SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS 9740/01

Paper 1 **25 August 2016**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

1 A local café, Toast Rox, sells its coffee in three sizes (regular, medium and large). Toast Rox customers get a 12.5% discount on their total bill if they buy at least 12 cups of coffee, regardless of size. The number of cups of coffee bought by three particular customers and the total amount they paid are shown in the following table.

Customer	Regular	Medium	Large	Amount paid
A	5	3	2	$20.90
B	3	4	1	$17.10
C	2	8	4	$28.00

Find the original price of each of the 3 sizes of coffee drink. [3]

2 **(i)** By using an algebraic method, solve the inequality

$$\frac{3x^2+14}{(x+1)(x+2)} \geq 2.$$ [4]

(ii) Hence, showing all your working clearly, solve the inequality

$$\frac{3x^2+14}{(|x|-1)(|x|-2)} \geq 2.$$ [2]

3 Referred to the origin O, points A and B have position vectors **a** and **b** respectively. Point C lies on the line segment AB, such that AC: CB = 2 : 1. Find, in terms of **a** and **b**, the position vector of C. [1]

If the angle between **a** and **b** is $60°$, show that the length of projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is

$$\frac{1}{3}\left(|\mathbf{a}|+|\mathbf{b}|\right).$$ [4]

4 **(a)** Two complex numbers z and w are such that

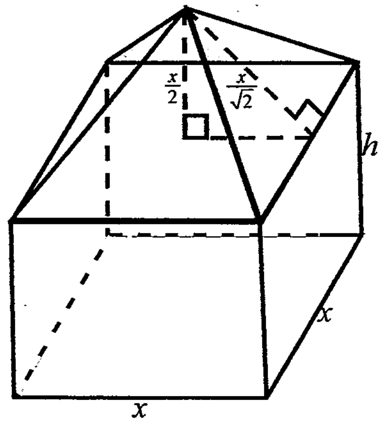
$$2w - z = 6\text{i} \quad \text{and} \quad wz = \frac{13}{2}.$$

Find w and z, giving each answer in the form $a + b\text{i}$, where a and b are real numbers. [4]

(b) The points P and Q represent the fixed complex numbers p and q respectively. It is given that $0 < \arg p < \arg q < \frac{\pi}{2}$, $|p| = 1$, $|q| = 2$, and $\arg q = 2\arg p$.

In a single Argand diagram, sketch and label the points P, Q, and the points R and S representing q^* and $q^* + 2p^2$ respectively, showing clearly any geometrical relationships. Identify the shape of the quadrilateral $OQSR$, where O is the origin. [4]

5 [It is given that volume of pyramid $= \frac{1}{3} \times$ base area $\times$ height.]



A model of a house is made up of the following parts.

- The roof is modelled by a pyramid with a square base of sides x cm and height $\frac{x}{2}$ cm. For each triangular side of the prism, the length of the perpendicular from the vertex to the base is $\frac{x}{\sqrt{2}}$ cm.
- The walls are modelled by rectangles with sides x cm and h cm as shown in the diagram.
- The base is a square with sides x cm.

All the parts are joined together as shown in the diagram. The model is made of material of negligible thickness. It is given that the volume of the model is a fixed value V cm^3 and the external surface area is at a minimum value, A cm^2. Use differentiation to find

(i) x, in the form $pV^{\frac{1}{3}}$, and

(ii) A, in the form $qV^{\frac{2}{3}}$,

leaving the values of p and q correct to 3 decimal places. [6]

6 A curve C has parametric equations

$$x = t^3 - kt, \quad y = 3\left(t^2 - k\right),$$

where k is a positive constant and t is a real parameter.

(i) Sketch C, labelling clearly the coordinates of any points of intersection with the axes. [2]

(ii) Find $\frac{dy}{dx}$ in terms of t and k. [2]

(iii) Find, in terms of k, the exact equation of the tangent to C at the point where $t = -\sqrt{\frac{k}{3}}$. [3]

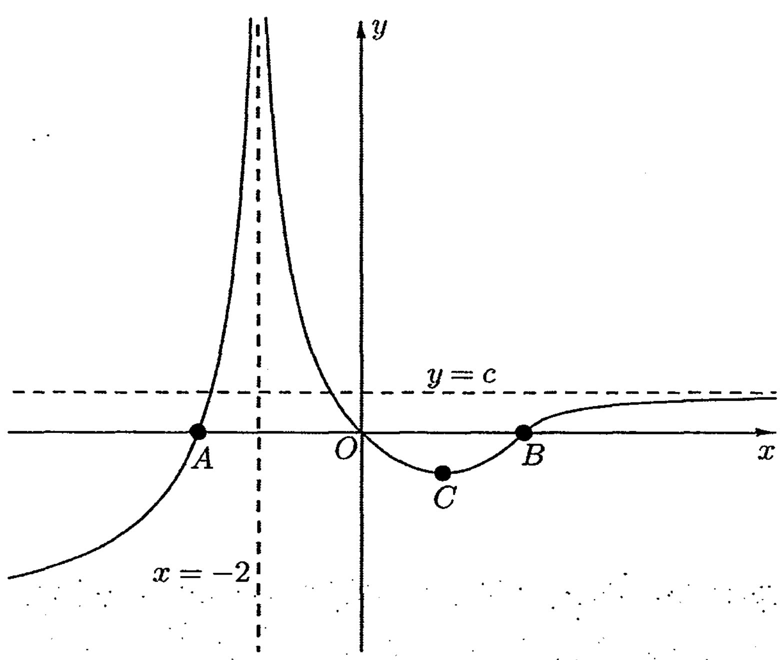
(iv) Given that the tangent found in part **(iii)** intersects C again at the point $\left(\frac{2}{3}k, k\right)$, find the value of k. [2]

7 (i) Given that $y = \ln(\sec x)$, show that $\dfrac{d^3y}{dx^3} = 2\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)$. [2]

(ii) Hence, by further differentiation, find the first two non-zero terms in the Maclaurin's series for y. [3]

(iii) The equation $\dfrac{1}{12}x^2 + \ln(\sec x) = \cos 2x$ has a positive root α close to zero. Use the result in part **(ii)** and the first three terms of the Maclaurin series for $\cos 2x$ to obtain an approximation to α, leaving your answer in surd form. [3]

8 The diagram below shows the curve with equation $y = f(x)$. The curve passes through the origin O, crosses the x-axis at the points A and B, and has a turning point at C. The coordinates of A, B and C are $(-4, 0)$, $(4, 0)$ and $\left(a, -\dfrac{b}{2}\right)$ respectively, where a and b are positive constants such that $a > 1$. The curve also has asymptotes $x = -2$ and $y = c$, where $c > 1$.



On separate diagrams, sketch the following curves, labelling clearly any asymptotes, axial intercepts and turning points in terms of a, b and c whenever necessary.

(a) $y = f(1-2x)$ [3]

(b) $y^2 = f(x)$ [3]

(c) $y = \dfrac{1}{f(x)}$ [4]

9 The gradient of a curve at the point (x, y) is given by the differential equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}-1=\frac{x-2}{y}.$$

(i) By using the substitution $y = z - x$, find the equation of the curve such that it has a minimum point at $(1, 1)$. [6]

(ii) Sketch the curve, indicating clearly the axial intercept(s) and the minimum point. [2]

10 (a) Using partial fractions, find $\int \frac{5x^2-2x+7}{(1-x)(2x^2+3)}\,\mathrm{d}x$. [6]

(b) (i) Differentiate $\sin\left(\mathrm{e}^{-x}\right)$ with respect to x. [1]

(ii) Obtain a formula for $\int_0^n \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ in terms of n, where $n > 0$. [3]

(iii) Hence find $\int_0^\infty \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ exactly. [2]

11 (a) Prove by the method of mathematical induction that

$$\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{n+1}{n^2(n+2)^2}=\frac{5}{16}-\frac{1}{4(n+1)^2}-\frac{1}{4(n+2)^2}.$$ [5]

(b) (i) By expressing $\frac{4n+5}{n(n+1)}$ in partial fractions, show that

$$\sum_{n=1}^{N}\left[\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)\right]=a+\frac{b}{(N+1)5^{N+1}},$$

for some real constants a and b to be determined exactly. [3]

(ii) State the sum to infinity of the series in part **(b)(i)**. [1]

(iii) Use your answer to part **(b)(i)** to find $\sum_{n=2}^{N-2}\left[\frac{4n+1}{n(n-1)}\left(\frac{1}{5^n}\right)\right]$ in terms of N. [2]

12 The curves C_1 and C_2 have equations $x^2+16(y-1)^2=16$ and $x^2-16(y-1)^2=16$ respectively.

(i) Verify that the point $(4,1)$ lies on both C_1 and C_2. [1]

(ii) Sketch C_1 and C_2 on the same diagram, labelling clearly any points of intersection with the axes and the equations of any asymptotes. [4]

(iii) The region R is bounded by the two curves C_1, C_2 and the positive x-axis. Find the numerical value of the volume of revolution formed when R is rotated completely about the x-axis. [3]

S is the region bounded by C_1.

(iv) Using the substitution $y=1+\cos\theta$, where $-\pi<\theta\leq\pi$, evaluate $\int_0^2 \sqrt{1-(y-1)^2}\,dy$ exactly. [4]

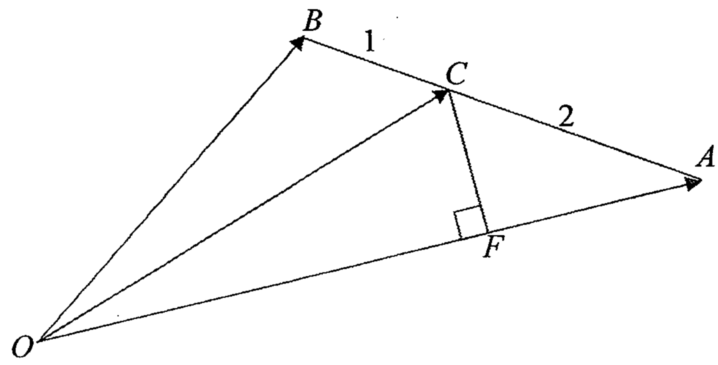
(v) Hence find the exact area of S. [2]

– END OF PAPER –

2016 SH2 H2 Mathematics Preliminary Examination Paper 1
Suggested Solutions

Qn No.	Solution
1	Let \$$x$ be the price of a regular cup of coffee. \$$y$ be the price of a medium cup of coffee. \$$z$ be the price of a large cup of coffee. $5x + 3y + 2z = 20.90$ (1) $3x + 4y + z = 17.10$ (2) There is a 12.5% discount given to customer C since she has bought more than 12 cups. Hence $2x + 8y + 4z = \frac{100}{87.5} \times 28 = 32$ (3) Using GC, $x = 1.80$, $y = 2.30$, $z = 2.50$

Qn No.	Solution
2 (i)	$\dfrac{3x^2+14}{(x+1)(x+2)} \geq 2$ $\Rightarrow \dfrac{3x^2+14}{(x+1)(x+2)} - 2 \geq 0,\ x \neq -1, -2$ $\Rightarrow \dfrac{3x^2+14-(2x^2+6x+4)}{(x+1)(x+2)} \geq 0$ $\Rightarrow \dfrac{x^2-6x+10}{(x+1)(x+2)} \geq 0$ $\Rightarrow (x+1)(x+2)(x^2-6x+10) \geq 0, \quad x \neq -1, -2.$ **Method 1: Completing the square** $\Rightarrow (x+1)(x+2)\left[(x-3)^2+1\right] \geq 0, \quad x \neq -1, -2.$ Since $(x-3)^2+1 \geq 0$ for all values of x, $\Rightarrow (x+1)(x+2) \geq 0$ and $x \neq -1, -2$. $\Rightarrow x < -2$ or $x > -1$. **Method 2: Using both the discriminant and coefficient of x^2** For $x^2-6x+10$, discriminant $= 6^2 - 4(1)(10) = -4 < 0$ and the coefficient of x^2 is positive. Hence the graph of $y = x^2-6x+10$ lies entirely above the x-axis, which implies that $x^2-6x+10 \geq 0$ for all values of x.
2 (ii)	$\dfrac{3x^2+14}{(\lvert x\rvert-1)(\lvert x\rvert-2)} \geq 2$ $\dfrac{3(-\lvert x\rvert)^2+14}{(-\lvert x\rvert+1)(-\lvert x\rvert+2)} \geq 2$ Therefore $-\lvert x\rvert < -2$ or $-\lvert x\rvert > -1$ $\Rightarrow \lvert x\rvert > 2$ or $\lvert x\rvert < 1$ $\Rightarrow x < -2$ or $-1 < x < 1$ or $x > 2$

Qn No.	Solution
3	 By ratio theorem, $\overrightarrow{OC} = \frac{2\mathbf{b}+\mathbf{a}}{3}$.
	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ = \frac{1}{2}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ Length of projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ $OF = \left\lvert \overrightarrow{OC}\cdot\frac{\overrightarrow{OA}}{\lvert\overrightarrow{OA}\rvert} \right\rvert$ $= \frac{1}{3}\left\lvert \frac{(2\mathbf{b}+\mathbf{a})\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert} \right\rvert$ $= \frac{1}{3}\left\lvert \frac{2\mathbf{b}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{a}}{\lvert\mathbf{a}\rvert} \right\rvert$ $= \frac{1}{3}\left\lvert \frac{2\mathbf{b}\cdot\mathbf{a}+\lvert\mathbf{a}\rvert^2}{\lvert\mathbf{a}\rvert} \right\rvert$ $= \frac{1}{3}\left(\frac{2\lvert\mathbf{a}\cdot\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert} + \lvert\mathbf{a}\rvert \right)$ $= \frac{1}{3}\left(\frac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ}{\lvert\mathbf{a}\rvert} + \lvert\mathbf{a}\rvert \right)$ $= \frac{1}{3}\left(\frac{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}{2\lvert\mathbf{a}\rvert} + \lvert\mathbf{a}\rvert \right)$ $= \frac{1}{3}\left(\lvert\mathbf{a}\rvert + \lvert\mathbf{b}\rvert\right)$.

Qn No.	Solution
4 (a)	**Method 1A** $2w-z=6\text{i} \quad \cdots\cdots(1)$ $wz=\frac{13}{2} \quad \cdots\cdots(2) \quad \Rightarrow w=\frac{13}{2z} \quad \cdots\cdots(3)$ Substituting (3) into (1), $2\left(\frac{13}{2z}\right)-z=6\text{i}$ $\frac{13}{z}-z=6\text{i}$ $13-z^2=6\text{i}z$ $z^2+6\text{i}z-13=0$ $z^2+6\text{i}z+(3\text{i})^2-(3\text{i})^2-13=0$ $(z+3\text{i})^2=4$ $z=2-3\text{i}$ or $-2-3\text{i}$ When $z=2-3\text{i}$, $w=\frac{13}{2(2-3\text{i})}=1+\frac{3}{2}\text{i}$ When $z=-2-3\text{i}$, $w=\frac{13}{2(-2-3\text{i})}=-1+\frac{3}{2}\text{i}$ $\therefore w=1+\frac{3}{2}\text{i}, z=2-3\text{i}$ or $w=-1+\frac{3}{2}\text{i}, z=-2-3\text{i}$ **Method 1B** $2w-z=6\text{i} \quad \cdots\cdots(1)$ $wz=\frac{13}{2} \quad \cdots\cdots(2) \quad \Rightarrow z=\frac{13}{2w} \quad \cdots\cdots(3)$ Substituting (3) into (1), $2w-\frac{13}{2w}=6\text{i}$ $4w^2-13=12\text{i}w$ $4w^2-12\text{i}w-13=0$ $4\left(w^2-3\text{i}w-\frac{9}{4}\right)+9-13=0$ $4\left(w-\frac{3\text{i}}{2}\right)^2-4=0$ $\left(w-\frac{3\text{i}}{2}\right)^2=1 \Rightarrow w=1+\frac{3\text{i}}{2}$ or $-1+\frac{3\text{i}}{2}$ When $w=1+\frac{3}{2}\text{i}$, $z=\frac{13}{2\left(1+\frac{3}{2}\text{i}\right)}=2-3\text{i}$ When $w=-1+\frac{3}{2}\text{i}$, $z=\frac{13}{2\left(-1+\frac{3}{2}\text{i}\right)}=-2-3\text{i}$ $\therefore w=1+\frac{3}{2}\text{i}, z=2-3\text{i}$ or $w=-1+\frac{3}{2}\text{i}, z=-2-3\text{i}$

Qn No.	Solution
4 (a)	**Method 2A**

$2w - z = 6\text{i} \quad \cdots\cdots(1) \Rightarrow z = 6\text{i} - 2w \cdots\cdots(3)$

$wz = \frac{13}{2} \quad \cdots\cdots(2)$

Substituting (3) into (2), $w(6\text{i} - 2w) = \frac{13}{2}$

$$w(6\text{i} - 2w) = \frac{13}{2}$$
$$4w^2 - 12\text{i}w = 13$$
$$4w^2 - 12\text{i}w - 13 = 0$$
$$4\left(w^2 - 3\text{i}w - \frac{9}{4}\right) + 9 - 13 = 0$$
$$4\left(w - \frac{3\text{i}}{2}\right)^2 - 4 = 0$$
$$\left(w - \frac{3\text{i}}{2}\right)^2 = 1 \Rightarrow w = 1 + \frac{3\text{i}}{2} \text{ or } -1 + \frac{3\text{i}}{2}$$

When $w = 1 + \frac{3}{2}\text{i}$, $z = \frac{13}{2\left(1 + \frac{3}{2}\text{i}\right)} = 2 - 3\text{i}$

When $w = -1 + \frac{3}{2}\text{i}$, $z = \frac{13}{2\left(-1 + \frac{3}{2}\text{i}\right)} = -2 - 3\text{i}$

$\therefore w = 1 + \frac{3}{2}\text{i}, z = 2 - 3\text{i}$ or $w = -1 + \frac{3}{2}\text{i}, z = -2 - 3\text{i}$

Method 2B

$2w - z = 6\text{i} \quad \cdots\cdots(1) \Rightarrow 2w = 6\text{i} + z \cdots\cdots(3)$

$wz = \frac{13}{2} \quad \cdots\cdots(2)$

Substituting (3) into (2), $(6\text{i} + z)z = 13$

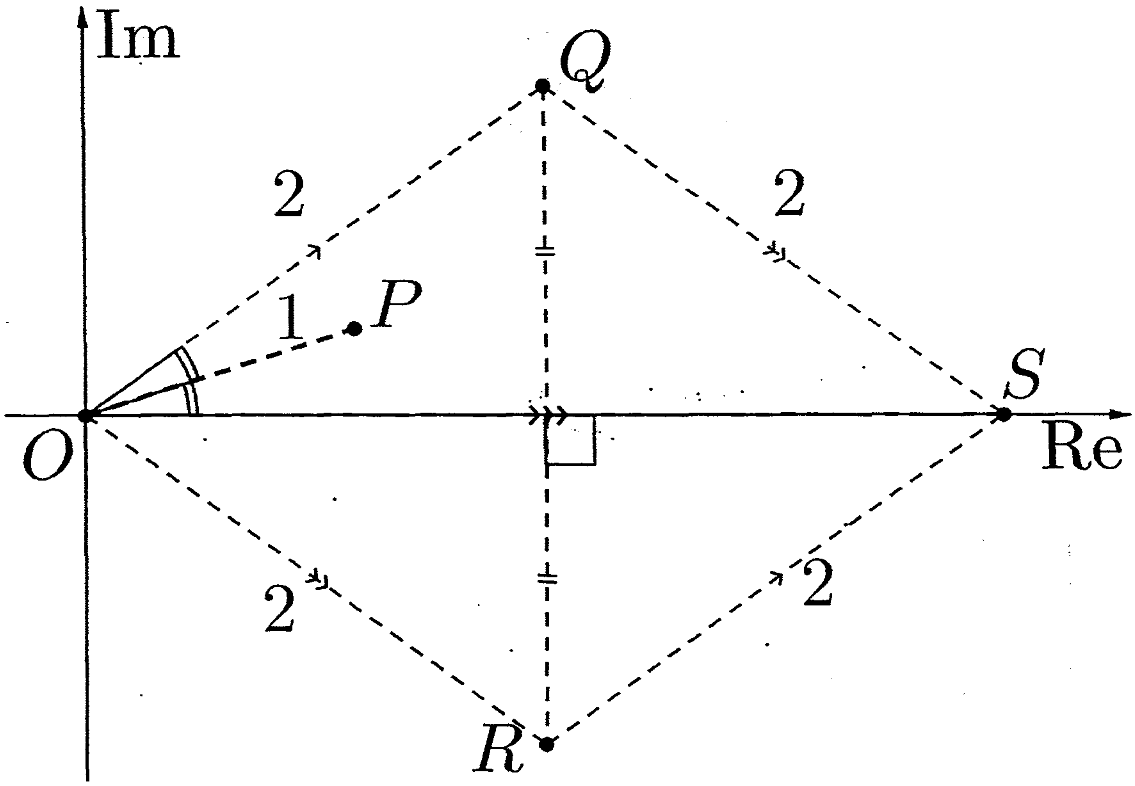
$$6\text{i}z + z^2 = 13$$
$$z^2 + 6\text{i}z - 13 = 0$$
$$z^2 + 6\text{i}z + (3\text{i})^2 - (3\text{i})^2 - 13 = 0$$
$$(z + 3\text{i})^2 = 4$$
$$z = 2 - 3\text{i} \text{ or } -2 - 3\text{i}$$

When $z = 2 - 3\text{i}$, $w = \frac{13}{2(2 - 3\text{i})} = 1 + \frac{3}{2}\text{i}$

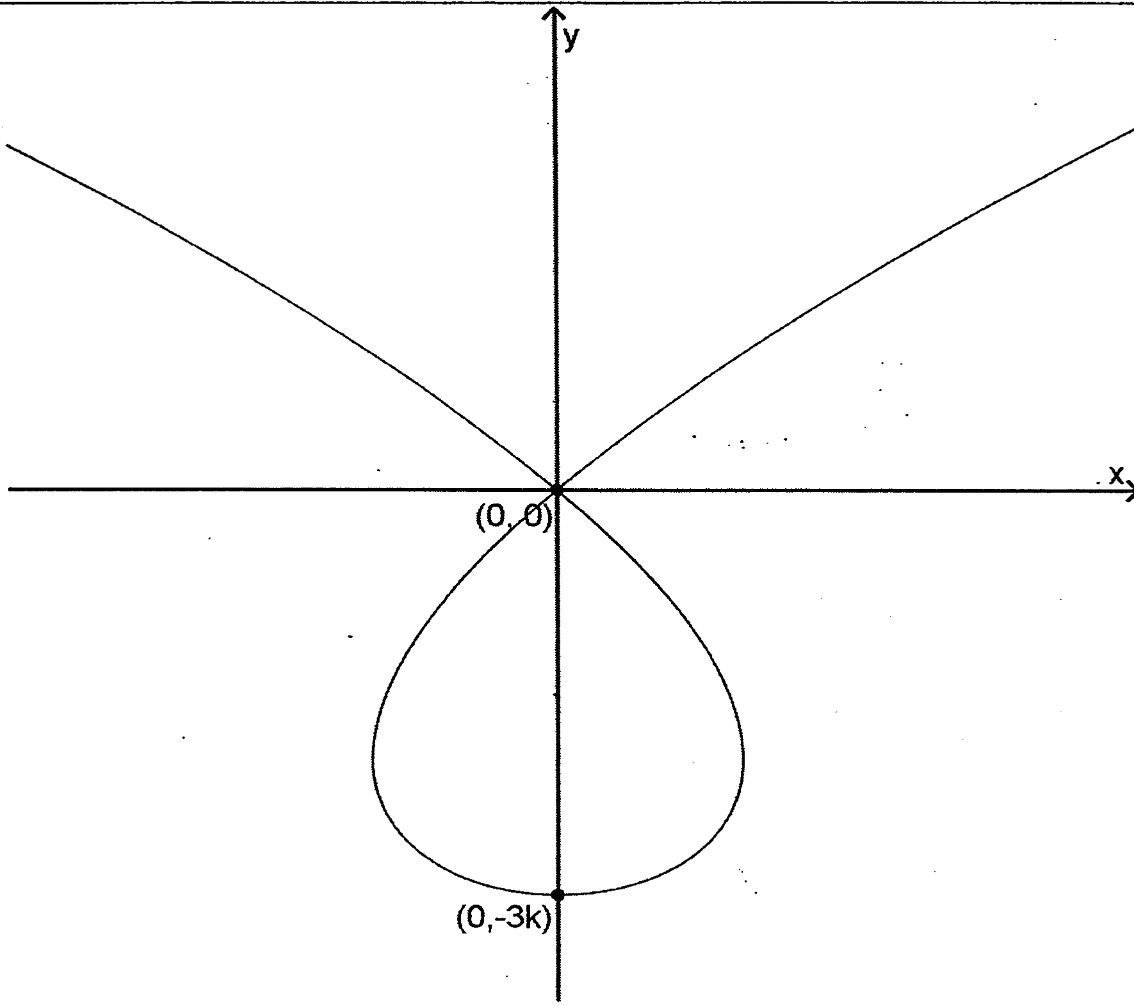
When $z = -2 - 3\text{i}$, $w = \frac{13}{2(-2 - 3\text{i})} = -1 + \frac{3}{2}\text{i}$

$\therefore w = 1 + \frac{3}{2}\text{i}, z = 2 - 3\text{i}$ or $w = -1 + \frac{3}{2}\text{i}, z = -2 - 3\text{i}$

Qn No.	Solution
4 (a)	**Method 3** $2w - z = 6\text{i} \quad \cdots\cdots(1)$ $wz = \frac{13}{2} \quad \cdots\cdots(2)$ $(2) \Rightarrow w = \frac{13}{2z} \quad \cdots\cdots(3)$ Substituting (3) into (1), $2\left(\frac{13}{2z}\right) - z = 6\text{i}$ $\frac{13}{z} - z = 6\text{i}$ $13 - z^2 = 6\text{i}z$ $z^2 + 6\text{i}z - 13 = 0$ Let $z = x + y\text{i}$. Then $(x+\text{i}y)^2 + 6\text{i}(x+\text{i}y) - 13 = 0$ $x^2 - y^2 + 2\text{i}xy + 6\text{i}x - 6y - 13 = 0$ $x^2 - y^2 - 6y - 13 + \text{i}(2xy + 6x) = 0$ Comparing imaginary parts, $2xy + 6x = 0$ $x(y+3) = 0$ $x = 0$ or $y = -3$ If $x = 0$, comparing real parts, $-y^2 - 6y - 13 = 0$ $y^2 + 6y + 13 = 0$ $(y+3)^2 + 4 = 0$ Therefore, there is no solution if $x = 0$. If $y = -3$, comparing real parts, $x^2 - 3^2 - 6(-3) - 13 = 0$ $x^2 - 4 = 0$ $x = \pm 2$ When $x = 2$, $z = 2 - 3\text{i}$, $w = \frac{13}{2(2-3\text{i})} = 1 + \frac{3}{2}\text{i}$ When $x = -2$, $z = -2 - 3\text{i}$, $w = \frac{13}{2(-2-3\text{i})} = -1 + \frac{3}{2}\text{i}$ $\therefore w = 1 + \frac{3}{2}\text{i}, z = 2 - 3\text{i}$ or $w = -1 + \frac{3}{2}\text{i}, z = -2 - 3\text{i}$

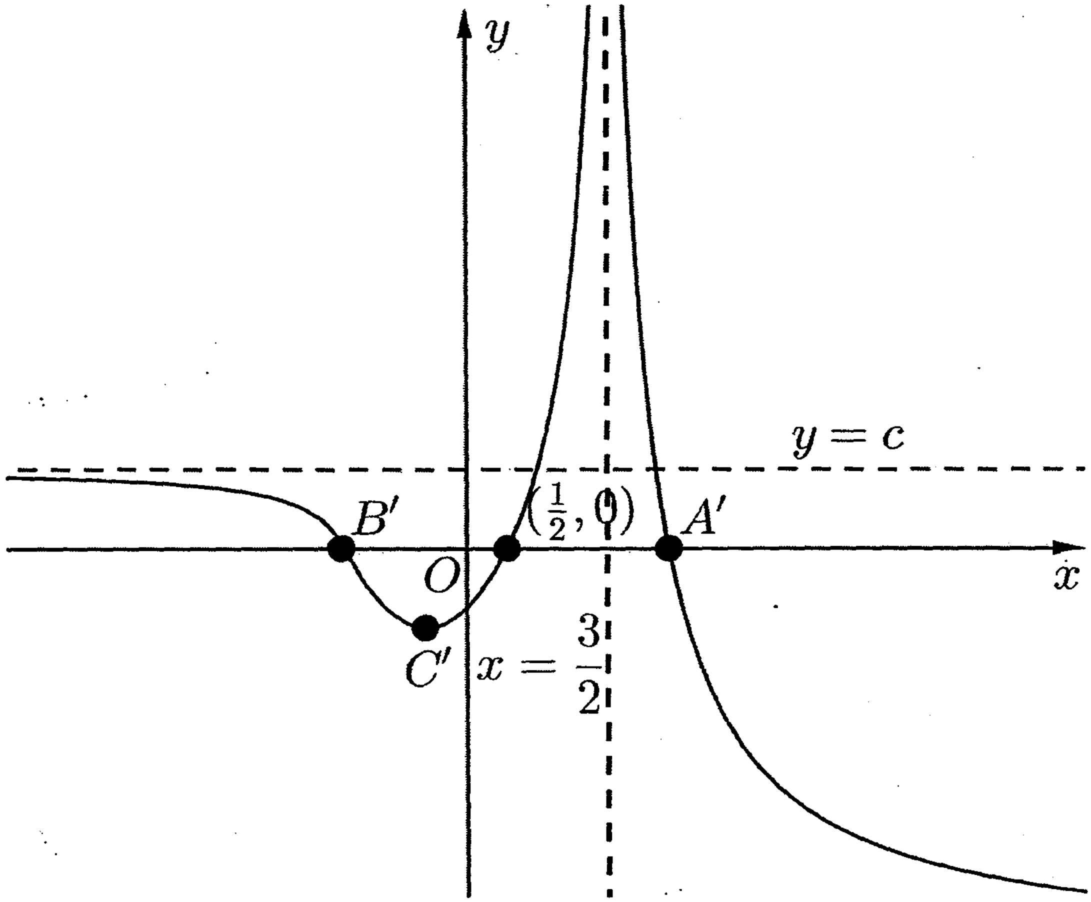
Qn No.	Solution
4 (b)	 Let $p = e^{i\alpha}$. Then $q = 2e^{i(2\alpha)}$. Let S represent $q^* + 2p^2 = 2e^{i(-2\alpha)} + 2e^{i(2\alpha)} = 2\,\mathrm{Re}\left(e^{i(2\alpha)}\right) = 2\cos 2\alpha$, i.e. S represents the point whose real coordinate is twice that of Q, and imaginary coordinate zero.

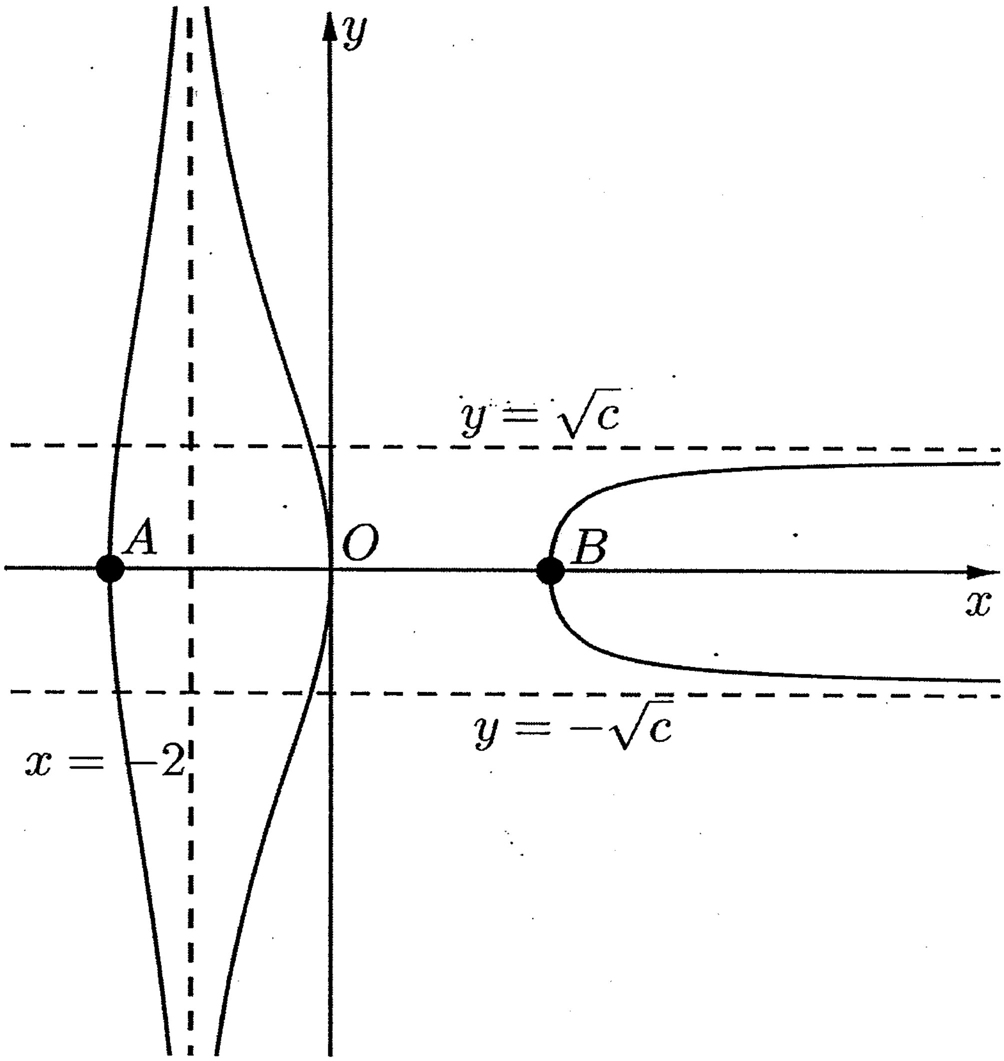
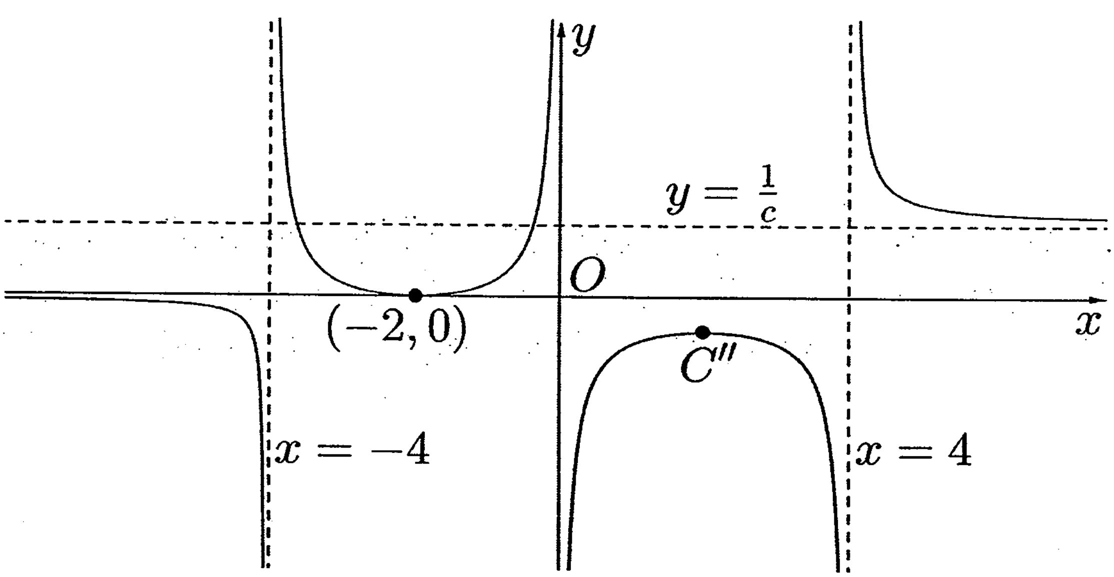
Qn No.	Solution
5	$V=x^2h+\frac{1}{3}x^2\left(\frac{1}{2}x\right)=x^2\left(h+\frac{1}{6}x\right)$ $h=\frac{V}{x^2}-\frac{1}{6}x$ $S=4xh+x^2+4\left(\frac{1}{2}\right)x\left(\frac{1}{\sqrt{2}}x\right)$ $=4xh+\left(1+\sqrt{2}\right)x^2$ Substituting $h=\frac{V}{x^2}-\frac{1}{6}x$: $S=4x\left(\frac{V}{x^2}-\frac{1}{6}x\right)+\left(1+\sqrt{2}\right)x^2$ $=\frac{4V}{x}+\left(\frac{1}{3}+\sqrt{2}\right)x^2$ For minimum surface area: $\frac{dS}{dx}=-\frac{4V}{x^2}+2\left(\frac{1}{3}+\sqrt{2}\right)x=0.$ $\frac{4V}{x^2}=2\left(\frac{1}{3}+\sqrt{2}\right)x$ $2V=\left(\frac{1}{3}+\sqrt{2}\right)x^3$ $6V=\left(1+3\sqrt{2}\right)x^3$ $x^3=\frac{6V}{1+3\sqrt{2}}$ $x=\left(\frac{6V}{1+3\sqrt{2}}\right)^{\frac{1}{3}}=1.0460049\left(V^{\frac{1}{3}}\right)$ $=1.046\left(V^{\frac{1}{3}}\right)$ (3d.p) Min $S=A$ $=\frac{4V}{1.0460049\left(V^{\frac{1}{3}}\right)}+\left(\frac{1}{3}+\sqrt{2}\right)\left(1.0460049\left(V^{\frac{1}{3}}\right)\right)^2$ $=\left(\frac{4}{1.0460049}+\left(\frac{1}{3}+\sqrt{2}\right)(1.0460049)^2\right)\left(V^{\frac{2}{3}}\right)$ $=5.736110\left(V^{\frac{2}{3}}\right)$ $=5.736\left(V^{\frac{2}{3}}\right)$ (3d.p) **Remark:** 1st or 2nd derivative test is not necessary as it is given in the question that the surface area had a minimum value.

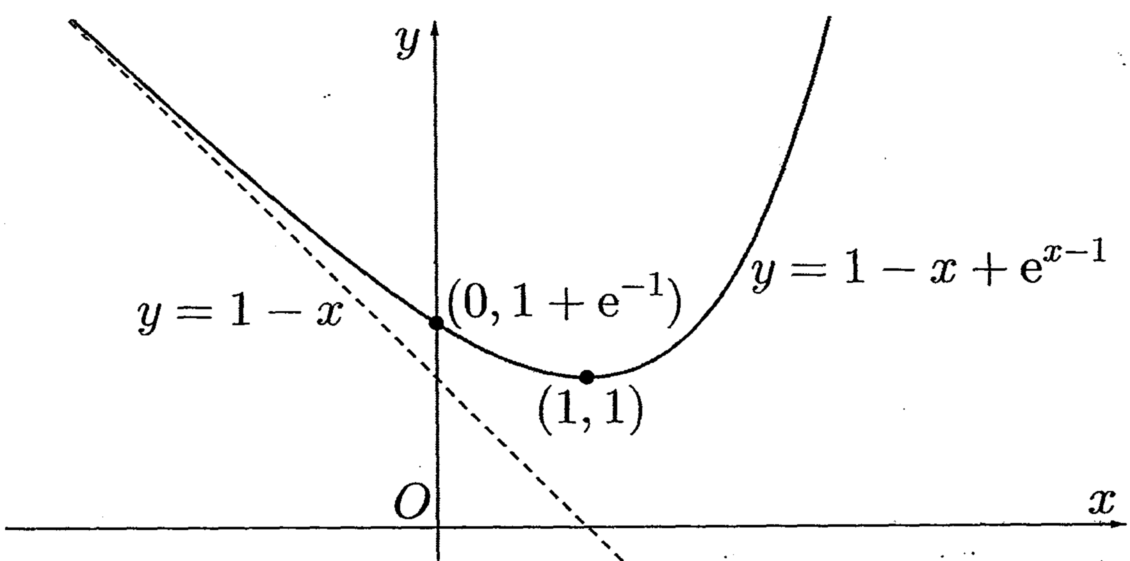
Qn No.	Solution
6 (i)	 To find y-intercept, let $x = 0$: $t^3 - kt = 0$; $t(t^2 - k) = 0$; $t = 0$ or $t = \pm\sqrt{k}$; $y = -3k$, $y = 0$; $\therefore (0,-3k)$, $(0,0)$ To find x-intercept, let $y = 0$: $3(t^2 - k) = 0$; $t = \pm\sqrt{k}$; $x = 0$; $\therefore (0,0)$
6 (ii)	$x = t^3 - kt \Rightarrow \frac{dx}{dt} = 3t^2 - k$ $y = 3(t^2 - k) \Rightarrow \frac{dy}{dt} = 6t$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{3t^2 - k}$

Qn No.	Solution
6 (iii)	When $t=-\sqrt{\frac{k}{3}}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6t}{3t^2-k}$ is undefined since $3t^2-k=3\left(\frac{k}{3}\right)-k=0$. Hence, the tangent is a vertical line with equation $x=t^3-kt$ $=t\left(t^2-k\right)$ $=\left(-\sqrt{\frac{k}{3}}\right)\left(\frac{k}{3}-k\right)$ $=\frac{2}{3}k\sqrt{\frac{k}{3}}$
6 (iv)	The tangent passes through the point $\left(\frac{2}{3}k,k\right)$, therefore $x=\frac{2}{3}k\sqrt{\frac{k}{3}}$ $\frac{2}{3}k=\frac{2}{3}k\sqrt{\frac{k}{3}}$ $k\left(1-\sqrt{\frac{k}{3}}\right)=0$ $k=0$ or $1-\sqrt{\frac{k}{3}}=0$ (NA since $k>0$) $\quad \frac{k}{3}=1$ $k=3$

Qn No.	Solution
7 (i)	$y=\ln(\sec x)$ $\frac{dy}{dx}=\frac{1}{\sec x}(\sec x\tan x)=\tan x$ $\frac{d^2y}{dx^2}=\sec^2 x$ $\frac{d^3y}{dx^3}=2\sec x(\sec x\tan x)$ $\frac{d^3y}{dx^3}=2(\sec^2 x)(\tan x)=2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)$ (shown)
7 (ii)	$\frac{d^4y}{dx^4}=2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right)+2\left(\frac{d^2y}{dx^2}\right)^2$ When $x=0$, $y=\ln(\sec 0)=\ln 1=0$ $\quad\frac{dy}{dx}=\tan 0=0$ $\frac{d^2y}{dx^2}=\sec^2 0=1$ $\quad\frac{d^3y}{dx^3}=2(1)(0)=0$ $\frac{d^4y}{dx^3}=2(0)(0)+2(1)^2=2$ Thus, Maclaurin series for y is $y\approx +0x+1\left(\frac{x^2}{2!}\right)+0x^3+2\left(\frac{x^4}{4!}\right)$ $=\frac{1}{2}x^2+\frac{1}{12}x^4$
7 (iii)	$\frac{1}{12}x^2+\ln(\sec x)=\cos 2x$ Using the above results and the first three terms of Maclaurin Series of $\cos 2x$, $\frac{1}{12}x^2+\frac{1}{2}x^2+\frac{1}{12}x^4\approx 1-\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4$ $\frac{7}{12}x^2+\frac{1}{12}x^4=1-2x^2+\frac{2}{3}x^4$ $\frac{7}{12}x^4-\frac{31}{12}x^2+1=0$ $7x^4-31x^2+12=0$ $(7x^2-3)(x^2-4)=0$ $x^2=\frac{3}{7}$ or $x^2=4$ $x=\pm\sqrt{\frac{3}{7}}$ or ± 2 Since α is positive and close to zero, $\alpha=\sqrt{\frac{3}{7}}$

Qn No.	Solution
8 (a)	 Coordinates of A', B' and C' are $\left(\frac{5}{2},0\right)$, $\left(-\frac{3}{2},0\right)$ and $\left(\frac{1-a}{2},\frac{b}{2}\right)$

Qn No.	Solution
8 (b)	
8 (c)	 Coordinates of C'' are $\left(a, -\frac{2}{b}\right)$.

Qn No.	Solution
9 (i)	$y=z-x \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$ The $\frac{1}{y}\frac{dy}{dx}-1=\frac{x-2}{y}$ becomes $\frac{1}{z-x}\left(\frac{dz}{dx}-1\right)-1=\frac{x-2}{z-x} \Rightarrow \frac{dz}{dx}-1-(z-x)=x-2$ $\Rightarrow \frac{dz}{dx}=z-1$ $\Rightarrow \int \frac{1}{z-1}\,dz=\int 1\,dx$ $\Rightarrow \ln\lvert z-1\rvert = x+c$ $\Rightarrow \lvert z-1\rvert = e^{x+c}=Ae^{x}$ $\Rightarrow z-1=\pm Ae^{x}=Be^{x}$ $\Rightarrow y=1-x+Be^{x}$ Since the solution curve passes through (1,1), then $1=1-1+Be \Rightarrow B=e^{-1}$. Hence the equation of curve is $y=1-x+e^{x-1}$.
9 (ii)	

Qn No.	Solution
10 (a)	

$$\frac{5x^2-2x+7}{(1-x)(2x^2+3)}=\frac{A}{1-x}+\frac{Bx+C}{2x^2+3}, \text{ where}$$

$$A=\frac{5-2+7}{2+3}=2,$$

$$5x^2-2x+7=2(2x^2+3)+(Bx+C)(1-x)$$

Comparing the coefficients of x^2 and x^0, we have
$4-B=5 \Rightarrow B=-1$, and
$6+C=7 \Rightarrow C=1$.

Hence we get $\dfrac{5x^2-2x+7}{(1-x)(2x^2+3)}=\dfrac{2}{1-x}+\dfrac{-x+1}{2x^2+3}$

Therefore,

$$\int \frac{5x^2-2x+7}{(1-x)(2x^2+3)}\,dx$$

$$=\int \frac{2}{1-x}+\frac{-x+1}{2x^2+3}\,dx$$

$$=\int \frac{2}{1-x}\,dx+\int \frac{-x+1}{2x^2+3}\,dx$$

$$=\int \frac{2}{1-x}\,dx-\int \frac{x}{2x^2+3}\,dx+\int \frac{1}{2x^2+3}\,dx$$

$$=-2\int \frac{-1}{1-x}\,dx-\frac{1}{4}\int \frac{4x}{2x^2+3}\,dx+\frac{1}{2}\int \frac{1}{x^2+\left(\sqrt{\frac{3}{2}}\right)^2}\,dx$$

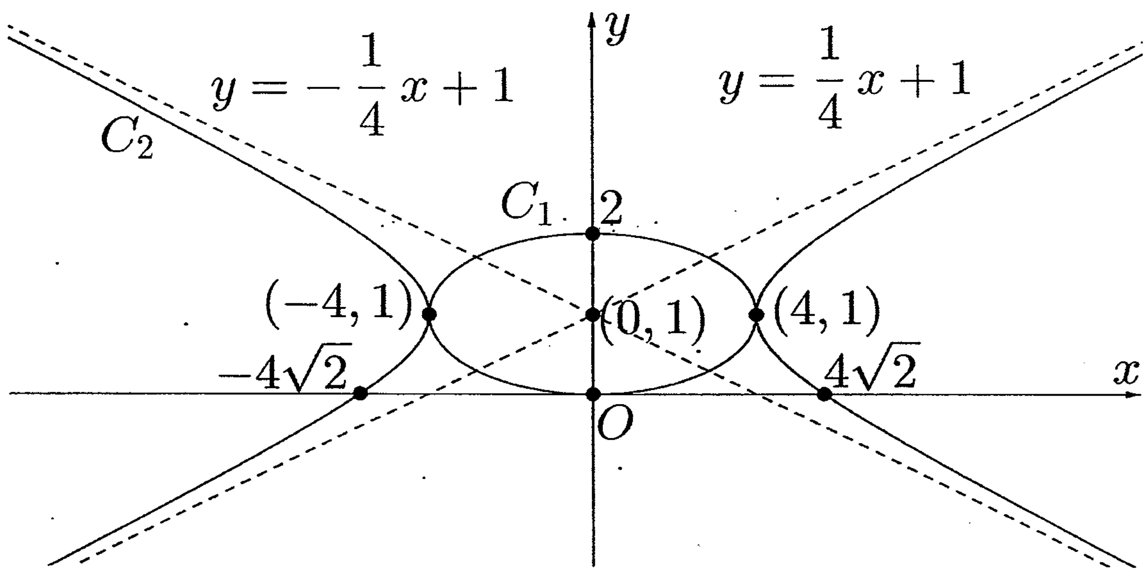
$$=-2\ln|1-x|-\frac{1}{4}\ln|2x^2+3|+\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{3}{2}}}\right)+c$$

$$=-2\ln|1-x|-\frac{1}{4}\ln|2x^2+3|+\frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{2}{3}}x\right)+c$$

Qn No.	Solution
10 (b) (i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\left(\mathrm{e}^{-x}\right)\right]=-\mathrm{e}^{-x}\cos\left(\mathrm{e}^{-x}\right)$
10 (b) (ii)	$\int_0^n \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ $=\int_0^n \mathrm{e}^{-x}\left(\mathrm{e}^{-x}\cos\left(\mathrm{e}^{-x}\right)\right)\mathrm{d}x$ $=\left[\mathrm{e}^{-x}\left(-\sin\left(\mathrm{e}^{-x}\right)\right)\right]_0^n-\int_0^n\left(-\sin\left(\mathrm{e}^{-x}\right)\right)\left(-\mathrm{e}^{-x}\right)\mathrm{d}x$ $=\left[\mathrm{e}^{-n}\left(-\sin\left(\mathrm{e}^{-n}\right)\right)\right]-\left[\mathrm{e}^{0}\left(-\sin\left(\mathrm{e}^{0}\right)\right)\right]+\int_0^n -\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ $=-\mathrm{e}^{-n}\sin\left(\mathrm{e}^{-n}\right)+\sin 1+\left[-\cos\left(\mathrm{e}^{-x}\right)\right]_0^n$ $=-\mathrm{e}^{-n}\sin\left(\mathrm{e}^{-n}\right)+\sin 1+\left[-\cos\left(\mathrm{e}^{-n}\right)+\cos\left(\mathrm{e}^{0}\right)\right]$ $=-\mathrm{e}^{-n}\sin\left(\mathrm{e}^{-n}\right)-\cos\left(\mathrm{e}^{-n}\right)+\sin 1+\cos 1$
10 (b) (iii)	$\int_0^\infty \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ $=\lim_{n\to\infty}\int_0^n \mathrm{e}^{-2x}\cos\left(\mathrm{e}^{-x}\right)\mathrm{d}x$ $=\lim_{n\to\infty}\left[-\mathrm{e}^{-n}\sin\left(\mathrm{e}^{-n}\right)-\cos\left(\mathrm{e}^{-n}\right)+\sin 1+\cos 1\right]$ $=\sin 1+\cos 1-1$

Qn No.	Solution
11 (a)	Let P_n be the statement $$\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{n+1}{n^2(n+2)^2}=\frac{5}{16}-\frac{1}{4(n+1)^2}-\frac{1}{4(n+2)^2}$$ for all $n\in\mathbf{Z}^+$. When $n=1$, $$\text{LHS}=\frac{2}{1^2\times3^2}=\frac{2}{9}$$ $$\text{RHS}=\frac{5}{16}-\frac{1}{4(1+1)^2}-\frac{1}{4(1+2)^2}$$ $$=\frac{5}{16}-\frac{1}{16}-\frac{1}{36}=\frac{2}{9}=\text{LHS}$$ Thus, P_1 is true. Assume P_k is true for some $k\in\mathbf{Z}^+$, i.e. $$\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{k+1}{k^2(k+2)^2}=\frac{5}{16}-\frac{1}{4(k+1)^2}-\frac{1}{4(k+2)^2}.$$ Consider P_{k+1}: To show $$\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{k+2}{(k+1)^2(k+3)^2}=\frac{5}{16}-\frac{1}{4(k+2)^2}-\frac{1}{4(k+3)^2}.$$ $$\text{LHS of }P_{k+1}=\frac{2}{1^2\times3^2}+\frac{3}{2^2\times4^2}+\cdots+\frac{k+1}{k^2(k+2)^2}+\frac{k+2}{(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+1)^2}-\frac{1}{4(k+2)^2}+\frac{k+2}{(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}+\frac{k+2}{(k+1)^2(k+3)^2}-\frac{1}{4(k+1)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}+\frac{4(k+2)-(k+3)^2}{4(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}+\frac{4k+8-k^2-6k-9}{4(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}+\frac{-k^2-2k-1}{4(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}+\frac{-(k+1)^2}{4(k+1)^2(k+3)^2}$$ $$=\frac{5}{16}-\frac{1}{4(k+2)^2}-\frac{1}{4(k+3)^2}$$ $=\text{RHS of }P_{k+1}$ P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n\in\mathbf{Z}^+$.

Qn No.	Solution
11 (b) (i)	$\frac{4n+5}{n(n+1)}=\frac{5}{n}-\frac{1}{n+1}$ $\sum_{n=1}^{N}\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)=\sum_{n=1}^{N}\left(\frac{5}{n}-\frac{1}{n+1}\right)\left(\frac{1}{5^{n+1}}\right)$ $=\sum_{n=1}^{N}\left(\frac{1}{n5^{n}}-\frac{1}{(n+1)5^{n+1}}\right)$ $= \frac{1}{1\cdot 5^1}-\frac{1}{2\cdot 5^2}$ $+\frac{1}{2\cdot 5^2}-\frac{1}{3\cdot 5^3}$ $\vdots$ $+\frac{1}{(N-1)5^{N-1}}-\frac{1}{N5^N}$ $+\frac{1}{N5^N}-\frac{1}{(N+1)5^{N+1}}$ $=\frac{1}{5}-\frac{1}{(N+1)5^{N+1}}$ (shown) $\therefore a=\frac{1}{5},\ b=-1$
11 (b) (ii)	$\sum_{n=1}^{\infty}\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)=\lim_{N\to\infty}\left[\frac{1}{5}-\frac{1}{(N+1)5^{N+1}}\right]=\frac{1}{5}$
11 (b) (iii)	**Method 1: Substitution of Indices** $\sum_{n=2}^{N-2}\left[\frac{4n+1}{n(n-1)}\left(\frac{1}{5^n}\right)\right]=\sum_{n+1=2}^{n+1=N-2}\left[\frac{4(n+1)+1}{(n+1)((n+1)-1)}\left(\frac{1}{5^{n+1}}\right)\right]$ $=\sum_{n=1}^{N-3}\left[\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)\right]$ $=\frac{1}{5}-\frac{1}{(N-2)5^{N-2}}$ **Method 2: Listing of Terms** $\sum_{n=2}^{N-2}\left[\frac{4n+1}{n(n-1)}\left(\frac{1}{5^n}\right)\right]$ $=\frac{9}{2\times1}\left(\frac{1}{5^2}\right)+\frac{13}{3\times2}\left(\frac{1}{5^3}\right)+\cdots+\frac{4N-7}{(N-2)(N-3)}\left(\frac{1}{5^{N-2}}\right)$ $=\frac{4\cdot1+5}{1\times2}\left(\frac{1}{5^2}\right)+\frac{4\cdot2+5}{2\times3}\left(\frac{1}{5^3}\right)+\cdots+\frac{4(N-3)+5}{(N-3)(N-2)}\left(\frac{1}{5^{N-2}}\right)$ $=\sum_{n=1}^{N-3}\left[\frac{4n+5}{n(n+1)}\left(\frac{1}{5^{n+1}}\right)\right]$ $=\frac{1}{5}-\frac{1}{(N-2)5^{N-2}}$

Qn No.	Solution
12 (i)	By substituting $(4,1)$ into the equations of C_1 and C_2, C_1: LHS $=(4)^2+16(1-1)^2=16=$ RHS C_2: LHS $=(4)^2-16(1-1)^2=16=$ RHS (justified)
12 (ii)	
12 (iii)	Volume of revolution of R about x-axis $=\pi\int_0^4\left[1-\sqrt{\frac{16-x^2}{16}}\right]^2 dx+\pi\int_4^{4\sqrt{2}}\left[1-\sqrt{\frac{x^2-16}{16}}\right]^2 dx$ $=2.17$ units^3 (to 3 s.f.)
12 (iv)	$\int_0^2\sqrt{1-(x-1)^2}dx$ $=\int_\pi^0\left(\sqrt{1-\cos^2\theta}\right)(-\sin\theta)\ d\theta$ $=\int_0^\pi\left(\sin^2\theta\right)\ d\theta$ $=\frac{1}{2}\int_0^\pi(1-\cos 2\theta)\ d\theta$ $=\frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^\pi$ $=\frac{\pi}{2}$
12 (v)	Area of $S=2\int_0^2 x\ dy$ $=2\int_0^2\sqrt{16-16(y-1)^2}dy$ $=8\int_0^2\sqrt{1-(y-1)^2}dy$ $=8\left(\frac{\pi}{2}\right)$ $=4\pi$ units^2

SENIOR HIGH 2 PRELIMINARY EXAMINATION
Higher 2

MATHEMATICS **9740/02**

Paper 2 **14 September 2016**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, registration number, subject tutorial group, on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in the brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 **(a)** Kenny took a loan of \$9600 from a friend, and arranged to pay his loan fully in a period of exactly 48 months. To fulfil this arrangement, he paid \$$a$ on the last day of the first month, and on the last day of each subsequent month, he paid \$$d$ more than in the previous month. However, due to financial difficulties, Kenny stopped his payments after his 40th payment, and as a result he still had exactly \$2400 left unpaid.

In which month did Kenny first pay at least \$130 on the last day of that month? [5]

(b) **(i)** Explain why the series $1+e^{-2x}+e^{-4x}+\cdots$ converges for any positive real number x, and express the sum to infinity in terms of x. [2]

(ii) Given that $x = 10$, find the least value of n such that $S-S_n < S(10^{-100})$, where S and S_n represent the sum to infinity and the sum of the first n terms of the series respectively. [3]

2 The functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2-4x+3, \text{ for } x \in \mathbb{R},\ x \le a \text{ and}$$

$$\mathrm{g}: x \mapsto \tan^{-1}(2x+1), \text{ for } x \in \mathbb{R},\ x > -2,$$

where a is a constant.

(a) If $a=2$, solve the equation $\mathrm{f}(x)=x$ exactly. [2]

(b) If $a=3$,

(i) give a reason why f has no inverse. [2]

(ii) Prove that the composite function gf exists and state the rule, domain and exact range of the composite function. [6]

3 The point A has coordinates $(2q, 0, 2)$, where q is a constant, and the planes p_1, p_2 have equations $x+y=4$ and $3x+2y-5z=7$ respectively.

(i) Find the coordinates of the foot of perpendicular from A to p_1. Express your answer in terms of q. [3]

(ii) The point B is the mirror image of A in p_1. If B lies in p_2, find the value of q. [4]

(iii) p_1 and p_2 intersect in a line l. Find a vector equation of l. [1]

Another plane p_3 has equation $\lambda x + z = \mu$, where λ and μ are constants.

(iv) Given that the three planes have no point in common, what can be said about the values of λ and μ? [2]

4 The complex number z satisfies the relation $|z-3|=5$.

(i) Illustrate this relation in an Argand diagram. [2]

(ii) Find the largest possible value of $\arg(z+3-3\mathrm{i})$. [3]

It is further given that z also satisfies the relation $|z-4\mathrm{i}|=|z-6+4\mathrm{i}|$.

(iii) Illustrate this relation in the same diagram as your sketch in part (i). Find the possible values of z exactly. [5]

Section B: Statistics [60 marks]

5 A school comprises a large number of students. A sample comprising 2% of the student population is to be selected to take part in a survey on their opinions about the school facilities.

(a) Describe briefly how this sample can be obtained via systematic sampling. [2]

(b) Give one advantage and one disadvantage of quota sampling in this context. [2]

6 The continuous random variable X has the distribution $\mathrm{N}(\mu, \sigma^2)$. It is known that $\mathrm{P}(X<17.7)=0.15$ and $\mathrm{P}(X>21.9)=0.2$. Calculate the values of μ and σ. [4]

7 A group of 15 student councillors comprises 6 from the House Committee, 5 from the Liaison Committee and 4 from the Welfare Committee. Two particular student councillors, Louis and Lionel, are from the House Committee and the Liaison Committee respectively.

The group stand in a circle to have a meeting. Find the number of possible arrangements if

(i) no two student councillors from the House Committee stand next to each other. [2]

(ii) student councillors from the same committee must stand next to one another and Louis and Lionel must stand next to each other. [2]

The group is to form a Task Force of 10 student councillors to organise a school activity. Find the number of possible ways the Task Force may be formed if the Task Force must include at least 1 student councillor from each of the 3 committees. [3]

8 The table below shows the ages of teak trees, x years, with trunk diameters, y inches. It can be assumed that the diameters of teak trees depend on their ages.

Age x (years)	11	15	28	45	52	57	75	81	88	97
Diameter y (inches)	7.5	11.5	16	19	20.5	21	21.5	21.9	22.2	22.22

(i) Draw a scatter diagram for these values, labelling the axes. [2]

(ii) It is desired to predict the diameters of very old trees (of over hundred years old). Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

(iii) Fit a model of the form $y = a - \frac{b}{x}$ to the data, and calculate the least squares estimates of a and b. Find the product moment correlation coefficient for this model. Use the equation that you have obtained to estimate the diameter of a 40 year-old teak tree, and comment on the reliability of your answer. [4]

9 It has been estimated that only 8% of the world's population has blue eyes. A group of 60 people are randomly selected from all over the world. The number of people in this group who have blue eyes is the random variable Y.

(i) State, in the context of this question, one assumption needed to model Y by a binomial distribution. [1]

Assume now that Y indeed follows a binomial distribution.

(ii) Find the probability that at least 5 but less than 21 people in the group will have blue eyes. [2]

(iii) Use a suitable approximation to find the probability that more than 9 people in the group have blue eyes. You should state the parameters of the distribution you have used. [3]

10 **(i)** Suppose a fair die is tossed twice. Calculate the probabilities that

(a) the sum of the scores of the two tosses is at least 8, and [1]

(b) the absolute difference between the scores of the two tosses is at least 4. [1]

In one round of a game, a player is to draw a ball, without replacement, from a box that contains 3 red balls and 4 white balls. If a red ball is drawn, the player will add the scores obtained from tossing a fair die twice. If a white ball is drawn, the player will take the absolute difference of the scores obtained from tossing a die twice.

The game ends if the sum of the scores is at least 8 or the absolute difference of the scores is at least 4. Else, the player will proceed to the second round of the game where the process of picking a ball from the box and tossing the die twice repeats.

(ii) Find the probability that the game ends at the first round. [2]

(iii) Suppose the game ends at the first round. Find the probability that a red ball is drawn. [2]

(iv) Find the probability that there are a total of 3 rounds of game played and exactly 2 white balls are selected. [3]

11 An accountant believes that the figures provided by a particular company for the amount of loans borrowed by its clients, $\$x$, are too low. He carries out an online survey for clients of this company. The responses from a random sample of 20 clients are summarised by

$$\sum x = 21350,\ \sum (x-\bar{x})^2 = 345900.$$

(i) Calculate unbiased estimates of the population mean and variance of the amount of loans borrowed by each client, correct to 1 decimal place. [2]

The company claims that its clients will borrow \$1000 on average.

(ii) Stating a necessary assumption, carry out a test at the 5% level of significance to determine whether the company has understated the mean amount of loans received by its clients. [5]

(iii) Explain, in the context of the question, the meaning of 'at the 5% level of significance'. [1]

The responses from another random sample of n clients are collected. The sample mean value for this sample is the same as the sample mean value for the previously collected sample.

(iv) Given that the standard deviation of X is 250, and that the assumption you have made in part **(ii)** holds, calculate the range of values of n for which the null hypothesis would not be rejected at the 5% level of significance. [3]

12 Cars join an immigration checkpoint queue in a 1-hour period, such that no two cars join the queue at the same instant in time.

(i) State, in the context of this question, an assumption needed for the number of cars joining an immigration checkpoint queue in a 1-hour period to be well modelled by a Poisson distribution. [1]

Assume now that the number of cars joining an immigration checkpoint queue in a 1-hour period is a random variable with the distribution Po(23). It is further given that the number of cars leaving the same immigration checkpoint queue in a 1-hour period is a random variable with the distribution Po(27).

(ii) It is given that in a period of n minutes, the probability that at least one car leaves the queue exceeds 0.9. Write down an inequality in n. Hence find the least integer value of n. [4]

(iii) At 0900 on a certain morning there are 19 cars in the queue. Use appropriate approximations to find the probability that by 1100 there are at most 12 cars in the queue, stating the parameters of any distributions that you use. (You may assume that the queue does not become empty during this period.) [5]

(iv) Explain why a Poisson model for the number of cars joining an immigration checkpoint queue would probably not be valid if applied to a time period of several hours. [1]

– END OF PAPER –

2016 SH2 H2 Mathematics Preliminary Examination Paper 2 Suggested Solutions

Qn No.	Solution

1 (a)

Since Kenny would have paid up his loan in full exactly in the 48th month,

$$S_{48} = \frac{48}{2}\left[2a + (48-1)d\right]$$

$$9600 = 24(2a + 47d)$$

$$400 = 2a + 47d \cdots\cdots (1)$$

Since Kenny had an outstanding payment of \$2400 after the 40th month, total amount paid by the 40th month $= \$9600 - 2400 = \$7200$. Therefore,

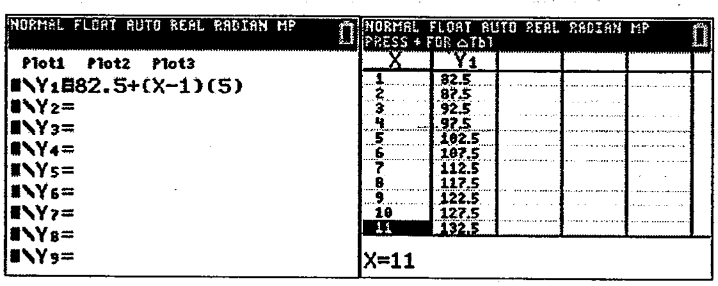
$$S_{40} = \frac{40}{2}\left[2a + (40-1)d\right]$$

$$7200 = 20(2a + 39d)$$

$$360 = 2a + 39d \cdots\cdots (2)$$

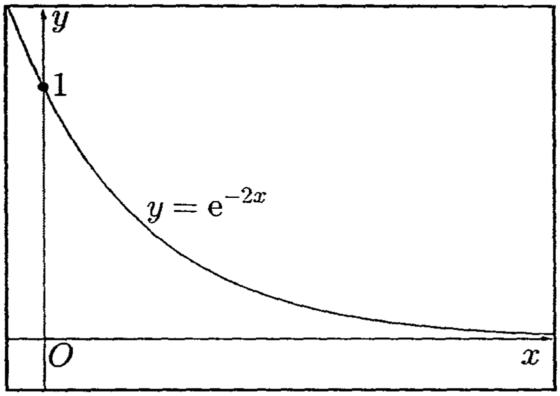
Solving (1) & (2), $a = 82.5, d = 5$.

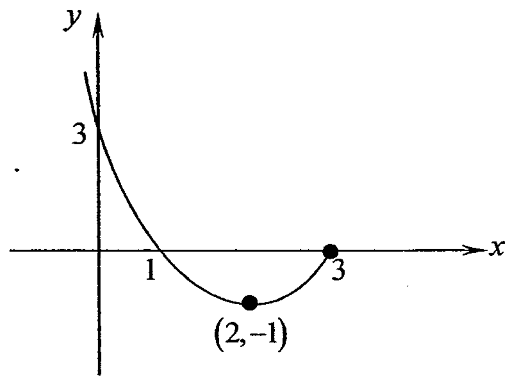
On the last day of the n^{th} month (for $1 \le n \le 40$), the amount paid by Kenny = \$82.5 + $(n-1)(5)$.



Therefore,
amount paid on last day of 10th month = \$127.5 < \$130,
amount paid on last day of 11th month = \$132.5 > \$130.

Therefore Kenny first paid at least \$130 on the last day of the **11th** month.

Qn No.	Solution
1 (b) (i)	Common ratio of $1+e^{-2x}+e^{-4x}+\cdots$ is $r=e^{-2x}$.  For $x>0$, $0<e^{-2x}<1$ (see above sketch). Therefore, the geometric series converges (since e^{-2x} is the common ratio). **OR** As $n\to\infty$, $e^{-2nx}\to 0$ (for $x>0$). Therefore $S_n=\dfrac{1-e^{-2nx}}{1-e^{-2x}}\to\dfrac{1}{1-e^{-2x}}$, i.e. the series is convergent. Sum to infinity $=\dfrac{1}{1-e^{-2x}}$
1 (b) (ii)	For $x=10$, $S_n=\dfrac{1-e^{-20n}}{1-e^{-20}}$, $S=\dfrac{1}{1-e^{-20}}$. $S-S_n<\dfrac{S}{10^{100}}$ $S_n>S-\dfrac{S}{10^{100}}$ $S_n>S\left(1-\dfrac{1}{10^{100}}\right)$ $\dfrac{1-e^{-20n}}{1-e^{-20}}>\dfrac{1}{1-e^{-20}}\left(1-\dfrac{1}{10^{100}}\right)$ $1-e^{-20n}>1-\dfrac{1}{10^{100}}$ $-e^{-20n}>-\dfrac{1}{10^{100}}$ $e^{-20n}<\dfrac{1}{10^{100}}$ $-20n<\ln\dfrac{1}{10^{100}}=-100\ln 10$. $n>5\ln 10=11.513$ Therefore, least value of $n=$ **12**

Qn No.	Solution
2(a)	$f(x)=x$ $x^2-4x+3=x$ $x^2-5x+3=0$ $x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2}$ $=\frac{5\pm\sqrt{13}}{2}$ Since $x\leq 2$, we have $x=\frac{5-\sqrt{13}}{2}$
2(b)(i)	 Any horizontal line $y=k$ where $k\in(-1,0]$ cuts the graph of $y=f(x)$ twice. Thus f is not one-one and hence its inverse does not exist. **OR** The line $y=0$ (or any appropriate value over $(-1,0]$ cuts the graph of $y=f(x)$ twice. Thus f is not-one and hence its inverse does not exist. **OR** $f(3)=f(1)=0$ but $3\neq 1$. Thus f is not one-one and its inverse does not exist.

Qn No.	Solution
2(b) (ii)	From the graph, $R_f = [-1, \infty)$. Moreover, $D_g = (-2, \infty)$. Since $R_f \subset D_g$, gf exists. **OR** $f(x) = x^2 - 4x + 3 = (x-2)^2 - 1$ For $x \leq 3$, $R_f = [-1, \infty)$. Moreover, $D_g = (-2, \infty)$. Since $R_f \subset D_g$, gf exists. $gf(x) = g\left(x^2 - 4x + 3\right)$ $\quad = \tan^{-1}\left(2x^2 - 8x + 7\right)$ Note that $D_{gf} = D_f$. Thus, $gf : x \mapsto \tan^{-1}\left(2x^2 - 8x + 7\right),\ x \in \square ,\ x \leq 3$. $R_{gf} = \left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$

Qn No.	Solution
3 (i)	Let the foot of perpendicular be N.

Method 1

Equation of the line that passes through A and perpendicular to p_1 is

$$l_A : \mathbf{r} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \gamma \in \mathbf{R}.$$

Since N lies on l_A, $\overrightarrow{ON} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for some $\gamma \in \mathbf{R}$.

$$\left[\begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 4 \Rightarrow 2q + 2\gamma = 4$$

$$\Rightarrow \gamma = 2 - q$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2-q \\ 2-q \\ 0 \end{pmatrix} = \begin{pmatrix} 2+q \\ 2-q \\ 2 \end{pmatrix}$$

Hence, N is the point $(2+q, 2-q, 2)$.

Method 2

Let C denote the point (0, 4, 2). Then C lies on p_1 since

LHS of eqn. of $p_1 = 0 + 4 = 4 =$ RHS of eqn. of p_1

$$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2q \\ 4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AN} = \text{Proj. vector of } \overrightarrow{AC} \text{ onto } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} -2q \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2 + 1^2 + 0^2}} \left(\frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{1^2 + 1^2 + 0^2}} \right)$$

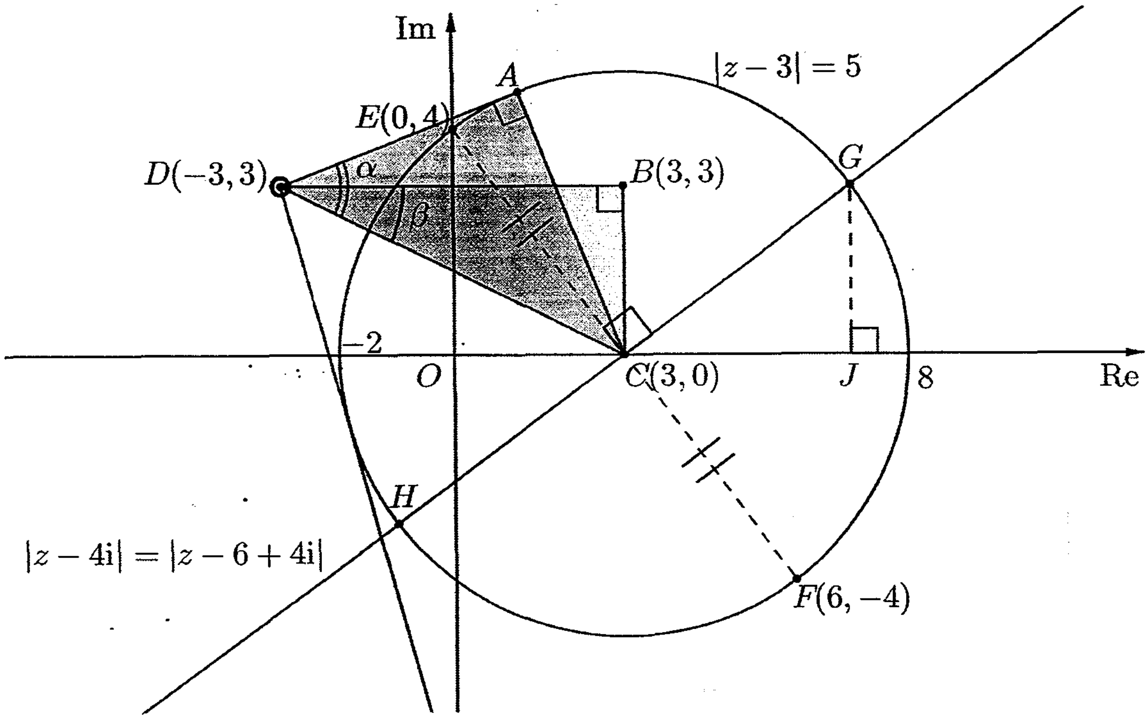
$$= \frac{4 - 2q}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (2 - q) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix} 2q \\ 0 \\ 2 \end{pmatrix} + (2-q) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2+q \\ 2-q \\ 2 \end{pmatrix} \gamma$$

Hence, N is the point $(2+q, 2-q, 2)$.

Qn No.	Solution
3 (ii)	Let **b** be the position vector of point B. By Ratio Theorem, $\begin{pmatrix}2q\\0\\2\end{pmatrix}+\mathbf{b}=2\begin{pmatrix}2+q\\2-q\\0\end{pmatrix}$ $\mathbf{b}=\begin{pmatrix}4\\4-2q\\2\end{pmatrix}$ Since B lies in p_2, $\begin{pmatrix}4\\4-2q\\2\end{pmatrix}\cdot\begin{pmatrix}3\\2\\-5\end{pmatrix}=7$ $12+8-4q-10=7$ $q=0.75$ **or** $\frac{3}{4}$
3 (iii)	Using GC, $l:\mathbf{r}=\begin{pmatrix}-1\\5\\0\end{pmatrix}+\theta\begin{pmatrix}5\\-5\\1\end{pmatrix},\theta\in\mathbf{R}$.
3 (iv)	$\begin{pmatrix}\lambda\\0\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-5\\1\end{pmatrix}=0\Rightarrow\lambda=-\frac{1}{5}$. $\begin{pmatrix}-1\\5\\0\end{pmatrix}\cdot\begin{pmatrix}\lambda\\0\\1\end{pmatrix}\neq\mu\Rightarrow\mu\neq\frac{1}{5}$.

<table>
<tr><th>Qn No.</th><th>Solution</th></tr>
<tr><td>4 (i), (iii) (1st part)</td><td>



</td></tr>
<tr><td>4 (ii)</td><td>

$CD=\sqrt{(3-(-3))^2+(0-3)^2}$
$=\sqrt{45}$

Largest value of $\arg(z+3-3\text{i})$
$=\alpha-\beta$
$=\sin^{-1}\left(\frac{AC}{CD}\right)-\tan^{-1}\left(\frac{BC}{BD}\right)$
$=\sin^{-1}\left(\frac{5}{\sqrt{45}}\right)-\tan^{-1}\left(\frac{1}{2}\right)$
$=0.37742$
$=0.377$ rad (to 3 s.f.)

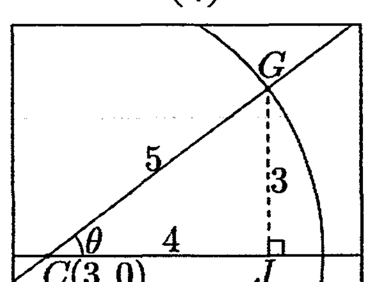
</td></tr>
<tr><td>4 (iii) (2nd part)</td><td>

<u>Method 1: Applying relationship between gradient of a line and the angle it makes with the positive horizontal axis</u>

Gradient of perpendicular bisector $=-\frac{1}{\left(-\frac{4}{3}\right)}=\frac{3}{4}$

Angle that perpendicular bisector makes with positive real axis, $\theta=\tan^{-1}\left(\frac{3}{4}\right)$

Hence,
$\frac{GJ}{5}=\sin\theta=\frac{3}{5}\Rightarrow GJ=3$
$\frac{CJ}{5}=\cos\theta=\frac{4}{5}\Rightarrow CJ=4$



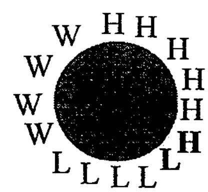
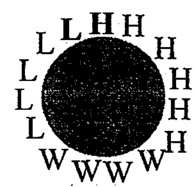
So G and H represent the complex numbers
$z=(3+4)+(0+3)\text{i}=7+3\text{i}$ (corresponding to G), and
$z=(3-4)+(0-3)\text{i}=-1-3\text{i}$ (corresponding to H) resp.

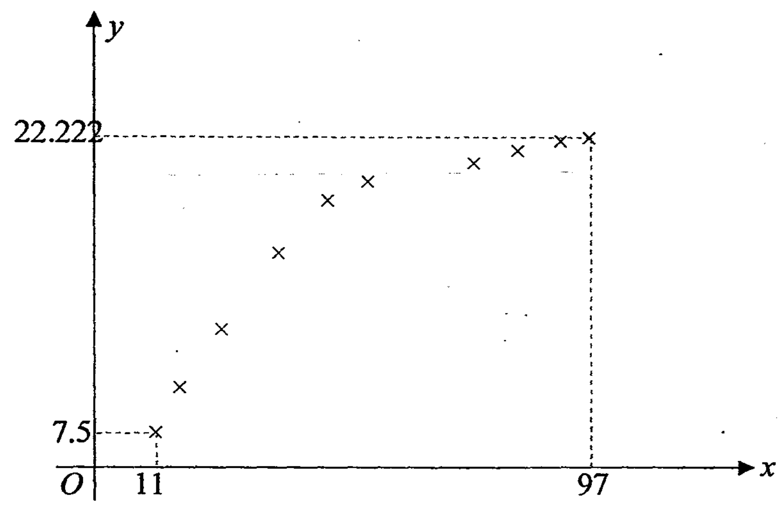
</td></tr>
</table>

Qn No.	Solution
4 (iii) (2nd part)	**Method 2: Using Similar Triangles** $\angle GCE = 90^\circ \Rightarrow \angle OCE + 90^\circ + \angle GCJ = 180^\circ$ $\Rightarrow \angle OCE = 90^\circ - \angle GCJ$ Also, $\angle OEC = 90^\circ - \angle OCE$ $= 90^\circ - (90^\circ - \angle GCJ)$ $= \angle GCJ$ Furthermore, $\angle COE = \angle GJC = 90^\circ$. Therefore, $\triangle COE \sim \triangle GJC$. Hence, $\frac{CO}{CE} = \frac{GJ}{GC} \Rightarrow \frac{3}{5} = \frac{GJ}{5} \Rightarrow GJ = 3$, and $\frac{OE}{CE} = \frac{CJ}{GC} \Rightarrow \frac{4}{5} = \frac{CJ}{5} \Rightarrow CJ = 4$ So coordinates of *G* are (3 + 4, 0 + 3), i.e. (7, 3) and similarly, coordinates of *H* are (3 – 4, 0 – 3), i.e. (–1, –3). Therefore, possible values of *z* are 7 + 3i and –1 –3i. **Method 3: Using Cartesian Equations** Equation of circle: $(x-3)^2 + y^2 = 5^2$ Equation of perpendicular bisector: $y - 0 = -\frac{1}{\left(-\frac{8}{6}\right)}(x-3) \Rightarrow y = \frac{3}{4}(x-3)$ Substituting, $(x-3)^2 + \left(\frac{3}{4}(x-3)\right)^2 = 5^2$ $\left(1 + \left(\frac{3}{4}\right)^2\right)(x-3)^2 = 5^2$ $\frac{25}{16}(x-3)^2 = 25$ $(x-3)^2 = 16$ $x - 3 = \pm 4$ $x = 7$ or -1. When $x = 7$, $y = \frac{3}{4}(7-3) = 3$ When $x = -1$, $y = \frac{3}{4}(-1-3) = -3$ Therefore, possible values of *z* are 7 + 3i and –1 –3i.

Qn No.	Solution
5 (a)	Assign a number from 1 to N to each of the students, where N represents the student population size OR obtain a list of the students from the administration office in order of their identification numbers or registration numbers. Next, determine the sampling interval size $k = \dfrac{1}{0.02} = 50$. Randomly select any student from the list, say the 1st student. Select every 50th student thereafter (i.e. 51st, 101th,...) until the required sample is obtained.
5 (b)	**Advantages**: • **<u>Representativeness of Sample</u>** Quota sampling allows the survey to capture the responses that represent various groups of students (e.g. different PM classes, or 1st CCAs); this may be preferred as certain homeroom or sports facilities may not be in as good a condition as others, and the representation of each group will ensure that the results will not be biased towards those who are often using these less functional facilities or towards those who are often using the more functional facilities. • **<u>Efficiency of Collecting the Sample</u>** Quota sampling may be more efficient as systematic sampling in this case requires the surveyor to identify the selected respondents and to contact them, which can be time consuming (e.g. student selected may be on MC on day of survey, selected students do not respond to online survey etc). **Disadvantages**: • **<u>Non-randomness/Selection Bias</u>** Quota sampling is non-random and may contain selection bias, where the surveyor chooses people who may appear friendlier or choose students in the canteen only at a selected time period. This results in certain students having no chance of being selected at all, which may affect the validity of the survey results. • **<u>Non-representativeness of Sample</u>** Quota sampling may result in a group (e.g. one entire cohort, or people coming later to the canteen etc.) being excluded entirely from the selection, which may result in the data collected being an inaccurate representation of the entire school population.

Qn No.	Solution
6	$X \sim N(\mu, \sigma^2)$ $P(X < 17.7) = 0.15$ $P\left(Z < \frac{17.7-\mu}{\sigma}\right) = 0.15$ $\frac{17.7-\mu}{\sigma} = -1.03643$ --- (1) $P(X > 21.9) = 0.2$ $P(X < 21.9) = 0.8$ $P\left(Z < \frac{21.9-\mu}{\sigma}\right) = 0.8$ $\frac{21.9-\mu}{\sigma} = 0.841621$ --- (2) Solving simultaneous equations (1) and (2): From (1): $\mu = 1.03643\sigma + 17.7$ From (2): $\mu = -0.841621\sigma + 21.9$ Using GC, $\mu = 20.0$ (3s.f) and $\sigma = 2.24$ (3s.f).

Qn No.	Solution
7 (i)	No. of ways $=\left(\frac{9!}{9}\right)\binom{9}{6}6!=2\,438\,553\,600$
7 (ii)	No. of ways $=\left(\frac{3!}{3}\right)(5!)(4!)(4!)(1)$ $=138\,240$ OR **<u>Case 1</u>** No. of ways $=(5!)(4!)(4!)$ $=138\,240/2$  **<u>Case 2</u>** No. of ways $=(5!)(4!)(4!)$ $=138\,240/2$ 
7 (last part)	**<u>Case 1</u>** : None from Liaison Committee No of ways $=1$ **<u>Case 2</u>** : None from Welfare Committee No of ways $=\binom{6}{6}\binom{5}{4}+\binom{6}{5}\binom{5}{5}=11$ or No of ways $=\binom{11}{10}=11$ Total Number of ways to select at least 3 men and 3 women $=\binom{15}{10}-1-11$ $=2991$

Qn No.	Solution
8 (i)	
8 (ii)	**Unsuitability of a Linear Model** A linear model predicts the average diameter will keep increasing indefinitely without any limit. Therefore a linear model is not appropriate. **Unsuitability of a Quadratic Model** A quadratic model predicts that the average diameter will eventually attain a maximum value, and thereafter decrease as the age increases, till it eventually takes on negative values. This is not possible, and therefore a quadratic model is not appropriate.
8 (iii)	Using the suggested model, the least square regression line is $y = 23.886 - \frac{185.346}{x} = 23.9 - \frac{185}{x}$ (to 3 s.f.) r-value = –0.994 (to 3 s.f.) When $x = 40$, $y = 23.886 - \frac{185.346}{40} = 19.3$ (to 3 s.f.) Since x = 40 is within the range of values of x, [11, 97] and the product moment correlation coefficient, –0.994, has an absolute value that is close to 1, suggesting a strong linear correlation between the variables y and $1/x$, therefore the estimate is reliable.

Qn No.	Solution
9 (i)	The eye colour of a person is independent of that of another person. **OR** Every person is equally likely to have blue eyes.
9 (ii)	$Y \sim \mathrm{B}(60, 0.08)$ Then $\mathrm{P}(5 \le Y < 21)$ $= \mathrm{P}(Y \le 20) - \mathrm{P}(Y \le 4)$ $= 0.530$ (to 3 s.f.)
9 (iii)	Since $n = 60$ (> 50) is large and $np = 4.8 < 5$, $Y \sim \mathrm{Po}(4.8)$ approximately. $\mathrm{P}(Y > 9) = 1 - \mathrm{P}(Y \le 9) = 0.0251$ (to 3 s.f.)

Qn No.	Solution
10 (i) (a)	Let E and F denote the event that the sum of scores is at least 8 and the event that the absolute difference between the scores is at most 4 respectively.

Sum of Scores

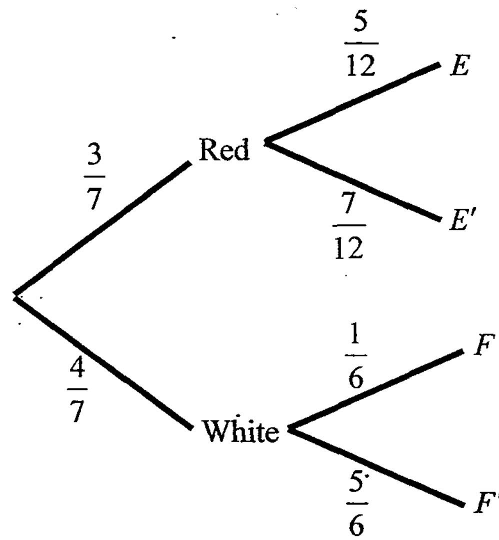
1st die \ 2nd die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

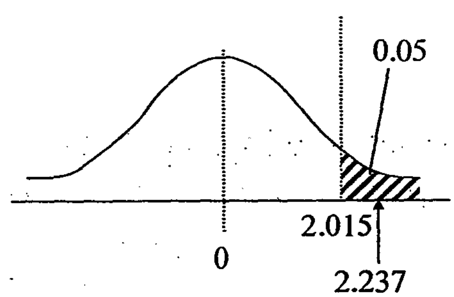
$$P(E)=\frac{1}{6}\times\frac{1}{6}\times 15=\frac{5}{12}$$

Qn No.	Solution
10 (i) (b)	**Absolute Difference between Scores**

1st die \ 2nd die	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

$$P(F)=\frac{1}{6}\times\frac{1}{6}\times 6=\frac{1}{6}$$

Qn No.	Solution
10 (ii)	 P(game ends at 1st round) = P(red and game ends at 1st round) +P(white and game ends at 1st round) $=\left(\frac{3}{7}\right)\left(\frac{5}{12}\right)+\left(\frac{4}{7}\right)\left(\frac{1}{6}\right)$ $=\frac{23}{84}$
10 (iii)	P(red ball selected | game ends at the first round) $=\frac{\text{P(red ball selected} \cap \text{game ends at the first round)}}{\text{P(game ends at the first round)}}$ $=\frac{\left(\frac{3}{7}\right)\left(\frac{5}{12}\right)}{\frac{23}{84}}=\frac{15}{23}$
10 (iv)	P(total of 3 rounds of game & exactly 2 white balls selected) = P(WWR) + P(WRW) + P(RWW) $=\left(\frac{4}{7}\right)\left(\frac{5}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{3}{5}\right)\left(\frac{5}{12}\right)+\left(\frac{4}{7}\right)\left(\frac{5}{6}\right)\left(\frac{3}{6}\right)\left(\frac{7}{12}\right)\left(\frac{3}{5}\right)\left(\frac{1}{6}\right)$ $+\left(\frac{3}{7}\right)\left(\frac{7}{12}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{3}{5}\right)\left(\frac{1}{6}\right)$ $=\frac{25}{504}+\frac{1}{72}+\frac{1}{72}$ $=\frac{13}{168}$

Qn No.	Solution
11 (i)	Unbiased estimate of μ, $\bar{x} = \frac{\sum x}{n} = \frac{21350}{20} = 1067.5$ Unbiased estimate of σ^2, $s^2 = \frac{1}{19}(345900)$ $= 18205.3$ (to 1 d.p.)
11 (ii)	H_0: $\mu = 1000$ H_1: $\mu > 1000$ Assume that the amounts of loans borrowed by the bank's clients follow a normal distribution. **OR** Assume that the amount of loans borrowed by each client follows a normal distribution. Level of Significance: 5% (upper-tailed) Under H_0, $T = \frac{\bar{X} - 1000}{S/\sqrt{20}} \sim t(19)$ Test Statistic: $t = \frac{\bar{x} - 1000}{s/\sqrt{20}}$ **<u>Method 1: Using critical region and observed test statistic, *t*</u>** Critical region: $t > 2.015$ $t = \frac{1067.5 - 1000}{s/\sqrt{20}} \approx 2.237$ $\left(s = \sqrt{18205.3}\right)$ Since $t = 2.237 > 2.015$, we reject H_0.  **<u>Method 2: Using *p*-value</u>** *p*-value $= 0.0187$ Since *p*-value $= 0.0187 < 0.05$, we reject H_0. We conclude that there is sufficient evidence at 5% level of significance that the company has understated the mean amount of loans borrowed by its clients.
11 (iii)	The meaning of 'at the 5% significance level' is that there is a probability of 0.05 that it was **wrongly concluded** that the company had understated the mean amount of loans borrowed by its clients.

Qn No.	Solution
11 (iv)	Test Statistic: $z=\dfrac{\overline{x}-1000}{250/\sqrt{n}}$ To not reject H_0, $z \le 1.6449$ $\dfrac{1067.5-1000}{250/\sqrt{n}} \le 1.6449$ $\sqrt{n} \le 6.092$ $n \le 37.1$ Since $n \in \mathbf{Z}^+$, $n \le 37$.

Qn No.	Solution
12 (i)	Any one of the following: [*Constant mean rate*] The mean number of cars joining the immigration checkpoint queue for any subinterval of the same length of time within 1 hour (e.g. minute) is constant. **OR** [*Independence of occurrence of event*] Cars join the immigration queue independently of one another, throughout the entire hour.
12 (ii)	Let X denote the random variable for the number of cars leaving an immigration checkpoint queue in a period of n minutes. $X \sim \text{Po}\left(\frac{27}{60}n\right)$ i.e. $X \sim \text{Po}(0.45n)$ $\text{P}(X \geq 1) > 0.9$ $1-\text{P}(X=0) > 0.9$ $\text{P}(X=0) < 0.1$ $\frac{e^{-0.45n}(0.45n)^0}{0!} < 0.1$ $e^{-0.45n} < 0.1$ $-0.45n < \ln(0.1)$ $n > \frac{\ln(0.1)}{-0.45} \approx 5.11$ Therefore, least integer n is 6.

Qn No.	Solution
12 (iii)	Let J and L denote the random variables for the number of cars joining and leaving an immigration checkpoint queue respectively in a 2-hour period. Then $J \sim \text{Po}(46)$ and $L \sim \text{Po}(54)$. Since $46 > 10$, $J \sim \text{N}(46,46)$ **approximately**. Since $54 > 10$, $L \sim \text{N}(54,54)$ **approximately**. Let W denote the number of people in the queue at 1100. Then $W = 19 + J - L$. $\text{E}(W) = \text{E}(19 + J - L)$ $= 19 + \text{E}(J) - \text{E}(L) = 19 + 46 - 54 = 11$, and $\text{Var}(W) = \text{Var}(19 + J - L)$ $= \text{Var}(J) + \text{Var}(L) = 46 + 54 = 100$ Therefore, $W = 19 + J - L \sim \text{N}(11,100)$ **approximately.** $\text{P}(19 + J - L \le 12) = \text{P}(19 + J - L \le 12.5)$ (by c.c.) $= 0.560$ (to 3.s.f) **<u>Alternatively,</u>** $J - L \sim \text{N}(-8,100)$ **approximately.** $\text{P}(J - L \le -7) = \text{P}(J - L \le -6.5)$ $= 0.560$ (to 3.s.f) **<u>Or equivalently,</u>** $L - J \sim \text{N}(8,100)$ **approximately.** $\text{P}(L - J \ge 7) = \text{P}(L - J \ge 6.5)$ $= 0.560$ (to 3.s.f)
12 (iv)	The mean number of cars joining the immigration checkpoint queue every hour may not be constant due to peak periods as there may be more cars heading to or returning from work.