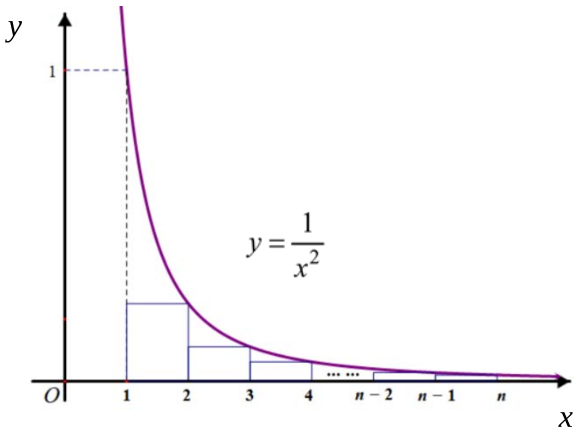
1



The diagram above shows the graph of $y = \frac{1}{x^2}$ for $x > 0$, together with a set of $(n-1)$ rectangles of unit width, starting at $x = 1$.

(i) By considering the areas of these rectangles, explain why

$$\frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,\mathrm{d}x\ ,$$

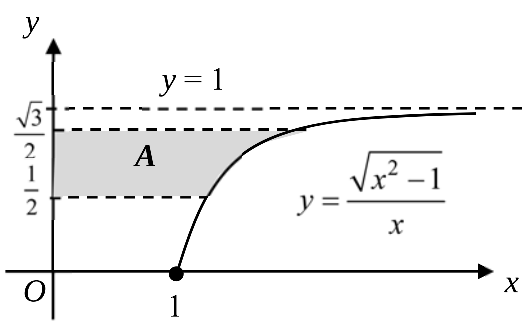
where n is an integer greater than 1.

[1]

(ii) Find $\int_1^n \frac{1}{x^2}\,\mathrm{d}x$ in terms of n. Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots < 2$. [2]

2 It is given that $\mathrm{f}(x) = x^3 \ln a + bx^2 + cx + d$, where a, b, c and d are constants. The curve with equation $y = \mathrm{f}(x)$ has a minimum point with coordinates $\left(\frac{5}{3}, \frac{320}{27}\right)$. When this curve is translated 1 unit in the negative x-direction, it has a maximum point with coordinates $(0, 12)$. Find the values of a, b, c, and d. [4]

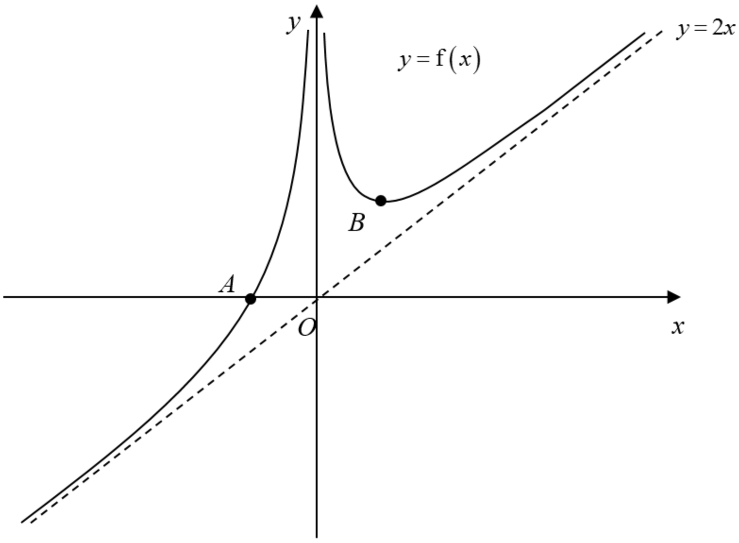
3 Find the exact area of the region A which is bounded by the curve $y = \frac{\sqrt{x^2 - 1}}{x}$, the horizontal lines $y = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$, and the y-axis as shown in the diagram.



[4]

4 **(a)** State a sequence of transformations which transform the curve with equation $\frac{(x-3)^2}{4^2}+\frac{(y+1)^2}{5^2}=1$ to the curve with equation $\frac{(x-6)^2}{8^2}+\frac{(y+3)^2}{5^2}=1$. [2]

(b)



The diagram shows the graph of $y=\mathrm{f}(x)$. The curve crosses the *x*-axis at *A*, and has a minimum point at *B*. The coordinates of *A* and *B* are $(a,\ 0)$ and $(b,\ 2)$ respectively, where $a<0$ and $b>0$. The line $y=2x$ and *y*-axis are the asymptotes to the curve.

Sketch on separate diagrams, the graphs of

(i) $y=\mathrm{f}'(x)$, [3]

(ii) $y=\frac{1}{\mathrm{f}(x)}$, [3]

showing clearly, in terms of *a* and *b* where possible, the equations of any asymptote(s), the coordinates of any turning point(s) and any point(s) where the curve crosses the x- and y-axes.

5 **(i)** Let $y=\ln(\mathrm{e}^x+1)$.

Show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [2]

(ii) By further differentiation of the result in part **(i)**, find the first four non-zero terms in the Maclaurin series for y. [5]

(iii) Hence, expand $\frac{\ln(\mathrm{e}^x+1)}{4-x^2}$ in ascending powers of *x* up to and including the term in x^3. Leave the coefficients of the series in exact form. [3]

6 **(a)** Find $\displaystyle\int \frac{2-x}{4+x^2}\,\mathrm{d}x$. [3]

(b) Use the substitution $x = \tan y$ to find the exact value of $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$. [4]

(c) Write down $\displaystyle\int x^2 \mathrm{e}^{x^3}\,\mathrm{d}x$. Hence find $\displaystyle\int x^5 \mathrm{e}^{x^3}\,\mathrm{d}x$. [4]

7 The function f is defined by

$$\mathrm{f} : x \mapsto x^2 + 4x - 5, \quad \text{for } x \le k, k \in \mathbb{R} .$$

(i) Find the largest exact value of k such that f^{-1} exists. For this value of k, define f^{-1} in a similar form. [4]

Another function g is defined by

$$\mathrm{g} : x \mapsto \begin{cases} 4 - x^2, & \text{for } 0 < x \le 2 \\ 2x - 4, & \text{for } 2 < x \le 4 \end{cases}$$

and that g(x) = g(x + 4) for all real values of x.

(ii) Sketch the graph of $y = \mathrm{g}(x)$ for $-1 < x \le 7$. [3]

(iii) Using the results in part **(i)** and **(ii)**, explain why composite function $\mathrm{f}^{-1}\mathrm{g}$ exists and find the exact value of $\mathrm{f}^{-1}\mathrm{g}(6)$. [4]

8 The logistic model for population growth states that the rate of growth of the population can be written as:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = cP\left(1 - \frac{P}{K}\right),$$

where c is the proportionality constant, P is the size of population (in billions) at time t (in years after 2010), K is the carrying capacity (in billions). The carrying capacity is the maximum population that the environment is capable of sustaining in the long run.

At the start of 2010, the population of the world was about 7 billion. Many scientists estimated the Earth has a maximum capacity of 10 billion people, based on the calculation of the earth's available resources. It is assumed that when the population of the world is 9 billion, the rate of growth of the population is $\dfrac{9}{1750}$.

(i) Show that $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{1}{1750}P(10-P)$. [2]

(ii) Use the logistic model to predict the world population at the start of 2020. [6]

(iii) After how many complete years will the world population first exceed 8.5 billion? [3]

(iv) Sketch a graph of P against t, where $t \ge 0$. [2]

9 The curve C has equation $y=\dfrac{x^2-4x+1}{2x+7}, x\in ¡ , x\neq -\dfrac{7}{2}$.

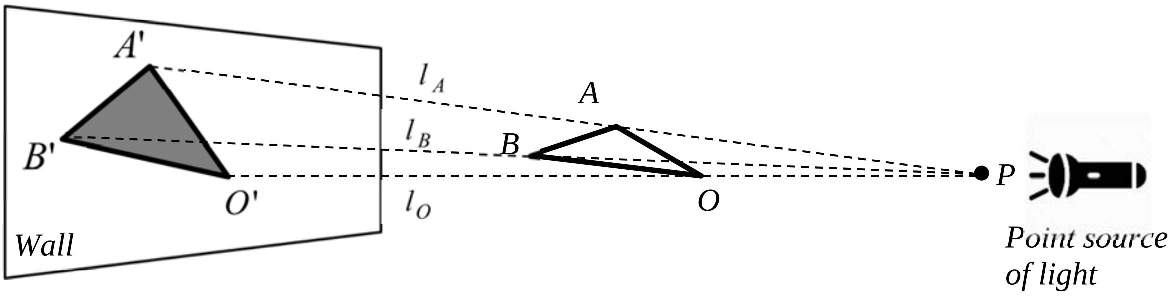
(i) Without using a calculator, find the set of values of y that C can take. [3]

(ii) Sketch C, stating the equations of any asymptotes, the coordinates of the points where the curve crosses the axes and the coordinates of the turning points. [4]

(iii) By drawing a suitable graph on the same diagram, solve the inequality $\dfrac{4x-x^2-1}{2x+7}>\dfrac{1}{(x+1)^2}$. [3]

10 In order to render the position of a shadow cast by an object on a wall in virtual reality, the concept of vector projection is used. When an object is placed between a point source of light and a wall, its shadow is projected onto the wall as shown in the diagram (not drawn to scale) below.

A triangular object *OAB* has *O* as the origin, $A(-23,\ 16,\ 10)$ and $B(-9.5,\ 6.5,\ 10.5)$ on the same plane where it is placed between the point source of light, $P(11,\ -22,\ -10)$, and the wall. Light rays l_A, l_B and l_O start at point *P*, passing through points *A*, *B* and *O* respectively and projecting their respective images A', B' and O' onto the wall.



Given that the coordinates of A' is $(-40,\ 35,\ 20)$ and the line that passes through *A'* and *B'* is parallel to the vector $10\mathbf{i}+11\mathbf{k}$,

(i) Find the coordinates of the point B'. [4]

It is given that the light ray l_O is perpendicular to the wall.

(ii) Find the equation of the plane $O'A'B'$ in scalar product form. Show that coordinates of O' is given by $(-22,\ 44,\ 20)$. [4]

(iii) Hence or otherwise, find the exact distance between point *P* and the wall. [2]

(iv) Are the planes *OAB* and $O'A'B'$ parallel? Justify your answer. [2]

11 It is given that $z = -\frac{1}{2}$ is a root of the equation

$$8z^3 + \left(4 - 4\sqrt{2}\right)z^2 + \left(2 - 2\sqrt{2}\right)z + 1 = 0.$$

The roots of the equation are denoted by z_1, z_2, z_3, where $\arg(z_1) < \arg(z_2) < \arg(z_3)$.

(i) Find z_1, z_2 and z_3 in the exact form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [6]

The complex number w has modulus $\sqrt{2}$ and argument $\frac{\pi}{24}$.

(ii) Find the modulus and argument of z_4, where $z_4 = \frac{w^2}{z_1}$. [3]

The complex numbers z_2, z_3 and z_4 are represented by the points Z_2, Z_3 and Z_4 respectively in an Argand diagram with origin O.

(iii) Mark, on an Argand diagram, the points Z_2, Z_3 and Z_4. [2]

(iv) By considering $\sin(A - B)$ with suitable values of A and B, show that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$. [1]

Hence or otherwise, find the exact area of the quadrilateral $OZ_3Z_4Z_2$. [2]

— END Of PAPER —

Section A: Pure Mathematics [40 marks]

1 **(i)** The sum of the first n terms of a sequence is given by $S_n = n(2n+7)$.
Show that $u_n = 4n+5$ and prove that the sequence is an Arithmetic Progression. [2]

(ii) Find $\displaystyle\sum_{n=1}^{N} \frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$ in terms of N, where u_n is the nth term of the arithmetic series in part **(i)**. [4]

(iii) Hence find the exact value of $\dfrac{1}{\sqrt{53}+\sqrt{49}}+\dfrac{1}{\sqrt{57}+\sqrt{53}}+\cdots+\dfrac{1}{\sqrt{361}+\sqrt{357}}$. [2]

2 The origin O, and the points A and B lie in the same plane where $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. It is given that $|\mathbf{a}|=1$, $|\mathbf{b}|=2$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\pi}{3}$ radians. The point F is the foot of the perpendicular from A to the line segment passing through O and B.

(i) Find $\overrightarrow{OF}$ in terms of $\mathbf{b}$ only. [4]

(ii) It is given that C is the point of reflection of A about the line segment passing through O and B. Find the position vector of C in terms of $\mathbf{a}$ and $\mathbf{b}$.
Hence, state with a reason, the shape of the quadrilateral $OABC$ and show that its exact area is $\sqrt{3}$. [4]

3 An airport proposes to collect passenger service fee from each traveller to fund an upgrading project for the airport. The upgrading project is expected to cost \$ 1, 000, 000.

The proposal is to collect a fee of \$5 per traveller from 1 January 2019, and increase this fee by \$2.50 on 1 January of every subsequent year. It is given that the airport handles 10, 000 travellers a year and that there is no change in the number of travellers every year.

(i) According to this proposal, find the earliest year in which the airport reaches its target of \$ 1, 000, 000. [4]

However, representatives from the airline industry strongly object to the above proposal. Consequently, the airport decides to allocate funds from its reserve investment revenue every year to finance the upgrading project instead, starting from the year 2019.

The allocated fund in 2019 is \$50, 000. Due to increases in other expenditures, the allocated fund is expected to decrease by 6% of the allocated fund in the previous year, such that the allocated fund is \$47, 000 in 2020, \$44, 180 in 2021 and so on.

(ii) Find the total allocated fund collected in the years from 2024 to 2030, giving your answer correct to nearest dollar. [3]

(iii) Will the airport eventually collect enough to fund the upgrading project? Justify your answer. [3]

4.

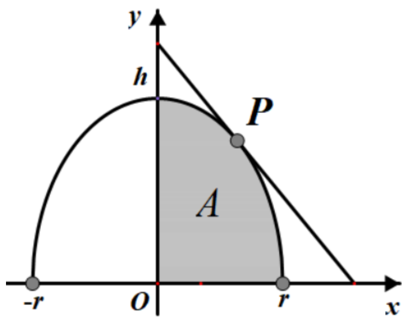


Figure 1

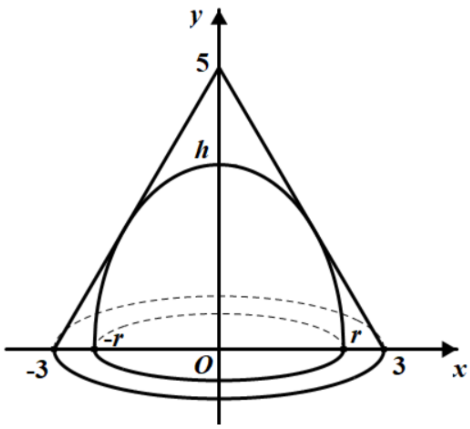


Figure 2

Figure 1 shows the upper-half of an ellipse with equation $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$, and region A bounded by the ellipse and the axes in the first quadrant.

(i) Show that the volume V of the solid of revolution generated by rotating region A through 2π radians about the y-axis, is $V=\frac{2}{3}\pi r^2 h$. [3]

(ii) The ellipse can also be represented by parametric equations $x=r\cos\theta$, $y=h\sin\theta$, where $0\le\theta\le\pi$. Show that the equation of the tangent to the ellipse at point P $(r\cos\alpha,\ h\sin\alpha)$, is $y=-\left(\frac{h}{r}\cot\alpha\right)x+h\ \text{cosec}\,\alpha$. [4]

Figure 2 shows a fixed right circular cone of height 5 and base radius 3, which contains an inscribed hemi-ellipsoid generated by rotating the region that is bounded by the ellipse $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$ and the x-axis through π radians about the y axis.

(iii) Given that the line $y=-\frac{5}{3}x+5$, the slanted edge of the cone joining points $(3,\ 0)$ and $(0,\ 5)$, is a tangent to the inscribed ellipse $\left(\frac{x}{r}\right)^2+\left(\frac{y}{h}\right)^2=1$, deduce that the base radius r and height h for the inscribed hemi-ellipsoid is related by the equation $\left(\frac{h}{5}\right)^2+\left(\frac{r}{3}\right)^2=1$. [3]

(iv) Find the exact value of height h for which the volume of the inscribed hemi-ellipsoid is a maximum as h varies. [4]

Section B: Probability and Statistics [60 marks]

5 In a supermarket game, ten randomly selected shoppers are allowed 15 minutes each to search for mystery items hidden amongst the supermarket shelves. Each shopper is allowed to find at most one mystery item. On average the probability that a shopper will find a mystery item is $\frac{1}{p}$ where $1 < p < 2$.

(i) State, in the context of this question, an assumption needed to model the number of shoppers who find a mystery item by a binomial distribution. [1]

Assume now that part **(i)** holds.

The probability that three of the ten shoppers find a mystery item is $\frac{15}{4}\left(\sqrt{2}-1\right)^7$.

(ii) Find the value of p. [2]

The supermarket conducts this game n times.

(iii) Find the least value of n such that there is a probability of more than 0.01 that in more than two of the games, three of the ten shoppers find a mystery item in a game. [3]

6. Five fair coins are tossed together in one throw. The number of tails and heads obtained in one throw are denoted by T and H respectively. The random variable X denotes $T - H$.

(i) Show that $\mathrm{P}(X = 1) = \frac{5}{16}$ and hence find the probability distribution of X. [3]

(ii) State the value of $\mathrm{E}(X)$ and find the value of $\mathrm{Var}(X)$. [2]

A player pays \$1 for one throw. He receives nothing if the difference between the number of tails and heads obtained is less than three, receives \$2 if the difference is equal to three, and receives \$$k$ if the difference is more than three.

(iii) Find the value of \$$k$ if the expectation of the player's profit is \$10. [2]

7 **(a)** For events A and B, it is given that $\mathrm{P}(B) = \frac{1}{3}$, $\mathrm{P}(A \cap B) = \frac{1}{5}$ and $\mathrm{P}(A' \cap B') = \frac{1}{6}$.

Find $\mathrm{P}(A \mid B')$. [3]

(b) A seven-digit number is formed by writing down the digits $1, 2, 2, 3, 4, 5, 5$ in some order. Find how many of such numbers can be formed if

(i) the two '5's are not next to each other, [2]

(ii) there are exactly three digits between the two '5's, [3]

(iii) the number is an odd number between $1\,000\,000$ and $2\,000\,000$. [3]

8 In a neuroscience study, researchers investigate the relationship between brain mass, x kilograms and Intelligence Quotient index, y. The table below shows the data of a random sample of 10 people.

x	1.196	1.342	1.399	1.476	1.493	1.504	1.521	1.568	1.582	1.601
y	73	82	98	109	114	119	128	138	142	148

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the intelligence quotient, y can be modelled by one of the formulae

$$y = a + bx \quad \text{or} \quad y = c + dx^2$$

where a, b, c and d are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) x and y, [1]

(b) x^2 and y. [1]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + dx^2$ is the better model. Hence, calculate the equation of the least squares regression line of the suitable model you have found. [3]

(iv) Use the regression line found in part **(iii)** to estimate the value of Intelligence Quotient index when the brain mass is 1500 grams.
Comment on the reliability of your answer. [2]

(v) An internet article claims that the neuroscience study shows that "heavier brain mass leads to higher intelligence quotient". Comment briefly on the validity of this statement. [1]

9 In a certain country, heights of males and females follow independent normal distributions. Heights of males have mean 175 cm and standard deviation 10 cm, and heights of females have mean μ cm and standard deviation σ cm.

(i) If the proportion of females shorter than 143 cm and the proportion of females taller than 183 cm are both equal to 0.01114 , write down the value of μ and show that σ is 8.75 , correct to 3 significant figures. [3]

(ii) If two females and one male are randomly chosen, find the probability that the sum of the heights of the two females differ from twice the height of the male by at least 30 cm. [4]

Flight attendants and fighter pilots have strict height requirements due to the nature of their professions. The national carrier of the country has a minimum height requirement of 155 cm and a maximum of 185 cm for female flight attendants. Its air force has a minimum height requirement of 160 cm and a maximum of 192.5 cm for its male fighter pilots.

(iii) Two females and one male are chosen at random. Find the probability that the male meets fighter pilot height requirement whereas only one female meets the female flight attendant height requirement. [3]

During a group interview conducted by the national carrier, all female candidates wear standard shoes with 5 cm heels. A random sample of 15 female candidates is chosen.

(iv) Find the probability that the average height of the sample of female candidates wearing standard shoes is greater than 170 cm. [3]

10 In a large busy restaurant, the mean time taken for a server to clear a table and set it for the next guest is 4.5 minutes. In order to improve the quality of service and maintain a clean environment, the restaurant manager introduced a new routine to clear tables. A random sample of 30 servers is taken and the time taken by each server to clear a table, x minutes, is recorded. The data are summarised as follows:

$$\sum x = 132 \text{ and } \sum x^2 = 583.96 \text{ .}$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 10% significance level, whether the mean time taken for a server to clear a table has changed. [5]

(iii) Using the results in part **(ii)**, state the range of values of α, where α% is the level of significance, at which a test would lead the manager to conclude that the mean time has not changed. [1]

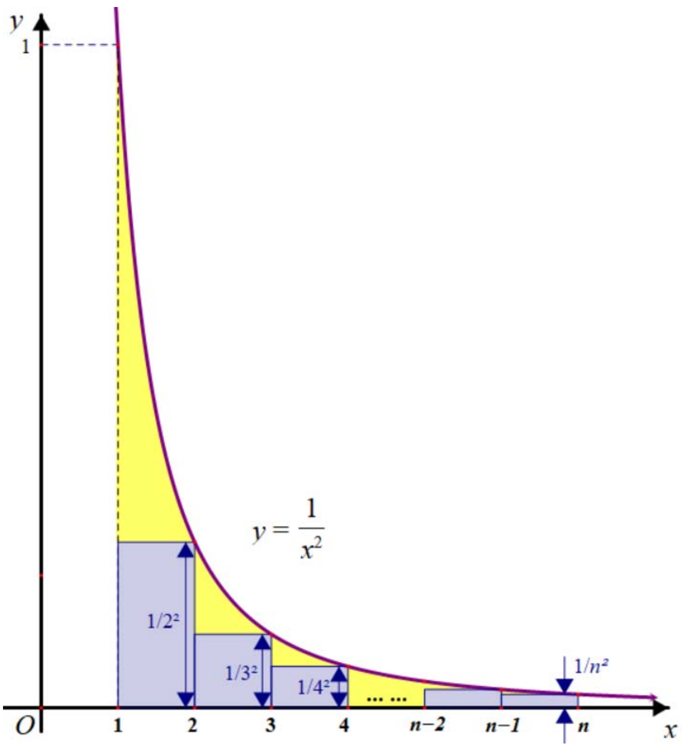
(iv) Without carrying out another test, state with reasons, the conclusion if the test at 5% significance level was to determine whether the mean time taken to clear a table is less than 4.5 minutes. [2]

After a few weeks, based on feedback from servers and guests, the manager modifies the routine. Another random sample of 30 servers is taken and their time taken to clear a table has mean $\overline{x}$ minutes and variance 1.01 minutes^2.

(v) Find the set of values of $\overline{x}$ for which a test at 1% significance level concludes that the mean time taken to clear a table is greater than 4.5 minutes, giving your answer to 3 decimal places. [4]

- END OF PAPER -

1(i)	Use given sketch of the graph of $y=\frac{1}{x^2}$, together with an array of vertically-aligned rectangles, each of width 1, inscribed beneath the curve :



From the diagram,

Area of first rectangle = length x breath $=\frac{1}{2^2}\times 1$

Area of second rectangle $=\frac{1}{3^2}\times 1$

…

Area of (n - 1)th rectangle $=\frac{1}{n^2}\times 1$

$$\text{Total area of } (n-1) \text{ rectangles} < \begin{pmatrix}\text{Area under the curve } y=\frac{1}{x^2} \\ \text{over the interval } 1\le x\le n\end{pmatrix}$$

$$\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}<\int_1^n \frac{1}{x^2}\,\mathrm{d}x$$

(ii)

$$\int_1^n \frac{1}{x^2}\,\mathrm{d}x=\left[\frac{x^{-1}}{-1}\right]_1^n$$

$$=\left[-\frac{1}{x}\right]_1^n$$

$$=\left(-\frac{1}{n}\right)-\left(-\frac{1}{1}\right)$$

$$=1-\frac{1}{n}$$

Since $\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,dx = 1-\frac{1}{n}$

Adding 1 to both sides,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2} < 1+\left(1-\frac{1}{n}\right)$$

As the difference between both sides of this inequality increases with n, as $n \to \infty$,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+... ... < \lim_{n\to\infty}\left(1+\left(1-\frac{1}{n}\right)\right)$$

$$= 1+(1-0) = 2 \quad \text{(shown)}$$

2. Marking Scheme:

$$f(x) = x^3 \ln a + bx^2 + cx + d$$

$$f'(x) = 3x^2 \ln a + 2xb + c$$

At $\left(\frac{5}{3}, \frac{320}{27}\right)$,

$$\left(\frac{5}{3}\right)^3 \ln a + b\left(\frac{5}{3}\right)^2 + \frac{5}{3}c + d = \frac{320}{27}$$

$$125\ln a + 75b + 45c + 27d = 320 \qquad -(1)$$

$$3\left(\frac{5}{3}\right)^2 \ln a + 2\left(\frac{5}{3}\right)b + c = 0$$

$$25\ln a + 10b + 3c = 0 \qquad -(2)$$

Let $g(x) = f(x+1) = (x+1)^3 \ln a + b(x+1)^2 + c(x+1) + d$

$$g'(x) = 3(x+1)^2 \ln a + 2(x+1)b + c$$

At $(0, 12)$,

$$\ln a + b + c + d = 12 \qquad -(3)$$

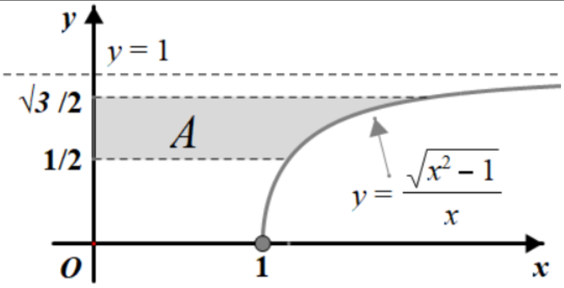
$$3\ln a + 2b + c = 0 \qquad -(4)$$

Using GC and solve, $\ln a = 1, b = -4, c = 5, d = 10$

$\therefore a = e, b = -4, c = 5, d = 10$

3. To obtain area of region A, consider integrating w.r.t. y.

$y = \frac{\sqrt{x^2-1}}{x}$ where $x, y > 0$	$\Leftrightarrow xy = \sqrt{x^2-1}$ $\Leftrightarrow x^2y^2 = x^2-1$ $\Leftrightarrow 1 = x^2(1-y^2)$ $\Leftrightarrow x^2 = \frac{1}{1-y^2}$ $\Leftrightarrow x = \frac{1}{\sqrt{1-y^2}}$



$$\therefore \text{ Area of } A = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-y^2}}\,\mathrm{d}y$$

$$= \left[\sin^{-1} y\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$$

$$= \frac{\pi}{3} - \frac{\pi}{6} \quad = \frac{\pi}{6}$$

4(a)

$$\frac{(x-3)^2}{4^2} + \frac{(y+1)^2}{5^2} = 1$$

$$\xrightarrow{\text{Replace } x \text{ with } \frac{x}{2}} \frac{\left(\left(\frac{x}{2}\right)-3\right)^2}{4^2} + \frac{(y+1)^2}{5^2} = 1$$

$$\xrightarrow{\text{Replace } y \text{ with } y+2} \frac{(x-6)^2}{2^2 4^2} + \frac{((y+2)+1)^2}{5^2} = 1$$

Stage 1: Scale parallel to the x-axis by factor 2.
Stage 2: Translate 2 units in the negative y-direction.
(Order does not matter in this case)

Alternative solutions:

$$\frac{(x-3)^2}{4^2} + \frac{(y+1)^2}{5^2} = 1$$

$$\xrightarrow{\text{Replace } x \text{ with } x+3} \frac{x^2}{4^2} + \frac{(y+1)^2}{5^2} = 1$$

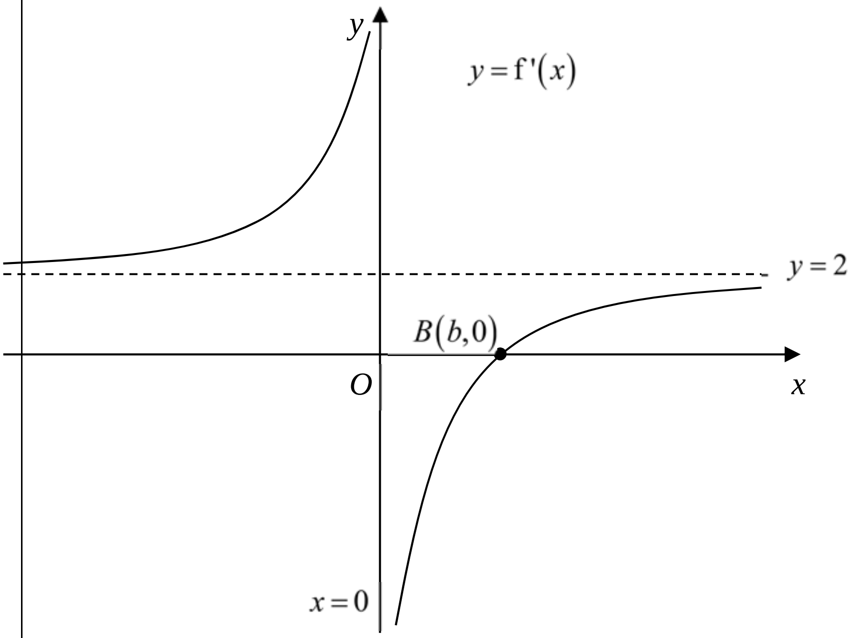
$$\xrightarrow{\text{Replace } x \text{ with } \frac{x}{2}} \frac{\left(\frac{x}{2}\right)^2}{4^2} + \frac{(y+1)^2}{5^2} = 1 \Rightarrow \frac{x^2}{8^2} + \frac{(y+1)^2}{5^2} = 1$$

$$\xrightarrow{\text{Replace } x \text{ with } x-6} \frac{(x-6)^2}{8^2} + \frac{(y+1)^2}{5^2} = 1$$

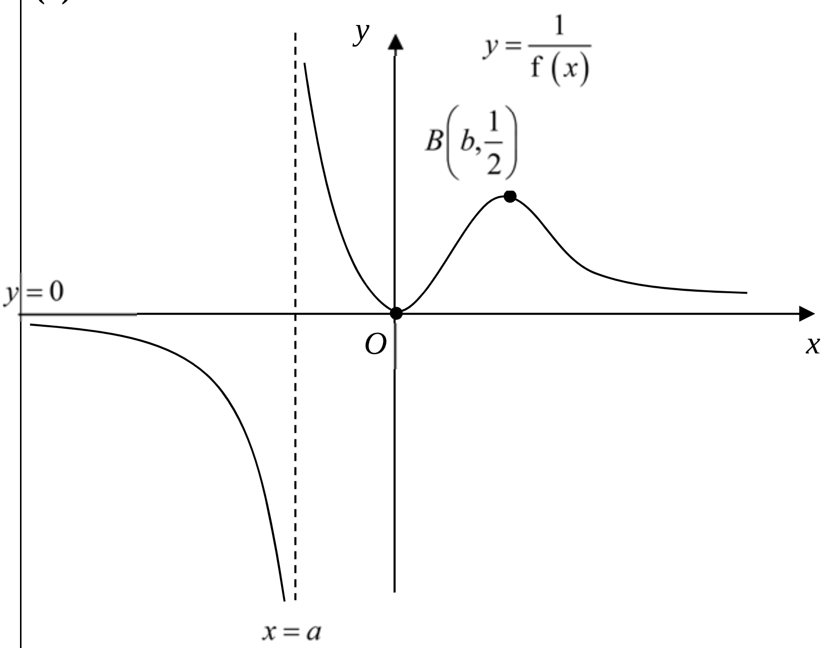
$$\xrightarrow{\text{Replace } y \text{ with } y+2} \frac{(x-6)^2}{8^2} + \frac{(y+3)^2}{5^2} = 1$$

Stage 1: Translate 3 units in the negative x-direction.
Stage 2: Scale parallel to x-axis by factor 2.
Stage 3: Translate 6 units in the positive x-direction.
Stage 4: Translate 2 units in the negative y-direction.

(b)(i)



(ii)



5(i) $y = \ln(e^x + 1)$

$$\frac{dy}{dx} = \frac{e^x}{e^x + 1}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(e^x + 1) - e^x(e^x)}{(e^x + 1)^2}$$

$$= \frac{e^x}{(e^x + 1)^2}$$

The expression

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=\frac{\mathrm{e}^x}{(\mathrm{e}^x+1)^2}-\frac{\mathrm{e}^x}{\mathrm{e}^x+1}+\left(\frac{\mathrm{e}^x}{\mathrm{e}^x+1}\right)^2$$

$$=\frac{\mathrm{e}^x-\mathrm{e}^x(\mathrm{e}^x+1)+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}$$

$$=\frac{\mathrm{e}^x-(\mathrm{e}^x)^2-\mathrm{e}^x+(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}$$

$$=0 \quad \text{(shown)}$$

<u>Alternative Method 1 (simpler)</u>

$$y=\ln(\mathrm{e}^x+1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^x}{\mathrm{e}^x+1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{e}^x(\mathrm{e}^x+1)-\mathrm{e}^x(\mathrm{e}^x)}{(\mathrm{e}^x+1)^2}$$

$$=\frac{\mathrm{e}^x}{\mathrm{e}^x+1}-\frac{(\mathrm{e}^x)^2}{(\mathrm{e}^x+1)^2}$$

$$=\frac{\mathrm{d}y}{\mathrm{d}x}-\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$$

$$\therefore \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0 \quad \text{(shown)}$$

<u>Alternative Method 2 (via implicit differentiation)</u>

$$y=\ln(\mathrm{e}^x+1)$$

$$\mathrm{e}^y=\mathrm{e}^x+1$$

Differentiating implicitly w.r.t. x, $\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x$

Differentiating implicitly w.r.t. x,

$$\left(\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\mathrm{e}^x$$

$$\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{e}^y\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$$

Multiplying throughout by e^{-y} produces

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0 \quad \text{(shown)}$$

(ii)

$$y = \ln(e^x + 1) \qquad\qquad \text{When } x = 0,\ y = \ln(e^0 + 1) = \ln 2$$

$$\frac{dy}{dx} = \frac{e^x}{e^x + 1} \qquad\qquad \frac{dy}{dx} = \frac{e^0}{e^0 + 1} = \frac{1}{2}$$

$$\frac{d^2y}{dx^2} = \frac{e^x}{(e^x + 1)^2} \qquad\qquad \frac{d^2y}{dx^2} = \frac{e^0}{(e^0 + 1)^2} = \frac{1}{4}$$

Q $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2 = 0$,

Differentiating this implicitly w.r.t. x produces

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$$

When $x = 0$, $\dfrac{d^3y}{dx^3} - \dfrac{1}{4} + 2\left(\dfrac{1}{2}\right)\left(\dfrac{1}{4}\right) = 0$

$$\Rightarrow \frac{d^3y}{dx^3} = 0$$

Differentiating the above once more implicitly w.r.t. x,

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + 2\left[\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)\right] = 0$$

When $x = 0$,

$$\frac{d^4y}{dx^4} - 0 + 2\left[\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)(0)\right] = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} = -\frac{1}{8}$$

$\therefore$ The Maclaurin series for y

$$= y\big|_{x=0} + \left(\frac{dy}{dx}\bigg|_{x=0}\right) x + \left(\frac{d^2y}{dx^2}\bigg|_{x=0}\right)\frac{x^2}{2!} + \left(\frac{d^3y}{dx^3}\bigg|_{x=0}\right)\frac{x^3}{3!} + \left(\frac{d^4y}{dx^4}\bigg|_{x=0}\right)\frac{x^4}{4!} + \ldots$$

$$= \ln 2 + \left(\tfrac{1}{2}\right) x + \left(\tfrac{1}{4}\right)\tfrac{x^2}{2!} + (0)\tfrac{x^3}{3!} + \left(-\tfrac{1}{8}\right)\tfrac{x^4}{4!} + \ldots$$

$$= \ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2 - \tfrac{1}{192}x^4 + \ldots$$

(iii)

$$\frac{\ln(e^x + 1)}{4 - x^2} = \ln(e^x + 1)\,(4 - x^2)^{-1}$$

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$, where $-1 < x < 1$.

$$(4-x^2)^{-1} = 4^{-1}\left(1+(-\tfrac{x^2}{4})\right)^{-1}$$
$$= \tfrac{1}{4}\left(1+(-1)(-\tfrac{x^2}{4}) + \tfrac{(-1)(-1-1)}{2!}(-\tfrac{x^2}{4})^2 + \ldots\right)$$
$$= \tfrac{1}{4}\left(1+\tfrac{1}{4}x^2 + \ldots\right)$$
$$= \tfrac{1}{4} + \tfrac{1}{16}x^2 \ldots$$

$$\frac{\ln(e^x+1)}{4-x^2} = \ln(e^x+1)\ (4-x^2)^{-1}$$
$$= \left(\ln 2 + \tfrac{1}{2}\ x + \tfrac{1}{8}x^2 \ldots\right)\ \left(\tfrac{1}{4} + \tfrac{1}{16}x^2 \ldots\right)$$
$$= (\ln 2)\cdot\tfrac{1}{4} + \left(\tfrac{1}{2}\cdot\tfrac{1}{4}\right)x + \left(\tfrac{1}{8}\cdot\tfrac{1}{4} + \tfrac{1}{16}\ln 2\right)x^2 + \left(\tfrac{1}{2}\cdot\tfrac{1}{16}\right)x^3 \ldots$$
$$= \tfrac{1}{4}\ln 2 + \tfrac{1}{8}\ x + \left(\tfrac{1}{32} + \tfrac{1}{16}\ln 2\right)\ x^2 + \tfrac{1}{32}x^3 \ldots$$

6(a)

$$\int \frac{2-x}{4+x^2}\,dx$$
$$= \int \frac{2}{4+x^2}\,dx + \int \frac{-x}{4+x^2}\,dx$$
$$= 2\int \frac{1}{2^2+x^2}\,dx + \left(-\frac{1}{2}\right)\int \frac{2x}{4+x^2}\,dx$$
$$= 2\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{2}\ln\left|4+x^2\right| + C$$
$$= \tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln\left|4+x^2\right| + C\ ,$$

where C is an arbitrary constant.

(b) $$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 y}}\sec^2 y\ dy$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\sec^2 y}}\sec^2 y\ dy$$
$$= \int_0^{\frac{\pi}{4}} \sec y\ dy$$
$$= \left[\ln\left|\sec y + \tan y\right|\right]_0^{\frac{\pi}{4}}$$
$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln\left|\sec 0 + \tan 0\right|$$
$$= \ln\left(\sqrt{2}+1\right)$$

(c) $\displaystyle\int x^2 e^{x^3}\ dx = \frac{1}{3}\int 3x^2 e^{x^3}\ dx = \frac{1}{3}e^{x^3} + C$

$$\int x^5 e^{x^3}\ dx = \int x^3\left(x^2 e^{x^3}\right) dx$$

$$= x^3\left(\frac{e^{x^3}}{3}\right) - \int\left(x^2 e^{x^3}\right) dx$$

$$= \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3} + C$$

7(i) Largest value of $k = -2$

Let $y = \mathrm{f}(x)$

$y = x^2 + 4x - 5$

$= (x+2)^2 - 9$

$x + 2 = \pm\sqrt{y+9}$

$x = -2 \pm \sqrt{y+9}$

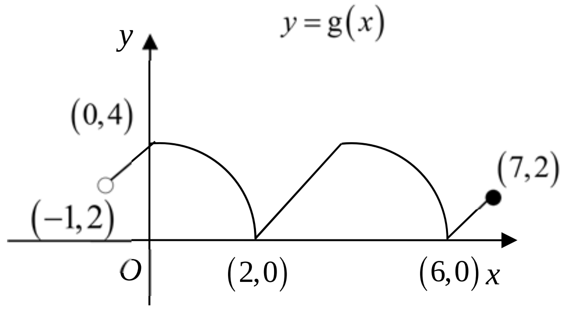
Since $x \le -2$, $x = -2 + \sqrt{y+9}$ is rejected.

$\therefore x = -2 - \sqrt{x+9}$

Since $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [-9, \infty)$,

$\mathrm{f}^{-1} : x \mapsto -2 - \sqrt{x+9}$, for $x \ge -9$

(ii)



(iii) $\mathrm{R}_{\mathrm{g}} = [0, 4]$

$\mathrm{D}_{\mathrm{f}^{-1}} = [-9, \infty)$ from part (i)

Since $\mathrm{R}_{\mathrm{g}} \subseteq \mathrm{D}_{\mathrm{f}^{-1}}$, $\mathrm{f}^{-1}\mathrm{g}$ exists.

$\mathrm{f}^{-1}\mathrm{g}(6) = \mathrm{f}^{-1}(0)$

$= -2 - \sqrt{9}$

$= -5$

8(i)	Given carrying capacity, $K = 10$ When $P = 9$, $\frac{dP}{dt} = \frac{9}{1750}$ Substituting into d.e. $\frac{dP}{dt} = cP(1-\frac{P}{K})$ $\frac{9}{1750} = c9\left(1-\frac{9}{10}\right)$ $\frac{1}{1750} = c\left(\frac{1}{10}\right)$ $c = \frac{1}{175}$ $\frac{dP}{dt} = \frac{1}{175}P\left(1-\frac{P}{10}\right)$ $\frac{dP}{dt} = \frac{1}{1750}P\left(10-P\right)$
(ii)	$\frac{dP}{dt} = \frac{1}{1750}P\left(10-P\right)$ $\int \frac{1}{P\left(10-P\right)}\frac{dP}{dt}\,dt = \int \frac{1}{1750}\,dt$ $\int \frac{1}{P\left(10-P\right)}\,dP = \frac{1}{1750}t + C$ <u>Method 1: By Partial Fractions</u> Let $\frac{1}{P\left(10-P\right)} = \frac{A}{P} + \frac{B}{10-P}$ So $A = \frac{1}{10}$ and $B = \frac{1}{10}$ $\frac{1}{10}\int \frac{1}{P} + \frac{1}{10-P}\,dP = \frac{1}{1750}t + C$ $\ln\left(P\right) - \ln\left(10-P\right) = \frac{1}{175}t + C$ since $0 < P < 10$ $\ln\left(\frac{P}{10-P}\right) = \frac{1}{175}t + C$ $\frac{P}{10-P} = e^{c}e^{\frac{1}{175}t}$ $\frac{P}{10-P} = Be^{\frac{1}{175}t}$ where $B = e^{c}$ $P = \frac{10Be^{\frac{1}{175}t}}{1 + Be^{\frac{1}{175}t}}$
	<u>Method 2: By Formula</u> $\int \frac{1}{10P - P^2}\,dP = \frac{1}{1750}t + C$ $\int \frac{1}{5^2 - (P-5)^2}\,dP = \frac{1}{1750}t + C$

$$\frac{1}{2(5)}\ln\left(\frac{5+(P-5)}{5-(P-5)}\right)=\frac{1}{1750}t+C \quad \text{since } 0 < P < 10$$

$$\ln\left(\frac{P}{10-P}\right)=\frac{1}{175}t+10C$$

$$\frac{P}{10-P}=e^{10c}e^{\frac{1}{175}t}$$

$$\frac{P}{10-P}=Be^{\frac{1}{175}t} \quad \text{where } B=e^{10c}$$

$$P=\frac{10Be^{\frac{1}{175}t}}{1+Be^{\frac{1}{175}t}}$$

Method 3: By Formula

$$\int\frac{1}{10P-P^2}\,dP=\frac{1}{1750}t+C$$

$$-\int\frac{1}{(P-5)^2-5^2}\,dP=\frac{1}{1750}t+C$$

$$-\frac{1}{2(5)}\ln\left|\frac{(P-5)-5}{(P-5)+5}\right|=\frac{1}{1750}t+C \quad \text{since } 0 < P < 10$$

$$\ln\left|\frac{P-10}{P}\right|=-\frac{1}{175}t-10C$$

$$\frac{P-10}{P}=\pm e^{-10c}e^{-\frac{1}{175}t}$$

$$\frac{P-10}{P}=Ae^{-\frac{1}{175}t} \quad \text{where } A=\pm e^{-10c}$$

$$P-10=PAe^{-\frac{1}{175}t}$$

$$P-PAe^{-\frac{1}{175}t}=10$$

$$P=\frac{10}{1+(-A)e^{-\frac{1}{175}t}}$$

$$P=\frac{10(-A)^{-1}e^{\frac{1}{175}t}}{(-A)^{-1}e^{\frac{1}{175}t}+1}$$

$$P=\frac{10Be^{\frac{1}{175}t}}{1+Be^{\frac{1}{175}t}} \quad \text{where } B=(-A)^{-1}$$

At start of 2010, let $t=0$, then $P=7$

$$7=\frac{10B}{1+B}$$

$$7+7B=10B$$

$$B=\frac{7}{3}$$

Therefore, $P=\dfrac{\frac{70}{3}e^{\frac{1}{175}t}}{1+\frac{7}{3}e^{\frac{1}{175}t}}=\dfrac{70e^{\frac{1}{175}t}}{3+7e^{\frac{1}{175}t}}$

At the start of 2020, t = 10.

$P=\dfrac{70e^{\frac{10}{175}}}{3+7e^{\frac{10}{175}}}$ = 7.1186

Hence population at start of 2010 is 7.12 billion.

Note:
Can also use initial t = 2010 with P = 7 and then find P when t = 2020 or initial t = 10 with P = 7 and then find P when t = 20. However, this will give a corresponding different values for B. Final value for P will be the same.

(iii) When P > 8.5,

Using GC on equation $P=\dfrac{70e^{\frac{1}{175}t}}{3+7e^{\frac{1}{175}t}}>8.5$

Or using expression from <u>Method 1</u>

Since at t = 0, P = 7, $\ln\left(\dfrac{P}{10-P}\right)=\dfrac{1}{175}t+\ln\dfrac{7}{3}$

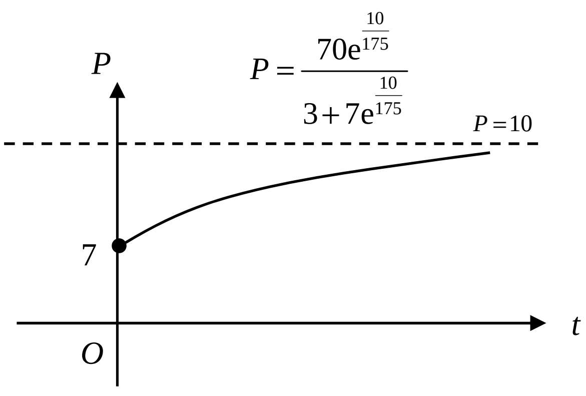
$$\ln\left(\frac{8.5}{10-8.5}\right)=\frac{1}{175}t+\ln\frac{7}{3}$$

$$t=175\left[\ln\left(\frac{8.5}{1.5}\right)-\ln\frac{7}{3}\right]$$

$$=155.2780591$$

Hence, the number of complete years needed is 156

(iv)



9(i) Discriminant method (recommended) :

(x, y) is on curve C.

$$y = \frac{x^2 - 4x + 1}{2x + 7} \quad \text{for some } x \in \mathbb{R}$$

$$y(2x+7) = x^2 - 4x + 1$$

$$x^2 - 4x - 2xy + 1 - 7y = 0$$

$$x^2 - (4 + 2y)x + (1 - 7y) = 0 \quad \text{for some } x \in \mathbb{R}$$

$\because$ Real roots exist for this quadratic eqn. in x,

$$b^2 - 4ac \geq 0$$

$$(4+2y)^2 - 4(1)(1-7y) \geq 0$$

$$16 + 16y + 4y^2 - 4 + 28y \geq 0$$

$$4y^2 + 44y + 12 \geq 0$$

$$y^2 + 11y + 3 \geq 0$$

$$\left(y + \frac{11}{2}\right)^2 + 3 - \left(\frac{11}{2}\right)^2 \geq 0$$

$$\left(y + \frac{11}{2}\right)^2 - \left(\frac{\sqrt{109}}{2}\right)^2 \geq 0$$

$$\left(y + \frac{11}{2} - \frac{\sqrt{109}}{2}\right)\left(y + \frac{11}{2} + \frac{\sqrt{109}}{2}\right) \geq 0$$

$$y \leq -\frac{\sqrt{109}}{2} - \frac{11}{2} \quad \text{or} \quad y \geq \frac{\sqrt{109}}{2} - \frac{11}{2}$$

Set of values of y that C can take

$$= \left\{ y \in \mathbb{R} : y \leq -\frac{\sqrt{109}}{2} - \frac{11}{2} \quad \text{or} \quad y \geq \frac{\sqrt{109}}{2} - \frac{11}{2} \right\}$$

Alternative method :

Finding stationary pts. on C via differentiation :

(**Not** recommended — steps more tedious)

$$y = \frac{x^2 - 4x + 1}{2x + 7}$$

$$\frac{dy}{dx} = \frac{(2x-4)(2x+7) - (x^2 - 4x + 1)(2)}{(2x+7)^2}$$

$$= \frac{4x^2 + 6x - 28 - (2x^2 - 8x + 2)}{(2x+7)^2}$$

$$= \frac{2x^2 + 14x - 30}{(2x+7)^2}$$

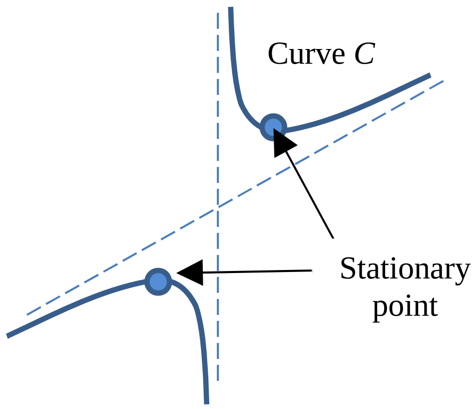
If y is stationary, then $\frac{dy}{dx} = 0$,

$$2x^2 + 14x - 30 = 0$$

$$x^2 + 7x - 15 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(1)(-15)}}{2(1)} = \frac{-7 \pm \sqrt{109}}{2}$$

Since curve C is the graph of a rational function of the form $y = \dfrac{ax^2 + bx + c}{dx + e} = Px + Q + \dfrac{R}{dx + e}$, with stationary points present, curve C would assume a general shape :



At $x = \dfrac{-7 \pm \sqrt{109}}{2}$,

$$y = \frac{x^2 - 4x + 1}{2x + 7}$$

$$= \frac{\left(\dfrac{-7 \pm \sqrt{109}}{2}\right)^2 - 4\left(\dfrac{-7 \pm \sqrt{109}}{2}\right) + 1}{\cancel{2}\left(\dfrac{-7 \pm \sqrt{109}}{\cancel{2}}\right) + 7}$$

$$= \frac{\frac{1}{4}(49 \mp 14\sqrt{109} + 109) - 2(-7 \pm \sqrt{109}) + 1}{\pm\sqrt{109}}$$

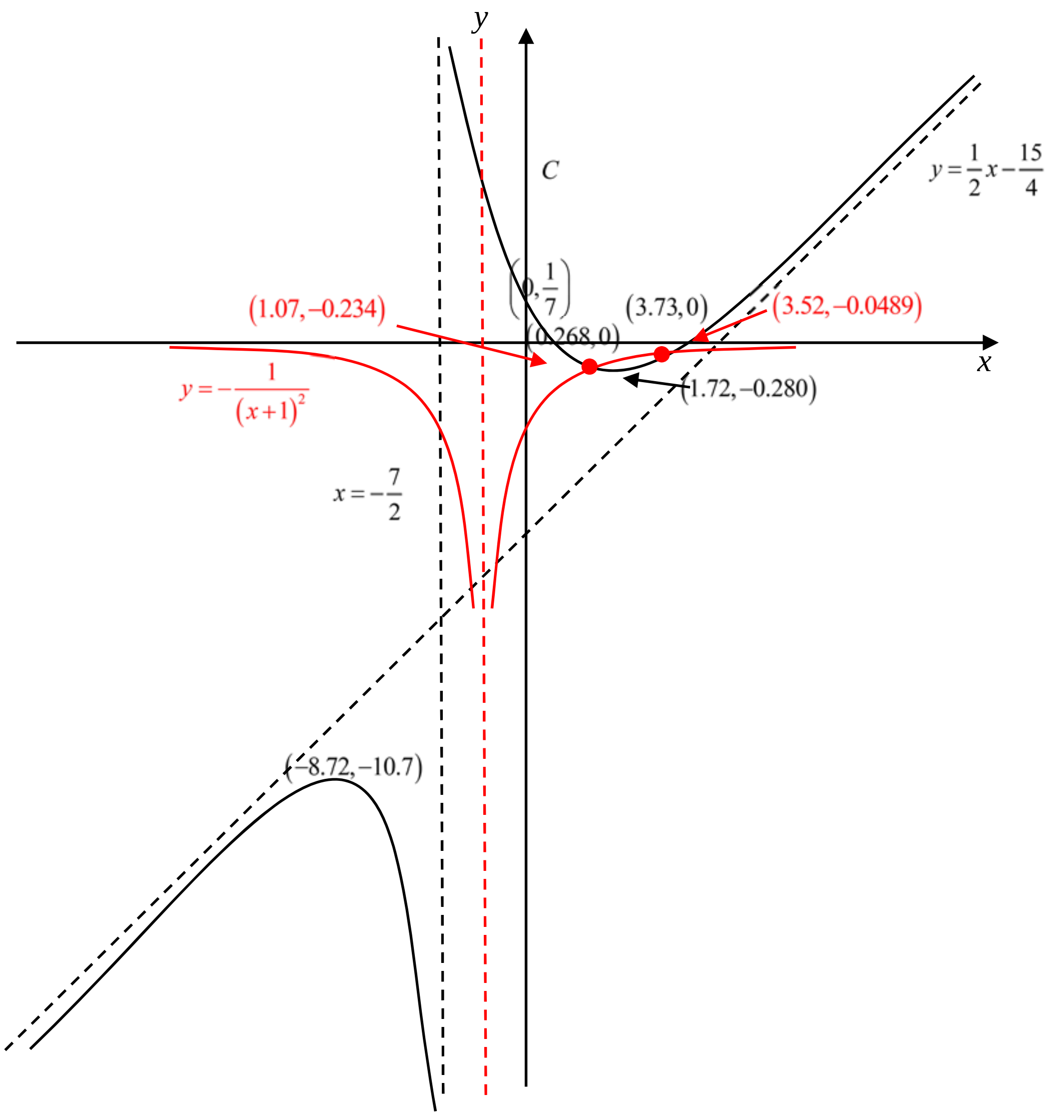
$$= \frac{\frac{1}{2}(79) + 15 \ \mp \frac{7}{2}\sqrt{109} \mp 2\sqrt{109}}{\pm\sqrt{109}}$$

$$= \frac{\frac{109}{2} \mp \frac{11}{2}\sqrt{109}}{\pm\sqrt{109}} \quad = \pm\frac{\sqrt{109}}{2} - \frac{11}{2}$$

Set of values of y that C can take

$$= \left\{ y \in \mathbb{R} : y \le -\frac{\sqrt{109}}{2} - \frac{11}{2} \text{ or } y \ge \frac{\sqrt{109}}{2} - \frac{11}{2} \right\}$$

(ii)



(iii) $\frac{x^2-4x+1}{2x+7}<-\frac{1}{(x+1)^2}$

Draw $y=-\frac{1}{(x+1)^2}$

Points of intersections

$(1.07,-0.234)$ and $(3.52,-0.0489)$

From the diagram, $x<-\frac{7}{2}$ or $1.07<x<3.52$.

10(i)

$\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP}$

$$= \begin{pmatrix} -9.5 \\ 6.5 \\ 10.5 \end{pmatrix} - \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} -20.5 \\ 28.5 \\ 20.5 \end{pmatrix}$$

$$l_B : \mathbf{r} = \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} -20.5 \\ 28.5 \\ 20.5 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$l_{A'B'} : \mathbf{r} = \begin{pmatrix} -40 \\ 35 \\ 20 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix}, \mu \in \mathbb{R}$$

B' is the point of intersection between lines l_B and $l_{A'B'}$

$$\overrightarrow{OB'} = \begin{pmatrix} 11 - 20.5\lambda \\ -22 + 28.5\lambda \\ -10 + 20.5\lambda \end{pmatrix} = \begin{pmatrix} -40 + 10\mu \\ 35 \\ 20 + 11\mu \end{pmatrix} \text{ for some } \lambda, \mu \in \mathbb{R}$$

$11 - 20.5\lambda = -40 + 10\mu \quad -(1)$

$-22 + 28.5\lambda = 35 \quad -(2)$

$-10 + 20.5\lambda = 20 + 11\mu \quad -(3)$

From eqn (2),

$\lambda = 2$

To check with eqn (1) and (2):

From (1), $11 - 20.5(2) = -40 + 10\mu \Rightarrow \mu = 1$

From (2), $-10 + 20.5(2) = 20 + 11\mu \Rightarrow \mu = 1$

$$\overrightarrow{OB'} = \begin{pmatrix} -30 \\ 35 \\ 31 \end{pmatrix}$$

Coordinates of *B'* is $(-30, 35, 31)$

(ii)

Since l_O passes through *O* and *P* and is perpendicular to the wall,

$$\mathbf{n} = \overrightarrow{OP} = \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$$

Equation of wall: $\mathbf{r} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} = \begin{pmatrix} -40 \\ 35 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} = -1410$

$$l_O : \mathbf{r} = s \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}, s \in \mathbb{R}$$

$$\overrightarrow{OO'} = \begin{pmatrix} 11s \\ -22s \\ -10s \end{pmatrix} \text{ for some } s \in \mathbb{R}$$

$$\begin{pmatrix} 11s \\ -22s \\ -10s \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} = -1410$$

$121s + 484s + 100s = -1410$

$s = -2$

$$\overrightarrow{OO'} = \begin{pmatrix} 11(-2) \\ -22(-2) \\ -10(-2) \end{pmatrix} = \begin{pmatrix} -22 \\ 44 \\ 20 \end{pmatrix}$$

Coordinates of O' is $(-22,\ 44,\ 20)$

(iii) Distance between point *P* and screen

$$\left|\overrightarrow{O'P}\right| = \left|\overrightarrow{OP} - \overrightarrow{OO'}\right| = \left|\begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} - \begin{pmatrix} -22 \\ 44 \\ 20 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 33 \\ -66 \\ -30 \end{pmatrix}\right| = \sqrt{33^2 + (-66)^2 + (-30)^2}$$

$$= \sqrt{6345}$$

(iv)

$$\mathbf{n} \text{ of plane } O'A'B' = \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$$

$$\mathbf{n} \text{ of plane } OAB = \overrightarrow{OA} \times \overrightarrow{OB}$$

$$= \begin{pmatrix} -23 \\ 16 \\ 10 \end{pmatrix} \times \begin{pmatrix} -9.5 \\ 6.5 \\ 10.5 \end{pmatrix}$$

$$= \begin{pmatrix} 103 \\ 146.5 \\ 2.5 \end{pmatrix}$$

Since $\begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix} \neq k \begin{pmatrix} 103 \\ 146.5 \\ 2.5 \end{pmatrix}$ for any *k*, planes *OAB* and $O'A'B'$ are not parallel.

Alternative method:
By using the fact that if a direction vector of plane *OAB* is not perpendicular to the normal of *O'A'B'*.

$$\overrightarrow{OA} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} -23 \\ 16 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ -22 \\ -10 \end{pmatrix}$$

$$= -705 \neq 0$$

Since $\overrightarrow{OA}$ (on the plane *OAB*) is not perpendicular to the normal of *O'A'B'*, the two planes are not parallel.

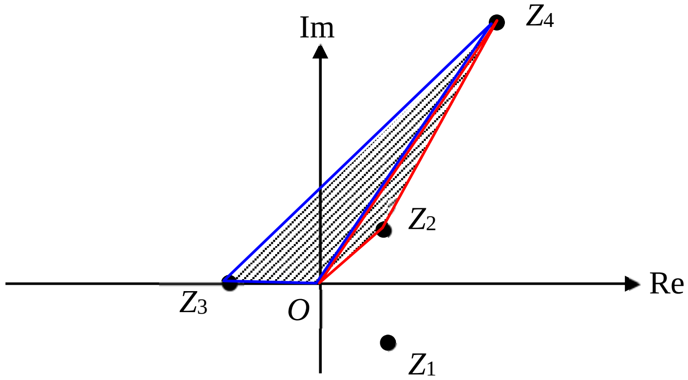
11(i)	$$\begin{array}{r} 8z^2-4\sqrt{2}z+2 \\ z+\frac{1}{2}\overline{\big)8z^3+\left(4-4\sqrt{2}\right)z^2+\left(2-2\sqrt{2}\right)z+1} \\ \underline{8z^3+\qquad 4z^2} \\ -4\sqrt{2}z^2+\left(2-2\sqrt{2}\right)z+1 \\ \underline{-4\sqrt{2}z^2\qquad -2\sqrt{2}z} \\ 2z\qquad +1 \\ \underline{2z\qquad +1} \\ 0 \end{array}$$ $$8z^3+\left(4-4\sqrt{2}\right)z^2+\left(2-2\sqrt{2}\right)z+1=0$$ $$\left(8z^2-4\sqrt{2}z+2\right)\left(z+\frac{1}{2}\right)=0$$ $$8z^2-4\sqrt{2}z+2=0$$ $$z=\frac{4\sqrt{2}\pm\sqrt{\left(4\sqrt{2}\right)^2-4(8)(2)}}{2(8)}$$ $$=\frac{4\sqrt{2}\pm\sqrt{32-64}}{16}$$ $$=\frac{4\sqrt{2}\pm\sqrt{-32}}{16}$$ $$=\frac{4\sqrt{2}\pm4\sqrt{2}\mathrm{i}}{16}$$ $$=\frac{\sqrt{2}\pm\sqrt{2}\mathrm{i}}{4}$$ $$z_1=\frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}},$$ $$z_2=\frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}},\ z_3=\frac{1}{2}\mathrm{e}^{\mathrm{i}\pi}$$
	$$w=\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{24}}$$ $$z_4=\frac{w^2}{z_1}$$ $$=\frac{\left(\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{24}}\right)^2}{\frac{1}{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}$$ $$=4\mathrm{e}^{\mathrm{i}\frac{\pi}{12}+\mathrm{i}\frac{\pi}{4}}$$ $$=4\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$$ Modulus = 4, argument = $\frac{\pi}{3}$

$$4e^{i\frac{\pi}{3}} = 4\cos\frac{\pi}{3} + 4\sin\frac{\pi}{3}i$$
$$= 2 + 2\sqrt{3}i$$

(iii)

$$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right)$$

(ii), (iv)



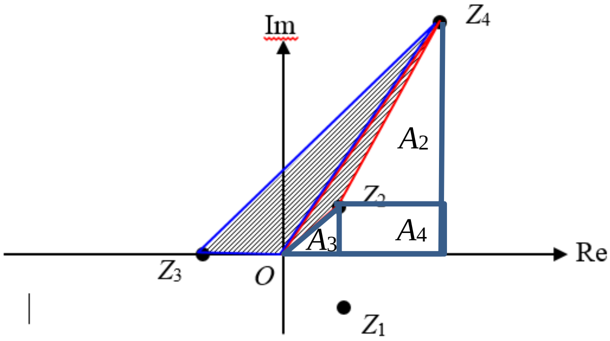
"Hence"

Area of quadrilateral $OZ_3Z_4Z_2$

= Area of triangle OZ_2Z_4 + Area of triangle OZ_4Z_3

$$= \frac{1}{2}\left(\frac{1}{2}\right)(4)\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{1}{2}\right)(4)\sin\left(\pi - \frac{\pi}{3}\right)$$
$$= \sin\left(\frac{\pi}{12}\right) + \sin\left(\frac{2\pi}{3}\right)$$
$$= \frac{\sqrt{2}}{4}\left(\sqrt{3} - 1\right) + \frac{\sqrt{3}}{2}$$

"Otherwise"



Area of quadrilateral $OZ_3Z_4Z_2$

= Area of big triangle – A_2 – A_3 – A_4

$$=\frac{1}{2}\left(2+\frac{1}{2}\right)2\sqrt{3}-\frac{1}{2}\left(\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4}\right)-\frac{1}{2}\left(2-\frac{\sqrt{2}}{4}\right)\left(2\sqrt{3}-\frac{\sqrt{2}}{4}\right)-\left(2-\frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4}\right)$$

$$=\frac{5}{2}\sqrt{3}-\frac{1}{16}-2\sqrt{3}+\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}-\frac{1}{16}-\frac{\sqrt{2}}{2}+\frac{2}{16}$$

$$=\frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)+\frac{\sqrt{3}}{2}$$

CATHOLIC JUNIOR COLLEGE
H2 MATHEMATICS
2018 JC2 PRELIMINARY EXAMINATION PAPER II SOLUTION

Section A: Pure Mathematics [40 marks]

1(i)

$$\begin{aligned} u_n &= S_n - S_{n-1} \\ &= n(2n+7)-(n-1)(2n+5) \\ &= 2n^2+7n-\left(2n^2+3n-5\right) \\ &= 4n+5 \end{aligned}$$

$$u_1 = 4(1)+5=9$$
$$S_1 = 1\left[2(1)+7\right]=9$$
$$u_1 = S_1$$

$$\begin{aligned} & u_n - u_{n-1} \\ &= (4n+5)-\left(4(n-1)+5\right) \\ &= 4n+5-(4n+1) \\ &= 4 \end{aligned}$$

Since $u_n - u_{n-1} = 4$ (constant), the sequence is AP.

(ii)

$$\begin{aligned} & \sum_{n=1}^{N} \frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}} \\ &= \sum_{n=1}^{N} \frac{\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)}{\left(\sqrt{u_{n+1}}+\sqrt{u_n}\right)\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)} \\ &= \sum_{n=1}^{N} \frac{\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right)}{u_{n+1}-u_n} \\ &= \frac{1}{4}\sum_{n=1}^{N}\left(\sqrt{u_{n+1}}-\sqrt{u_n}\right) \\ &= \frac{1}{4}\Big\{\left(\cancel{\sqrt{u_2}}-\sqrt{u_1}\right)+ \\ &\qquad \left(\cancel{\sqrt{u_3}}-\cancel{\sqrt{u_2}}\right)+ \\ &\qquad\qquad \text{M} \\ &\qquad \left(\cancel{\sqrt{u_N}}-\cancel{\sqrt{u_{N-1}}}\right)+ \\ &\qquad \left(\sqrt{u_{N+1}}-\cancel{\sqrt{u_N}}\right)\Big\} \\ &= \frac{1}{4}\left(\sqrt{u_{N+1}}-\sqrt{u_1}\right) \\ &= \frac{1}{4}\left(\sqrt{4N+9}-3\right) \end{aligned}$$

(iii)

$$\frac{1}{\sqrt{53}+\sqrt{49}}+\frac{1}{\sqrt{57}+\sqrt{53}}+\cdots+\frac{1}{\sqrt{361}+\sqrt{357}}$$
$$=\frac{1}{\sqrt{4(11)+9}+\sqrt{4(11)+5}}+\frac{1}{\sqrt{4(12)+9}+\sqrt{4(12)+5}}+\cdots+\frac{1}{\sqrt{4(88)+9}+\sqrt{4(88)+5}}$$
$$=\sum_{n=1}^{88}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}-\sum_{n=1}^{10}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$$
$$=\frac{1}{4}\left(\sqrt{4(88)+9}-3\right)-\frac{1}{4}\left(\sqrt{4(10)+9}-3\right)$$
$$=4-1$$
$$=3$$

2(i)

Equation of line segment passing through O and B: $\mathbf{r}=\lambda\mathbf{b},\lambda\in\mathbb{R}$

F lies on the line: $\overrightarrow{OF}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}$

$$\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\lambda\mathbf{b}-\mathbf{a}$$
$$\overrightarrow{AF}\cdot\mathbf{b}=0$$
$$(\lambda\mathbf{b}-\mathbf{a})\cdot\mathbf{b}=0$$
$$\lambda\mathbf{b}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}=0$$
$$\lambda|\mathbf{b}|^2-|\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{3}=0$$
$$\lambda(2)^2-2\left(\frac{1}{2}\right)=0$$
$$\lambda=\frac{1}{4}$$
$$\overrightarrow{OF}=\frac{1}{4}\mathbf{b}$$

Alternative:
$$\left|\overrightarrow{OF}\right|=\cos\frac{\pi}{3}=0.5$$
$$\therefore\overrightarrow{OF}=\frac{0.5}{2}\mathbf{b}=\frac{1}{4}\mathbf{b}$$

(ii)

F is the midpoint of A and C
$$\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)$$
$$\overrightarrow{OC}=2\overrightarrow{OF}-\overrightarrow{OA}$$
$$=\frac{1}{2}\mathbf{b}-\mathbf{a}$$
Hence,
Since the diagonals AC and OB are perpendicular, $OA=OC=1$ and $AB=CB$, $OACB$ is a kite.

$$\begin{aligned}
\text{Area of } OACB &= 2 \times \text{Area of } OAB \\
&= \left|\overrightarrow{OA} \times \overrightarrow{OB}\right| \\
&= |\mathbf{a} \times \mathbf{b}| \\
&= |\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{3} \\
&= 2\left(\frac{\sqrt{3}}{2}\right) \\
&= \sqrt{3}
\end{aligned}$$

3(i)	Let the amount received in \$ be u_n, where n is the number of years from 2018. $u_1 = 10,000 \times 5.00 = 50,000$ $u_2 = 10,000 \times 7.50 = 75,000$ $u_3 = 10,000 \times 10.00 = 100,000$ ... forms an AP with a = 50,000, d = 25,000 For $S_n \geq 1,000,000$, $\frac{n}{2}\left[2(50,000)+(n-1)(25,000)\right] \geq 1,000,000$ $n\left[(500)+(n-1)(125)\right] \geq 10000$ $n\left[125n+375\right] \geq 10000$ $n\left[n+3\right] \geq 80$ From GC, $n \geq 7.5692$ or $n \leq -10.569$ $n \geq 8$ or $n \leq -11$ rejected as n is positive In the year of 2026 or 2027 or 2026 Dec or 2027 Jan.
(ii)	Let the amount received in \$ be v_n, where n is the number of years from 2018. $v_1 = 50,000$ $v_2 = 50,000 \times 0.94 = 47,000$ $v_3 = (50,000 \times 0.94) \times 0.94 = 44,180$ … forms a GP with a = 50, 000, r = 0.94 The total allocated fund received in the years 2024 to 2030, $v_6 + v_7 + ... + v_{12}$ $= S_{12} - S_5$ $= \frac{50000\left(1-(0.94)^{12}\right)}{1-0.94} - \frac{50000\left(1-(0.94)^{5}\right)}{1-0.94}$ $= 1000000\left((0.94)^5 - (0.94)^{12}\right)$ $= 214986.42$ $= 214986$ The total allocated fund received in the years 2024 to 2030 is \$214, 986. Accept answer to 3SF i.e. \$215, 000.
(iii)	In the long run, as $n \to \infty$, $S_\infty = \frac{50000}{1-0.94}$ $= 833,333.33$ The airport would not be able to collect the fund for expansion project as the amount received will never reach \$1, 000, 000.

4(i) Volume of required solid of revolution generated by rotating about the y-axis, $V = \int_0^h \pi x^2 \, \mathrm{d}y$.

$$\left(\tfrac{x}{r}\right)^2 + \left(\tfrac{y}{h}\right)^2 = 1$$

$$\Rightarrow \; x^2 = r^2\left(1 - \tfrac{y^2}{h^2}\right)$$

$$V = \int_0^h \pi r^2 \left(1 - \tfrac{y^2}{h^2}\right) \, \mathrm{d}y$$

$$= \pi r^2 \left[y - \tfrac{y^3}{3h^2} \right]_0^h$$

$$= \pi r^2 \left[\left(h - \tfrac{h^3}{3h^2}\right) - 0 \right]$$

$$= \tfrac{2}{3}\pi r^2 h \qquad \text{(shown)}$$

(ii) Given $x = r\cos\theta$, $y = h\sin\theta$, where $0 \le \theta \le \pi$.

$$\tfrac{\mathrm{d}x}{\mathrm{d}\theta} = -r\sin\theta \, , \quad \tfrac{\mathrm{d}y}{\mathrm{d}\theta} = h\cos\theta \, ,$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \bigg/ \frac{\mathrm{d}y}{\mathrm{d}\theta}$$

$$= \frac{h\cos\theta}{-r\sin\theta} = -\tfrac{h}{r}\cot\theta$$

At point $P(r\cos\alpha, \; h\sin\alpha)$, $\theta = \alpha$

gradient of tangent $= -\tfrac{h}{r}\cot\alpha$.

Equation of the tangent to the curve at P :

$$y - h\sin\alpha = -\tfrac{h}{r}\cot\alpha \; (x - r\cos\alpha) \, ,$$

$$y = -\left(\tfrac{h}{r}\cot\alpha\right)x + h\cot\alpha\cos\alpha + h\sin\alpha ,$$

$$y = -\left(\tfrac{h}{r}\cot\alpha\right)x + h\left(\tfrac{(\cos\alpha)^2}{\sin\alpha} + \sin\alpha\right),$$

$$y = -\left(\tfrac{h}{r}\cot\alpha\right)x + h\left(\tfrac{(\cos\alpha)^2 + (\sin\alpha)^2}{\sin\alpha}\right),$$

$$y = -\left(\tfrac{h}{r}\cot\alpha\right)x + h \ \operatorname{cosec}\alpha \, . \quad \text{(shown)}$$

Alternatively,

Equation of the tangent to the curve at P :

$$y = -\tfrac{h}{r}\cot\alpha \, x + C \, , \qquad \text{for some constant } C$$

Q $P(r\cos\alpha, \; h\sin\alpha)$ is on this tangent line,

$$h\sin\alpha = -\tfrac{h}{r}\cot\alpha \; (r\cos\alpha) + C$$

$$C = h\cot\alpha\cos\alpha + h\sin\alpha$$

$$= h\left(\tfrac{\cos^2\alpha}{\sin\alpha} + \sin\alpha\right) \; = h\left(\tfrac{\cos^2\alpha + \sin^2\alpha}{\sin\alpha}\right)$$

$$= h\left(\tfrac{1}{\sin\alpha}\right) \; = h \ \operatorname{cosec}\alpha$$

$$\therefore \; y = -\left(\tfrac{h}{r}\cot\alpha\right)x + h \ \operatorname{cosec}\alpha \, . \qquad \text{(shown)}$$

(iii) When the hemi-ellipsoid is inscribed within the cone, there is a point on the ellipse whose tangent line matches that of the cone's slanted edge.

i.e. General equation of tangent line *

$y = -\left(\frac{h}{r}\cot\alpha\right)x + h\ \mathrm{cosec}\alpha$ matches $y = -\frac{5}{3}x + 5$,

for some α.

Comparing coefficients produces

$\frac{h}{r}\cot\alpha = \frac{5}{3}$ and $h\ \mathrm{cosec}\alpha = 5$

$\Rightarrow$ $\cot\alpha = \frac{5r}{3h}$ and $\mathrm{cosec}\alpha = \frac{5}{h}$

[Obtain relationship between variables r and h, by eliminating parameter α.]

Since $1 + (\cot\alpha)^2 = (\mathrm{cosec}\alpha)^2$,

$$1 + \left(\tfrac{5r}{3h}\right)^2 = \left(\tfrac{5}{h}\right)^2$$

Multiplying by $\left(\frac{h}{5}\right)^2$ on both sides produces

$$\left(\frac{h}{5}\right)^2 + \left(\frac{r}{3}\right)^2 = 1 \quad \text{(shown)}$$

<u>Alternative method</u>

Substituting (3,0) and (0,5) into the general eqn. of tangent line produces :

$0 = -\left(\frac{h}{r}\cot\alpha\right)(3) + h\ \mathrm{cosec}\alpha \quad \Rightarrow \quad 3\frac{h}{r}\frac{\cos\alpha}{\sin\alpha} = \frac{h}{\sin\alpha}$

$\Rightarrow \quad \cos\alpha = \frac{r}{3}$, and

$5 = \cancel{-\left(\frac{h}{r}\cot\alpha\right)(0)} + h\ \mathrm{cosec}\alpha \quad \Rightarrow \quad \sin\alpha = \frac{h}{5}$.

Since $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$, $\left(\frac{h}{5}\right)^2 + \left(\frac{r}{3}\right)^2 = 1$ (shown).

(iv) Hemi-ellipsoid volume $V = \frac{2}{3}\pi r^2 h$.

$\left(\frac{h}{5}\right)^2 + \left(\frac{r}{3}\right)^2 = 1 \quad \Rightarrow \quad r^2 = 3^2\left(1 - \frac{h^2}{5^2}\right) = 9\left(1 - \frac{h^2}{25}\right)$

$\therefore V = \frac{2}{3}\pi \times 9\left(1 - \frac{h^2}{25}\right)h$

$= 6\pi\left(h - \frac{h^3}{25}\right)$.

When volume V is maximum, it is also stationary w.r.t. h.

$\therefore \frac{\mathrm{d}V}{\mathrm{d}h} = 6\pi\left(1 - \frac{3h^2}{25}\right) = 0$.

$\Rightarrow h^2 = \frac{25}{3}$, $h = \frac{5}{\sqrt{3}}$ $(\because h > 0)$.

$\therefore \frac{\mathrm{d}^2V}{\mathrm{d}h^2} = 6\pi\left(-\frac{6h}{25}\right)$

$= -\frac{36}{25}\pi h$

At $h = \frac{5}{\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -\frac{36}{25}\pi\left(\frac{5}{\sqrt{3}}\right) < 0$.

$\therefore$ By the 2nd derivative test, volume V

attains a maximum at $h = \frac{5}{\sqrt{3}}$.

Alternatively, use the 1st derivative test :

h	$\left(\frac{5}{\sqrt{3}}\right)^-$	$\frac{5}{\sqrt{3}}$	$\left(\frac{5}{\sqrt{3}}\right)^+$
$\frac{dV}{dh}$	> 0	0	< 0
Slope			

$\therefore$ By the 1st derivative test, volume V attains a maximum at $h = \frac{5}{\sqrt{3}}$.

Section B: Probability and Statistics [60 marks]

5(i) The probability, $\frac{1}{p}$, of a shopper finding a mystery item is a constant for all shoppers.

Or

The event where a shopper finds a mystery item is independent of all other shoppers finding a mystery item.

(ii) Let X be the random variable denoting the number of shoppers who found a mystery item out of 10 shoppers.

Then $X \sim \mathrm{B}(10, \frac{1}{p})$

Given $\mathrm{P}(X = 3) = \frac{15}{4}(\sqrt{2} - 1)^7$

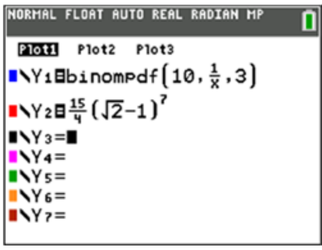
$${}^{10}C_3\left(\frac{1}{p}\right)^3\left(1 - \frac{1}{p}\right)^7 = \frac{15}{4}(\sqrt{2} - 1)^7$$

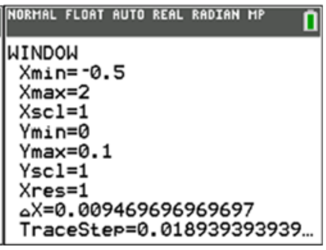
$$\frac{120}{p^{10}}(p - 1)^7 = \frac{15}{4}(\sqrt{2} - 1)^7$$

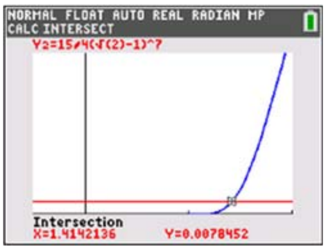
Hence, $p = \sqrt{2}$

Alternative Method – using graph

$\mathrm{P}(X = 3) = \frac{15}{4}(\sqrt{2} - 1)^7$







From GC, p = 1.4142136 or 22.303621

Since $0 < p < 2$, hence p = 1.41 (to 3 s.f.)

(iii) Let Y be the random variable denoting the number of games where three shoppers found a mystery item out of 10 shoppers out of n games.

$Y \sim \text{B}(n, \frac{15}{4}(\sqrt{2}-1)^7)$

$\text{P}(Y > 2) = 1 - \text{P}(Y \le 2) > 0.01$

Using GC,

n	$1-\text{P}(Y \le 2)$
56	0.0098
57	0.0103
58	0.0108

Hence the least value of n is 57.

OR

$\text{P}(Y > 2) = 1 - \text{P}(Y \le 2) > 0.01$

$\text{P}(Y \le 2) < 0.99$

Using GC,

n	$\text{P}(Y \le 2)$
56	0.9902
57	0.9897
58	0.9892

Hence the least value of n is 57.

6(i) X = number of tails – number of heads in one throw.

Using probability to obtain the p.d. table

$$P(X=-5)=P(0\text{ tail}-5\text{ heads})=\left(\frac{1}{2}\right)^5=\frac{1}{32}$$

$$P(X=-3)=P(1\text{ tail}-4\text{ heads})=\left(\frac{1}{2}\right)^1\times\left(\frac{1}{2}\right)^4\times\frac{5!}{4!}=\frac{5}{32}$$

$$P(X=-1)=P(2\text{ tails}-3\text{ heads})=\left(\frac{1}{2}\right)^2\times\left(\frac{1}{2}\right)^3\times\frac{5!}{2!3!}=\frac{10}{32}$$

$$P(X=1)=P(3\text{ tails}-2\text{ heads})=\left(\frac{1}{2}\right)^3\times\left(\frac{1}{2}\right)^2\times\frac{5!}{3!2!}=\frac{10}{32}$$

$$P(X=3)=P(4\text{ tails}-1\text{ head})=\left(\frac{1}{2}\right)^4\times\left(\frac{1}{2}\right)^1\times\frac{5!}{4!}=\frac{5}{32}$$

$$P(X=5)=P(5\text{ tails}-0\text{ head})=\left(\frac{1}{2}\right)^5=\frac{1}{32}$$

	0T -5H	1T -4H	2T -3H	3T -2H	4T -1H	5T -0H
$X=x$	-5	-3	-1	1	3	5
$P(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Alternative Method

Taking $X \sim B(5,\frac{1}{2})$

$$P(X=1)=\frac{5}{16}$$

	0T -5H	1T -4H	2T -3H	3T -2H	4T -1H	5T -0H
$X=x$	-5	-3	-1	1	3	5
$P(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

(ii) By symmetry, $E(X)=0$

Using p.d. table

$$\text{Var}(X)=E(X^2)-[E(X)]^2$$

$$=\frac{(-5)^2+(-3)^2(5)+(-1)^2(10)+1^2(10)+3^2(5)+5^2}{32}-[0]^2$$

$$=\frac{160}{32}=5$$

(iii) Using the table in (i)

$\|X\| = \|x\|$	1	3	5
Winnings \$	0	2	k
$P(X = x)$	$\frac{20}{32}$	$\frac{10}{32}$	$\frac{2}{32}$

Expected winnings

$$= 0 \times \frac{20}{32} + 2 \times \frac{10}{32} + k \times \frac{2}{32}$$

$$= \frac{20 + 2k}{32} = \frac{10 + k}{16}$$

Expected profit $\frac{10 + k}{16} - 1 = 10$

$$k = 11 \times 16 - 10 = 166$$

Alternative Method

Winnings \$	-1	1	k - 1
$P(X = x)$	$\frac{20}{32}$	$\frac{10}{32}$	$\frac{2}{32}$

Expected winnings

$$= (-1) \times \frac{20}{32} + 1 \times \frac{10}{32} + (k - 1) \times \frac{2}{32}$$

$$= \frac{1}{16}k - \frac{3}{8}$$

So, $\frac{1}{16}k - \frac{3}{8} = 10$

$$k - 6 = 160$$

$$k = 166$$

7(a) Given $P(B) = \frac{1}{3}$. $P(A \cap B) = \frac{1}{5}$ and $P(A' \cap B') = \frac{1}{6}$

$P(A \mid B')$

$$= \frac{P(A \cap B')}{P(B')}$$

$$= \frac{P(A \cup B) - P(B)}{1 - \frac{1}{3}}$$

$$= \frac{1 - P(A' \cap B') - \frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{1 - \frac{1}{6} - \frac{1}{3}}{\frac{2}{3}} = \frac{3}{4}$$

(b) $1, 2, 2, 3, 4, 5, 5$

Note that there are three distinct digits
and two repeated "2" and two repeated "5"

(bi) Method 1: Using complement

No. of digits
= all possible – two "5"s together

$$= \frac{7!}{2!2!} - \frac{6!}{2!}$$

$$= 900$$

Method 2: Slotting in method

X X X X X
↑ ↑ ↑ ↑ ↑ ↑

No. of ways to arrange three distinct digits and two "2"s

$$= \frac{5!}{2!} = 60$$

No. of ways to choose 2 slots for the two "5"s $= {}^6C_2 = 15$

So, total number $= 60 \times 15 = 900$

(bii) $\boxed{\text{"5" \_ \_ \_ "5"}}$ _ _

Case 1: If the two "2"s are between the two "5"s
= select one other digit to put in between the two "5"
$\times$ no. of ways to arrange the two "2"s and one other digit between the two "5"s
$\times$ no. of ways to arrange one block and 2 distinct digits

$$= {}^3C_1 \times \frac{3!}{2!} \times 3! = 54$$

Case 2: If one "2"s is between the two "5"s
= select two digits to put in between the two "5"
$\times$ no. of ways to arrange one "2"s and two other digits between the two "5"s
$\times$ no. of ways to arrange one block and 2 distinct digits

$$= {}^3C_2 \times 3! \times 3! = 108$$

Case 3: If no "2"s is between the two "5"s
= select three digits to put in between the two "5"s
$\times$ no. of ways to arrange 3 distinct digits between two "5"s
$\times$ no. of ways to arrange one block and two "2"s

$$= {}^3C_3 \times 3! \times \frac{3!}{2!} = 18$$

So, total = 54 + 108 + 18 = 180

Alternative solution

No. of ways to select and arrange three digits to put between two "5" = ${}^5P_3$

No. of ways to arrange one block of 5 digits and the remaining 2 individual digits = 3!

To remove duplication from two "2"s = $\frac{1}{2!}$

Total no. of ways $= \frac{{}^5P_3 \times 3!}{2!} = 180$

(biii) 1 _ _ _ _ _ _ _ _ _
↑
{3, 5}

Seven-digit number must start with "1" and end with digits 3 or 5
Case 1: Seven-digit number ends with "3"
= arrange the remaining digits with repeated "5" and "2"

$= \dfrac{5!}{2!2!} = 30$

Case 2: Seven-digit number ends with "5"

= arrange the remaining digits with repeated "2"

$= \dfrac{5!}{2!} = 60$

Total no. of ways $= 30 + 60 = 90$

Alternative solution

Number must start with "1" = 1

No. of ways to select the last odd digit out of 3 choices i.e. {3, 5, 5} = ${}^{3}C_{1}$

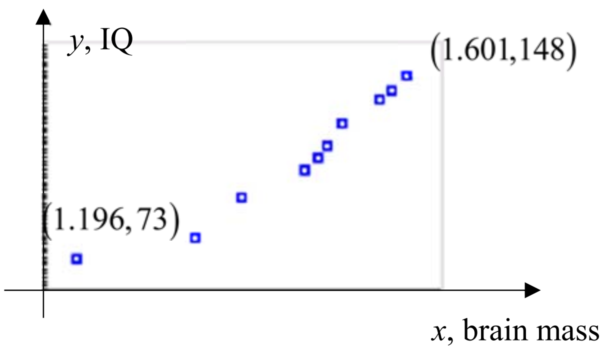
No. of ways to arrange the remaining 5 digits = 5!

To remove double counting from two "2" and two "5"

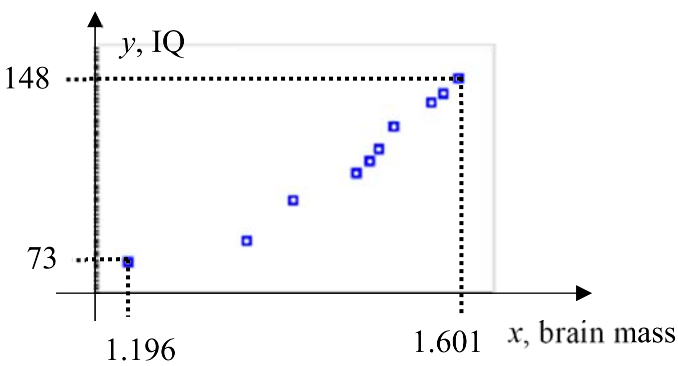
$= \dfrac{1}{2!2!}$

Total no. of ways $= \dfrac{1 \times 5! \times {}^{3}C_{1}}{2!2!} = 90$

8(i)



OR



(ii)(a) Between x and y: $r = 0.9645001408 \approx 0.965$ (3 s.f)

(b) Between x^2 and y: $r = 0.9745220219 \approx 0.975$ (3 s.f.)

(iii) From **(i)**, since as x increases, y increases at a increasing rate, the points on the scatter diagram take the shape of the graph of $y = c + dx^2$.

And

From **(ii)**, the product moment correlation coefficient between x and y^2 is closer to 1, as compared to that between x and y,

	$\therefore$ the model $y = c + dx^2$ is the better model. $y = -36.5807 + 69.911x^2$ $y = -36.6 + 69.9x^2$
(iv)	$y = -36.5807 + 69.911(1.5)^2$ $= 120.71905$ ≈ 121 Since the value of 1.5 is within the range of values of x and the value of r is close to 1, this estimate is reliable.
(v)	This statement may not be valid as strong positive linear correlation between brain mass and IQ as shown by the data does not imply causation. There are other factors affecting intelligence quotient other than brain mass as well.

9(i)	Let M and F be the random variable the height of a random female and male respectively. Then $M \sim \mathrm{N}(175, 10^2)$ and $F \sim \mathrm{N}(\mu, \sigma^2)$
	Given $\mathrm{P}(F < 143) = 0.01114$ and $\mathrm{P}(F < 183) = 0.98886$ Since $\mathrm{P}(F > 183) = 1 - 0.98886 = 0.01114$ Hence by symmetry, $\mu = \frac{183 + 143}{2} = 163$ Using $\mathrm{P}(F < 143) = 0.01114$ $\frac{143 - 163}{\sigma} = -2.285560213$ $\sigma = 8.750589849 \approx 8.75$ (shown)
(ii)	$F_1 + F_2 - 2M \sim \mathrm{N}(2 \times 163 - 2 \times 175, 2 \times 8.75^2 + 2^2 \times 10^2)$ $F_1 + F_2 - 2M \sim \mathrm{N}(-24, 553.125)$ $\mathrm{P}(\lvert F_1 + F_2 - 2M \rvert > 30)$ $= 1 - \mathrm{P}(-30 < F_1 + F_2 - 2M < 30)$ $= 0.4101527381$ ≈ 0.410
(iii)	Probability $= \mathrm{P}(155 < F < 185) \times [1 - \mathrm{P}(155 < F < 185)] \times 2 \times \mathrm{P}(160 < M < 192.5)$ $= 0.2707250364$ ≈ 0.271
(iv)	Let X be the random variable denoting height of a random female candidate wearing standard shoes. So $X = F + 5$ Since F is normally distributed, then $\bar{X} \sim \mathrm{N}(163 + 5, \frac{8.75^2}{15})$ $\mathrm{P}(\bar{X} > 170) = 0.1880099665 \approx 0.188$
	<u>Alternative solution 1</u> $X = F + 5 \Rightarrow \bar{X} = \bar{F} + 5$ $\bar{F} \sim \mathrm{N}(163, \frac{8.75^2}{15})$ $\mathrm{P}(\bar{X} > 170) = \mathrm{P}(\bar{F} > 165) = 0.1880099665 \approx 0.188$ **<u>Alternative solution 2</u>**

$F_1 + F_2 + ... + F_{15} \sim N(15 \times 163, 15 \times 8.750589849^2)$

$F_1 + F_2 + ... + F_{15} \sim N(2445, 1148.592341)$

$P\left(\frac{F_1 + F_2 + ... + F_{15} + 15 \times 5}{15} > 170\right)$

$= P(F_1 + F_2 + ... + F_{15} > 2475)$

$= 0.1880260552 \approx 0.188$

Alternative solution 3

$F_1 + F_2 + ... + F_{15} + 15 \times 5$

$\sim N(15 \times 163 + 75, 15 \times 8.750589849^2)$

$F_1 + F_2 + ... + F_{15} + 75 \sim N(2520, 1148.592341)$

$P(F_1 + F_2 + ... + F_{15} + 15 \times 5 > 170 \times 15)$

$= P(F_1 + F_2 + ... + F_{15} + 75 > 2550)$

$= 0.1880260552 \approx 0.188$

10(i)

Unbiased estimate of population mean, $\hat{\mu} = \frac{\sum x}{n} = \frac{132.00}{30} = 4.4$ (exact)

Unbiased estimate of population variance,

$$s^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right]$$

$$s^2 = \frac{1}{29}\left[583.96 - \frac{(132.00)^2}{30}\right]$$

$= 0.1089655172$

≈ 0.109

(ii)

Let X be the random variable denoting the time taken for a server to clear a table, and μ be the population mean time taken.

$H_0 : \mu = 4.5$ mins, the mean time taken is 4.5 mins

$H_1 : \mu \neq 4.5$ mins, the mean time taken is not 4.5 mins

Under H_0, since $n = 30$ is large enough, by Central Limit Theorem

$\bar{X} \sim N(4.5, \frac{0.1089655172}{30})$ approximately

Test statistic: $Z = \frac{\bar{X} - 4.5}{\sqrt{\frac{0.1089655172}{30}}} \sim N(0,1)$

At 10% level, reject H_0 if p-value < 0.1

p-value = 0.0970621091 < 0.10

Since p-value < 0.01, we reject H_0.

	OR At 10% level, reject H_0 if $z_{\text{test}} \le -1.64485$ or $z_{\text{test}} \ge 1.64485$ $z_{\text{test}} = \dfrac{4.4-4.5}{\sqrt{\dfrac{0.1089655172}{30}}} = -1.65926627$ Since $z_{\text{test}} = -1.659266271 < -1.64485$, we reject H_0. Conclude at 10% level that there is sufficient evidence that the mean time taken has changed from 4.5 minutes.
(iii)	Since p-value = 0.0970621091, To not reject H_0, $\text{p-value} > \dfrac{\alpha}{100}$ So, $0.0970621091 > \dfrac{\alpha}{100}$ Hence $0 < \alpha < 9.71$
(iv)	In this case, $H_0: \mu = 4.5$ mins, $H_1: \mu < 4.5$ left-tail test. **<u>p-value method</u>** Using (ii), $\text{p-value} = \dfrac{0.0970621091}{2} = 0.0485310546 < 0.05$ OR At 10% level, reject H_0 if $z_{\text{test}} \le -1.28155$ Using (ii), $z_{\text{test}} = -1.6592 \le -1.28155$ So reject H_0, conclude at 5% level that there is sufficient evidence that the mean taken to clear a table is less than 4.5 minutes.
(v)	$s^2 = \dfrac{n}{n-1}(\text{sample variance}) = \dfrac{30}{29}(1.01) \approx 1.044827586$ $H_0: \mu = 4.5$ $H_1: \mu > 4.5$ Under H_0, since $n = 40$ is large enough, by Central Limit Theorem $\bar{X} \sim N(4.5, \dfrac{1.01}{29})$ approximately Test-statistic, $Z = \dfrac{\bar{X}-4.5}{\sqrt{\dfrac{1.01}{29}}}$ To reject H_0 at 1% level for right-tailed test $\Rightarrow z_{test} > 2.326347877$

$$\Rightarrow \frac{\overline{x}-4.5}{\sqrt{\frac{1.01}{29}}} > 2.326347877$$

$$\Rightarrow \overline{x} > 4.934146542$$

$$\Rightarrow \overline{x} > 4.934$$

Hence, $\{\overline{x} \in R : \overline{x} > 4.934 \text{ minutes}\}$