find the equation of a suitable regression line for this model. [3]

(iv) Use the equation of your regression line to estimate the concentration of the drug in the bloodstream when $t = 150$, correct to 2 decimal places. Comment on the reliability of the estimate obtained. [2]

12 Min Ho has just learnt how to use two different methods to mow a piece of lawn in his house garden.

Method A: This is a two-stage process that involves cutting the grass with a strimmer and then collecting the grass by raking it up. The time, X minutes, taken to cut the grass has the distribution $N(30, 4.8^2)$. Once the grass is cut, the time, Y minutes, taken to collect the grass has the distribution $N(20, 3.1^2)$.

Method B: This method uses a mower with a rechargeable battery that will cut and collect the grass at the same time. The time, S minutes, taken to do this has the distribution $N(38, 2.6^2)$. In addition to this, the battery has to be recharged once before the cut, and this time is fixed at 15 minutes.

(i) Find the probability that Min Ho takes more than 45 minutes to mow the lawn using Method A. [3]

(ii) Find the probability that using Method A to mow the lawn is faster than using Method B by more than 5 minutes. [4]

Assume that Min Ho mows the piece of lawn in his house garden on a weekly basis. Over a particular period of ten consecutive weeks, Min Ho uses Method A for the first four weeks and Method B for the next six weeks. Find the probability the average time taken to mow the lawn in a week is greater than 50 minutes. [4]

H2 Mathematics
JC2 Preliminary Examinations Paper 2
Solutions

1	Solution

$$\frac{3}{4x+3} \le \frac{x}{x+1}$$

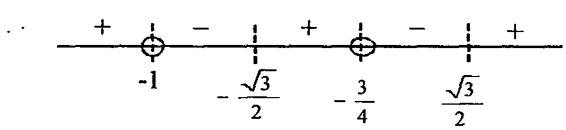
$$\frac{3}{4x+3} - \frac{x}{x+1} \le 0$$

$$\frac{3x+3-4x^2-3x}{(4x+3)(x+1)} \le 0$$

$$\frac{-4x^2+3}{(4x+3)(x+1)} \le 0 \quad ---(*)$$

$$\frac{4x^2-3}{(4x+3)(x+1)} \ge 0$$

$$\frac{(2x-\sqrt{3})(2x+\sqrt{3})}{(4x+3)(x+1)} \ge 0$$



Hence $x < -1$ or $-\frac{\sqrt{3}}{2} \le x < -\frac{3}{4}$ or $x \ge \frac{\sqrt{3}}{2}$

For $\frac{3}{4e^x+3} > \frac{e^x}{e^x+1}$, making use of the result in above part,

$-1 < e^x < -\frac{\sqrt{3}}{2}$ or $-\frac{3}{4} < e^x < \frac{\sqrt{3}}{2}$

(no solns since e^x is always positive)

Hence, $e^x < \frac{\sqrt{3}}{2} \Rightarrow x < \ln\left(\frac{\sqrt{3}}{2}\right)$

2 **Solution**

(i)

Hour	Start of hour	End of hour
1	1	$1+3=4$
2	4	$4+12=16$
3	16	$16+48=64$
4	...	...

$$4(4)^{n-1} > 200\,000$$
$$(4)^{n-1} > 50000$$
$$n-1 > 7.80482$$
$$n > 8.80842$$

Number of complete hours = 9

<u>Alternative Solution</u>

$$4(4)^{n-1} > 200\,000$$

n	Total
8	$65536 < 200\,000$
9	$262144 > 200\,000$
10	$11048576 > 200\,000$

Number of complete hours = 9

(ii)

Day	Start of day	End of day
1	$200000-x$	$1.02(200000-x)$
2	$1.02(200000-x)$ $-x$	$1.02[1.02(200000-x)$ $-x]$ $=1.02^2(200000)$ $-1.02x-1.02^2x$
3	...	...

At the end of day n, the number comments

$=1.02^n(200000)-(1.02x+1.02^2x+...+1.02^nx)$ --- (*)

	$=1.02^n(200000)-x\left(\frac{1.02(1.02^n-1)}{0.02}\right)$ $=1.02^n(200000)-51x(1.02^n-1)$
(iii)	$1.02^{30}(200\,000)-51x(1.02^{30}-1)<0$ $x>\frac{1.02^{30}(200\,000)}{51(1.02^{30}-1)}$ $x\geq 8755$ (to nearest integer)
	Day 1: no. of comments removed $=15000$ Day 2: no. of comments removed $=15000(0.9)$ Day 3: no. of comments removed $=15000(0.9)^2$ As $n\to\infty$, no. of comments removed $=\frac{15000}{1-0.9}=150\,000$ Software Y is unable to remove all the comments because eventually it is only able to remove 150 000 comments.

3	**Solution**
(i)	Method 1:

$$\int \frac{1}{x(5-x)}\,dx = \int \frac{1}{10}\,dt \quad ---(*)$$

Doing partial fractions

$$\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x}$$

$$= \frac{A(5-x)+B(x)}{x(5-x)}$$

$$A = \frac{1}{5}$$

$$B = \frac{1}{5}$$

$$\int \frac{1}{5x} + \frac{1}{5(5-x)}\,dP = \int \frac{1}{10}\,dt$$

$$\frac{1}{5}\left[\ln|x| - \ln|5-x|\right] = \frac{1}{10}t + c$$

$$\ln\left|\frac{x}{5-x}\right| = \frac{1}{2}t + c$$

$$\frac{x}{5-x} = Ae^{\frac{1}{2}t}, \text{ where } A = \pm e^{c}$$

Given $x = 1$ when $t = 0$, $\frac{1}{5-1} = Ae^{0} \Rightarrow A = \frac{1}{4}$

$$x = \frac{5}{4}e^{\frac{1}{2}t} - \frac{1}{4}xe^{\frac{1}{2}t}$$

$$x(4 + e^{\frac{1}{2}t}) = 5e^{\frac{1}{2}t}$$

$$x = \frac{5e^{\frac{1}{2}t}}{4 + e^{\frac{1}{2}t}}$$

(i)	Method 2: $\int \frac{1}{x(5-x)}\,dx = \int \frac{1}{10}\,dt$ ---(*) $\int \frac{1}{\frac{25}{4}-(x-\frac{5}{2})^2}\,dx = \int \frac{1}{10}\,dt$ $\frac{1}{2\left(\frac{5}{2}\right)}\ln\left\|\frac{\frac{5}{2}+(x-\frac{5}{2})}{\frac{5}{2}-(x-\frac{5}{2})}\right\| = \frac{1}{10}t+c$ $\ln\left\|\frac{x}{5-x}\right\| = \frac{1}{2}t+c$ $\frac{x}{5-x} = Ae^{\frac{1}{2}t}$, where $A = e^{\pm c}$ Given $x=1$ when $t=0$, $\frac{1}{5-1} = Ae^0 \Rightarrow A = \frac{1}{4}$ $x = \frac{5}{4}e^{\frac{1}{2}t} - \frac{1}{4}xe^{\frac{1}{2}t}$ $x(4+e^{\frac{1}{2}t}) = 5e^{\frac{1}{2}t}$ $x = \frac{5e^{\frac{1}{2}t}}{4+e^{\frac{1}{2}t}}$ or $x = \frac{5}{4e^{-\frac{1}{2}t}+1}$
(ii)	When $x=2$, $\frac{5e^{\frac{1}{2}t}}{4+e^{\frac{1}{2}t}} = 2$ $8+2e^{\frac{1}{2}t} = 5e^{\frac{1}{2}t}$ $3e^{\frac{1}{2}t} = 8$ $e^{\frac{1}{2}t} = \frac{8}{3}$ $t = 2\ln\left(\frac{8}{3}\right)$ It takes $t = 2\ln\left(\frac{8}{3}\right)$ years.
(iii)	As $t \to \infty$, $x \to 5$. $\therefore$ The population of wild boars will increase and stabilise at 500 eventually.

4	**Solution**
(i)	$l:\ \mathbf{r}=\begin{pmatrix}1\\0\\-7\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\4\end{pmatrix}$, where λ is a real parameter. $p:\ \mathbf{r}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $\sin\theta=\dfrac{\left\|\begin{pmatrix}-2\\1\\4\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-2\\1\\4\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\|}=\dfrac{6}{\sqrt{21}\sqrt{2}}$ $\therefore\theta=67.8°$ (1 dec pl)
(ii)	For the point of intersection between l and p, $\begin{pmatrix}1-2\lambda\\\lambda\\-7+4\lambda\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $1-2\lambda+7-4\lambda=2$ $\lambda=1$ The position vector of point of intersection is $\begin{pmatrix}1\\0\\-7\end{pmatrix}+\begin{pmatrix}-2\\1\\4\end{pmatrix}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}$. Coordinates of point of intersection are $(-1,1,-3)$.
(iii)	The line perpendicular to p passing through $(1,0,-7)$ is $\mathbf{r}=\begin{pmatrix}1\\0\\-7\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \mu\in\mathbb{R}$ $\begin{pmatrix}1+\mu\\0\\-7-\mu\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=2$ $1+\mu+7+\mu=2$

	$2\mu = -6$ $\mu = -3$ $\overrightarrow{ON} = \begin{pmatrix} 1-3 \\ 0 \\ -7+3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$
(iv)	<u>Method 1:</u> Let the coordinates of A be $(1,0,-7)$. Let A' be the reflected point of A in p. Using ratio theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$ $\Rightarrow \overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = 2\begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}$ The reflected line contains the point A' and point of intersection between l and p. The direction vector of the reflected line is $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ $\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \mathbb{R}$
(iv)	<u>Method 2:</u> Let the coordinates of A be $(1,0,-7)$.Let A' be the reflected point of A in p. Let the coordinates of B be $(-1,1,-3)$. $\overrightarrow{BN} = \dfrac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$ $\overrightarrow{BA'} = 2\overrightarrow{BN} - \overrightarrow{BA}$ $= 2\begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$ $\therefore \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}, \alpha \in \mathbb{R}$

5	**Solution**
(i)	Systematic sampling
(ii)	(slower, more difficult to collect) Systematic sampling is a more tedious process to select the employees, whereas quota sampling is quick and easy. Another possible reason: might miss out a certain group of people due to different reporting times.
(iii)	The interviewer could consider transport mode of the employees as the stratum. A possible quota for each stratum is as follows: By private transport: 10; By public transport: 10; By walking: 10 The interviewer can then stand at the entrance of the building and select the sample until the above quota is met.

Quota table from 5(iii):

By private transport	By public transport	By walking
10	10	10

6	**Solution**
	$\mathrm{E}(X)=1.93 \quad \mathrm{Var}(X)=1.4$ Since n is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1.93, \frac{1.4}{n}\right)$ approximately. Given that $\mathrm{P}\left(\overline{X}>2\right)<0.24$ --- (*)
	Method 1: Using GC to set up table when $n=142$, $\mathrm{P}\left(\overline{X}>2\right)=0.24041$ (>0.24) when $n=143$, $\mathrm{P}\left(\overline{X}>2\right)=0.23964$ (<0.24) when $n=144$, $\mathrm{P}\left(\overline{X}>2\right)=0.23887$ (<0.24) $\therefore$ least n is 143.
	Method 2: Using algebraic method via standardization $\mathrm{P}\left(\overline{X}\le 2\right)>0.76$ $\mathrm{P}\left(Z\le \frac{2-1.93}{\sqrt{1.4/n}}\right)>0.76$

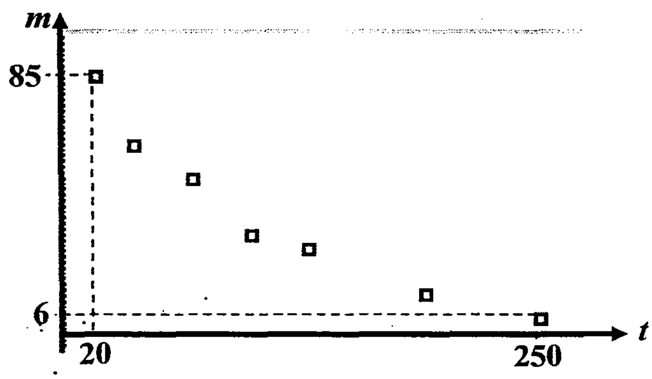
	From GC, $$\frac{2-1.93}{\sqrt{1.4/n}} > 0.70630 \quad ---(**)$$ $$\sqrt{n} > \frac{0.70630}{0.07}\sqrt{1.4}$$ $$\sqrt{n} > 11.939$$ $$n > 142.53$$ $\therefore$ least n is 143.

7	**Solution**
(i)	*Method 1:* Required probability $= 1 - \frac{^{13}C_4}{^{24}C_4} - \frac{^{11}C_4}{^{24}C_4} = 0.902$ (3 sig fig) *Method 2:* Required probability $= 1 - \frac{13\times12\times11\times10}{24\times23\times22\times21} - \frac{11\times10\times9\times8}{24\times23\times22\times21} = 0.902$ (3 sig fig) *Method 3:* Required probability $= \frac{^{11}C_1 \times {^{13}C_3} + {^{11}C_2} \times {^{13}C_2} + {^{11}C_3} \times {^{13}C_1}}{^{24}C_4}$ $= 0.902$ (3 sig fig)
(ii)	*Method 1:* Required probability $= \frac{^{11}C_2 \times 2! \times {^{22}C_2} \times 2!}{^{24}C_4 \times 4!} = 0.199$ (3 sig fig) *Method 2:* Required probability $= \frac{11\times10\times22\times21}{24\times23\times22\times21} = 0.199$ (3 sig fig)

8	**Solution**
(i)	Unbiased estimate of the population mean $\bar{x}=\dfrac{573.39}{13}=44.10692308=44.1$ (3 s.f.) Unbiased estimate of the population variance $s^2=\dfrac{42.22}{12}=3.518333333=3.52$ (3 s.f.)
(ii)	The battery life of a PI-99 calculator is assumed to be normally distributed.
(iii)	Let X be the r.v. denoting the battery life of a randomly chosen PI-99 calculator. Let μ be the population mean battery life of the PI-99 calculators. $H_0:\ \mu=k$ $H_1:\ \mu\neq k$ where H_0 is the null hypothesis and H_1 is the alternative hypothesis.
(iv)	To test at 5% level of significance. Under H_0, the test statistic is $T=\dfrac{\bar{X}-k}{\frac{S}{\sqrt{13}}}\sim t_{(12)}$. Since the null hypothesis is not rejected, t-value falls outside critical region. $\therefore\ -2.178812 < t-\text{value} < 2.178812$ $-2.178812 < \dfrac{\bar{x}-k}{s/\sqrt{13}} < 2.178812$ --- (*) $\bar{x}-2.178812\left(\dfrac{s}{\sqrt{13}}\right) < k < \bar{x}+2.178812\left(\dfrac{s}{\sqrt{13}}\right)$ where $\bar{x}=44.10692$ and $s=\sqrt{3.51833}$ $\therefore\ 42.97 < k < 45.24$ The required set is $\{k\in\mathbb{R}: 42.97 < k < 45.24\}$

9	**Solution**
(i)	The average number of faults detected by each system (for the track and the train) is constant from one day to another.
(ii)	Let X be the r.v. denoting the total number of faults detected by the two systems in a periods of 10 days. $X \sim \text{Po}\left((0.25+0.15)\times 10\right)$, i.e. $X \sim \text{Po}(4)$ $\therefore\ \text{P}\left(X \le 4\right) = 0.6288369 = 0.629$ (3 sig fig)
(iii)	Let Y be the r.v. denoting the total number of faults detected by the two systems in a period of n days. $Y \sim \text{Po}(0.4n)$ Given $\text{P}\left(Y=0\right) < 0.05$, *Method 1: Algebraic method* $e^{-0.4n} < 0.05$ **(o.e.** $\left(e^{-0.25n}\right)\left(e^{-0.15n}\right) < 0.05$) $n > 7.489$ $\therefore$ the smallest number of days required is 8. *Method 2: GC table* When $n=7$, $\text{P}\left(Y=0\right) = 0.06081$ (>0.05) When $n=8$, $\text{P}\left(Y=0\right) = 0.04076$ (<0.05) When $n=9$, $\text{P}\left(Y=0\right) = 0.02732$ (<0.05) $\therefore$ the smallest number of days required is 8.
(iv)	Let W and V be the r.v. denoting the number of faults detected on the track and on the track in a period of 10 days respectively. $W \sim \text{Po}(2.5)$ and $V \sim \text{Po}(1.5)$ Required probability $= \text{P}\left(W \ge 3 \mid V+W \le 4\right)$ $= \dfrac{\text{P}\left(W \ge 3 \cap V+W \le 4\right)}{\text{P}\left(V+W \le 4\right)}$ $= \dfrac{\text{P}(W=3)\text{P}(V=0)+\text{P}(W=3)\text{P}(V=1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\left(V+W \le 4\right)}$ $= \dfrac{\text{P}(W=3)\text{P}(V \le 1)+\text{P}(W=4)\text{P}(V=0)}{\text{P}\left(V+W \le 4\right)}$ $= 0.237$ (3 sig fig)

10	**Solution**
(i)	Ben's performance (i.e. whether he loses or wins) in a game is independent of any other games that he plays with Alex.
(ii)	Let X be the r.v. denoting the number of games that Ben loses out of 10 games. $X \sim \text{B}(10, 0.7)$ $\text{P}(X > 5) = 1 - \text{P}(X \le 5)$ $= 0.84973$ ≈ 0.850 (3 sig fig)
(iii)	Let Y be the r.v. denoting the number of games that Ben loses out of n games. $Y \sim \text{B}(n, 0.3)$ $\text{P}(Y > 8) \le 0.01$ $1 - \text{P}(Y \le 8) \le 0.01$ Using GC, When $n = 13$, $\text{P}(Y > 8) = 0.00403$ (< 0.01) When $n = 14$, $\text{P}(Y > 8) = 0.00829$ (< 0.01) When $n = 15$, $\text{P}(Y > 8) = 0.01524$ (> 0.01) $\therefore$ the greatest value of n is 14.
(iv)	Let W be the r.v. denoting the number of games that Ben loses out of 50 games. $W \sim \text{B}(50, 0.3)$ As $n = 50$ is large, $np = 15 \ (> 5)$ and $nq = 35 \ (> 5)$, $\therefore W \sim \text{N}(15, 10.5)$ approximately $\text{P}(10 \le W \le 20) = \text{P}(9.5 \le W \le 20.5)$ ---(*) $= 0.910$ (3 sig fig)

11	**Solution**
(i)	 From the scatter diagram, a curvinlinear correlation is observed between m and t (i.e. as t increases, m decreases at a decreasing rate), and hence a linear model with equation of the form $m=a+bt$ cannot be used to model the relationship between m and t.
(ii) (a)	Product moment correlation coefficient between m and $t^2=-0.8454$.
(b)	Product moment correlation coefficient between m and $\ln t\ =-0.9961$.
(iii)	Since the absolute value of the correlation coefficient between m and $\ln t$ (i.e. case (b)) is **<u>closer</u>** to 1, this indicates that the linear correlation between the variables m and $\ln t$ is **<u>stronger</u>** as compared to that between the variables for case (a). $\therefore$ case (b) is the better model for the relationship between m and t. $m=179.026-31.2175\ln t$ $\Rightarrow\ m=179-31.2\ln t$ (3 sig fig)
(iv)	When $t=150$, $m=179.026-31.2175\ln 150=22.61$ (2 dec pl) The estimate obtained is reliable, because the given value of $t\ =150$ lies within the given sample data range for t and the product moment correlation coefficient between m and $\ln t$ is very close to -1, hence indicating a strong negative linear correlation between the variables m and $\ln t$.

12	**Solution**
(i)	$X+Y \sim \mathrm{N}(50, 32.65)$ $\mathrm{P}(X+Y>45)=0.809224=0.809$ (3 sig fig)
(ii)	$\mathrm{E}(X+Y-S)=12$ $\mathrm{Var}(X+Y-S)=39.41$ $\therefore\ X+Y-S \sim \mathrm{N}(12, 39.41)$ P(method A is faster than method B by more than 5 mins) $=\mathrm{P}(S+15-(X+Y)>5)$ $=\mathrm{P}(X+Y-S<10)$ $=0.375020=0.375$ (3 sig fig)
(iii)	Let $A=X+Y$ and $B=S+15$. $A \sim \mathrm{N}(50, 32.65)$ and $B \sim \mathrm{N}(53, 2.6^2)$ Let $W=\dfrac{A_1+A_2+A_3+A_4+B_1+\ldots+B_6}{10}$ $\therefore\quad \mathrm{E}(W)=\dfrac{50\times 4+53\times 6}{10}=51.8$ & $\quad \mathrm{Var}(W)=\dfrac{32.65\times 4+2.6^2\times 6}{10^2}=1.7116$ $\therefore W \sim \mathrm{N}(51.8, 1.7116)$ Required probability $=\mathrm{P}(W>50)$ $=0.915566=0.916$ (3 sig fig)