Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2018 Year 6 Preliminary Examination
Higher 2

MATHEMATICS 9758/01

Paper 1

13 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphic calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

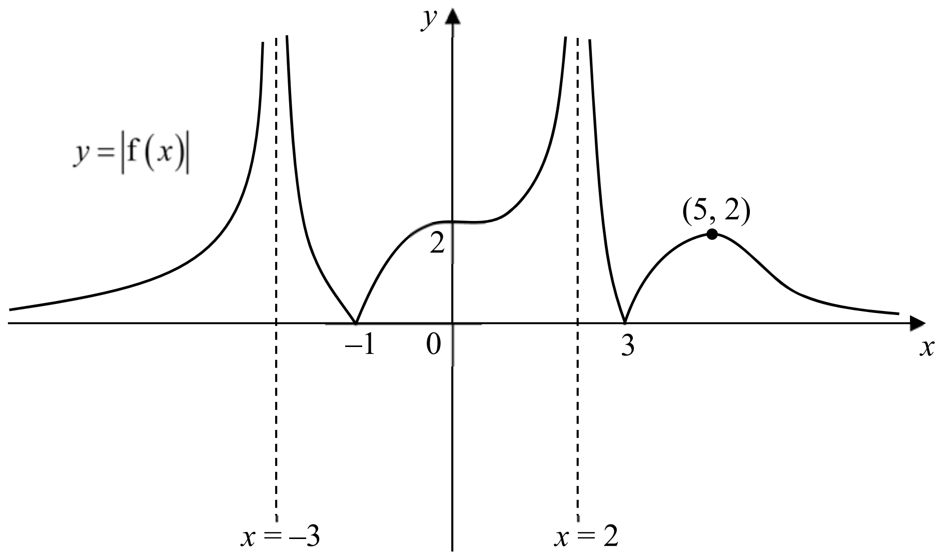
At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

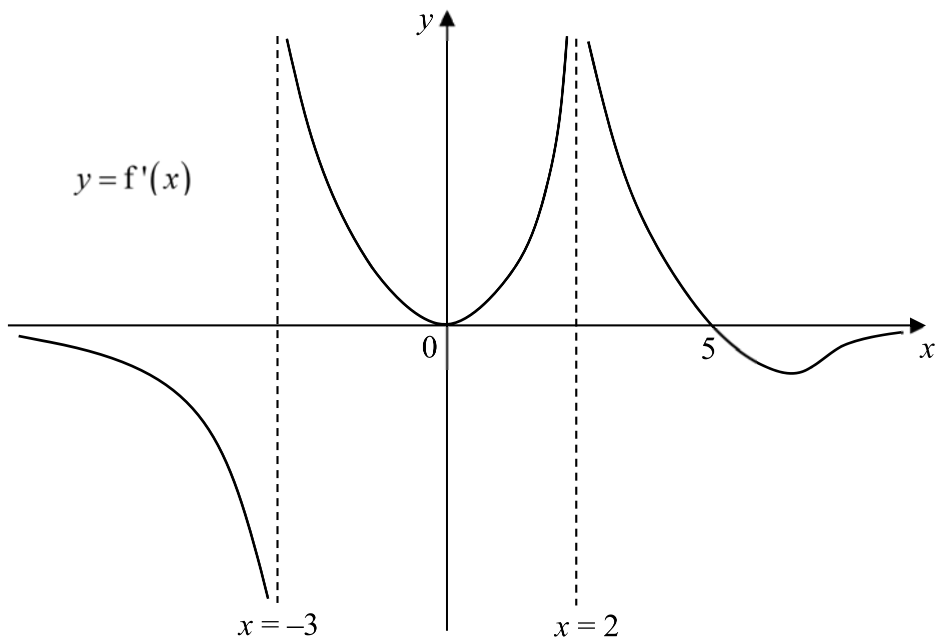
This document consists of **8** printed pages.

1. Relative to the origin O, the points A, B, C and D have position vectors $\overrightarrow{OA}=\begin{pmatrix}p\\q\\r\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}1\\2\\1\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\overrightarrow{OD}=\begin{pmatrix}0\\-3\\0\end{pmatrix}$ respectively.

The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular to each other. The vector $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ is a normal to the plane containing O, A and C. The point A lies on the line passing through the point D parallel to the vector $\overrightarrow{OB}$. Find the values of p, q and r. [6]

2. The diagrams show the graphs of $y=|\mathrm{f}(x)|$ and $y=\mathrm{f}'(x)$.





The graph of $y=|\mathrm{f}(x)|$ has stationary points at $(0,2)$ and $(5,2)$, and the equations of asymptotes are $x=-3$ and $x=2$. The equations of asymptotes of $y=\mathrm{f}'(x)$ are $x=-3$ and $x=2$.

On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(x)$; [3]

(ii) $y=\dfrac{1}{\mathrm{f}'(x)}$. [3]

3. In this question, all physical quantities are of appropriate units which you need not state.

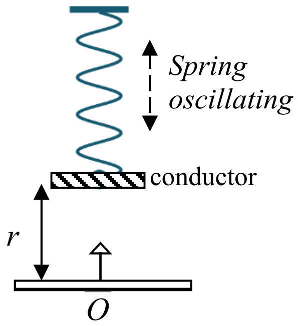
In a science experiment, charged particles from a fixed position O are bombarded towards a charged conductor. The electric potential energy (E) of each particle is given by

$$E = \frac{0.9qQ}{r}$$

where q is the charge of the particle, Q is the charge of conductor and r is the distance between them.

(i) Assuming q and Q are constants, express $\frac{\mathrm{d}E}{\mathrm{d}t}$ in terms of q, Q, r and $\frac{\mathrm{d}r}{\mathrm{d}t}$. [1]

It is further given that the conductor is attached to a spring that moves in an oscillating manner such that the distance between the conductor and O is given by $r = 0.8 + 0.6\cos\left(\frac{1}{2}t\right)$ as shown in the diagram below.



(ii) State the range of values of r. [1]

At a certain instant, a particle with a constant charge of 1.5 is of distance 1.1 units from the conductor, which has a constant charge of 4.0.

(iii) Find the exact time of this instant, and the rate at which the electrical potential energy of the particle is increasing at this instant. [5]

4. Relative to the origin O, the points A, B and C are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$ respectively, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors which are mutually non-parallel.

The plane Π and the line l have the following equations

$$\Pi : \mathbf{r}=\lambda\mathbf{a}+\mu\mathbf{b} \quad \text{where } \lambda, \mu \in \text{¡} \quad \text{and}$$

$$l : \mathbf{r}=\mathbf{a}+\beta\mathbf{c} \quad \text{where } \beta \in \text{¡}$$

respectively.

(i) Find the point(s) of intersection between Π and l given that the points O, A, B and C are

(a) coplanar,

(b) not coplanar. [3]

(ii) State the geometrical meaning of $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. [1]

In the rest of the question, O, A, B and C are not coplanar.

(iii) The vector $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. State the geometrical meaning of $|\mathbf{c}\cdot\mathbf{p}|$. [1]

(iv) It is given that $\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}0\\0\\2\end{pmatrix}$. Find the volume of the pyramid $OABC$. [3]

5. **(i)** By using the substitution $x=2\sin^2\theta$, find the exact value of $\int_0^1 \sqrt{2x-x^2}\,\mathrm{d}x$. [5]

(ii) Hence, find the exact value of $\int_0^1 \frac{x(1-x)}{\sqrt{2x-x^2}}\,\mathrm{d}x$. [3]

6. A curve C has parametric equations

$$x = 1 - \sin 2t, \quad y = \cos t, \text{ for } 0 \le t < 2\pi.$$

(i) Find $\dfrac{dy}{dx}$ in terms of t. [2]

(ii) Find the stationary points of C. [2]

(iii) There are four points where the normals to C are parallel to the x-axis. Deduce that the four points form a rectangle. State the length and breadth of the rectangle. [5]

7. Given that $y = \mathrm{f}(x)$ where $\sqrt{1-x^2}\dfrac{dy}{dx} = \dfrac{y}{2}$ and $\mathrm{f}(0) = 1$.

(i) Find the Maclaurin series for y up to and including the term in x^3. [5]

(ii) The Maclaurin series for $e^{y'}$, up to and including the term in x, is $p + qx$, where y' denotes the first derivative of y. Find the exact values of p and q. [4]

8. **(a)** Given that $z^* = \dfrac{(1+a\mathrm{i})^3}{(1-\mathrm{i})^2}$, where a is a positive real number.

(i) Find a such that $|z| = 4$. [2]

(ii) Without the use of a calculator, express z in the form $re^{i\theta}$, where r and θ are exact values, $r > 0$ and $-\pi < \theta \le \pi$. [3]

(b) Given that $\mathrm{f}(z) = z^4 + 2z^3 + (\pi^2 + e^2)z^2 + pz + q$, where $p \ge 0$ and $q \ge 0$, and that $\mathrm{i}\pi$ satisfies the equation $\mathrm{f}(z) = 0$.

(i) Find the exact values of p and q. [2]

(ii) Hence find the remaining roots of $\mathrm{f}(z) = 0$ in the form $\alpha + \beta\mathrm{i}$, where α and β are exact real numbers. [4]

9. A geometric series G has common ratio r and an arithmetic series A has first term a and common difference d, where a and d are non-zero. The first three terms of G are equal to the third, fourth and sixth terms of A respectively. The sum of the first four terms of A is -6.

(i) Show that $r^2 - 3r + 2 = 0$ and find the value of r. [2]

(ii) Deduce the values of a and d. [4]

(iii) Determine the least value of m such that the difference between the m^{th} terms of G and A is more than 10000. [2]

Let a_n denote the n^{th} term of A. The sequence x_n is defined as $x_n = (2k)^{a_n}$ where k is a constant.

(iv) Find the range of values of k such that the series $\sum_{n=1}^{\infty} x_n$ converges. [3]

10. A group of scientists have recently discovered the growth of a rare species of plant in a wildlife reserve area in central Africa that is well known for its severe climate changes at times. The scientists also discovered that the species of plant can be used in the cure for several human terminal illnesses.

Let N denote the number of the species of plant at time t years after its discovery.

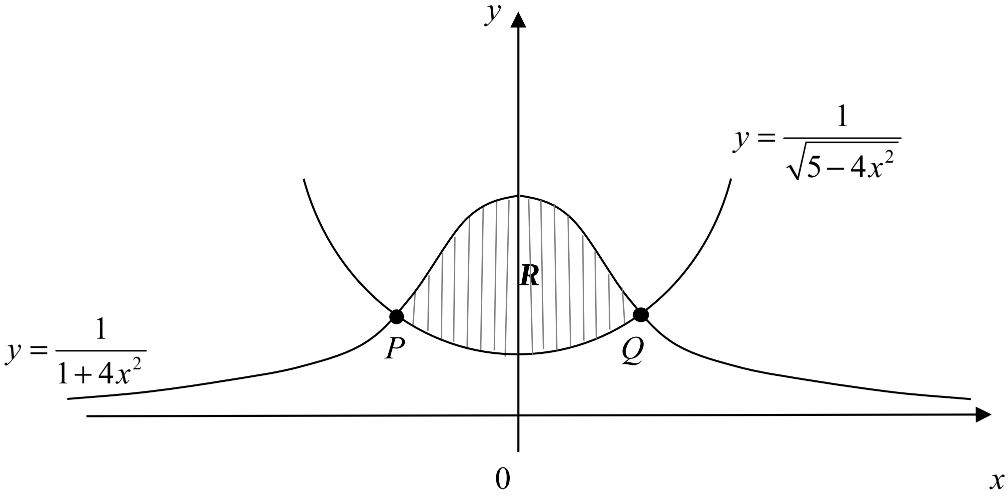
(i) Based on studies on other similar species of plant in other areas, the scientists believe that the rate of increase of N is proportional to the product of N and $(1000 - N)$. Given that they expect the number of plants to increase at a rate of 125 per year when there are 500 plants in the area, deduce a differential equation relating N and t. [2]

(ii) Given further that there are initially 200 of this rare species of plant in the area, find an expression for N in terms of t and sketch of the graph of N against t. [7]

(iii) The scientists target to have the number of the species of plants in the area to exceed 1200 by the end of 5 years. Determine if this target can be achieved. [1]

(iv) Based on the context given, provide two reasons why the proposed model of growth for this rare species of plant may not be suitable. [2]

11. Alice is tasked to design a new logo for a toy company. The logo consists of the finite region R bounded by the curves $y=\dfrac{1}{1+4x^2}$, $y=\dfrac{1}{\sqrt{5-4x^2}}$ and between the points P and Q as shown below.



(i) Find the x-coordinates of P and Q and show that the exact value of the area of R is given by

$$\frac{\pi}{4}-\sin^{-1}\left(\frac{1}{\sqrt{5}}\right).$$ [6]

Alice then decides to explore different regions bounded by various transformations of the above two curves.

(ii) Without evaluating the actual integrals, deduce the exact areas of the finite regions bounded by the following curves. Justify your answers.

(a) $y=\dfrac{1}{4x^2-8x+5}$ and $y=\dfrac{1}{\sqrt{1+8x-4x^2}}$ [2]

(b) $y=\dfrac{2}{1+4x^2}$ and $y=\dfrac{1}{\sqrt{\frac{5}{4}-x^2}}$ [2]

In designing a related 3-dimensional logo for the company, Alice considers rotating the above finite region R about the y-axis.

(iii) Find the volume of the 3-dimensional logo formed by rotating R 180° about the y-axis. [3]

END OF PAPER

Name	( )	Class	

RIVER VALLEY HIGH SCHOOL
2018 Year 6 Preliminary Examination
Higher 2

MATHEMATICS 9758/02

Paper 2

17 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphic calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

1. A curve C has equation $y=\dfrac{x^2-3x+3}{x-2}$ where $1\le x\le 3$ and $x\ne 2$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and comment on the gradient of C. [3]

(ii) Sketch the graph of C. [2]

(iii) By adding a suitable graph to the sketch of C, solve the inequality

$$\frac{x^2-4x+5}{x-2}\ge 3\left(x-2\right)^2,\ \text{ for } 1\le x\le 3,\ x\ne 2.$$ [3]

(iv) The function g is defined by $\mathrm{g}(x)=\dfrac{x^2-3x+3}{x-2}$ for $1\le x\le 3$ and $x\ne 2$ such that $\mathrm{g}(x)=\mathrm{g}(x+3)$. Sketch the graph of $y=\mathrm{g}(x)$ for $-1<x<5$. [2]

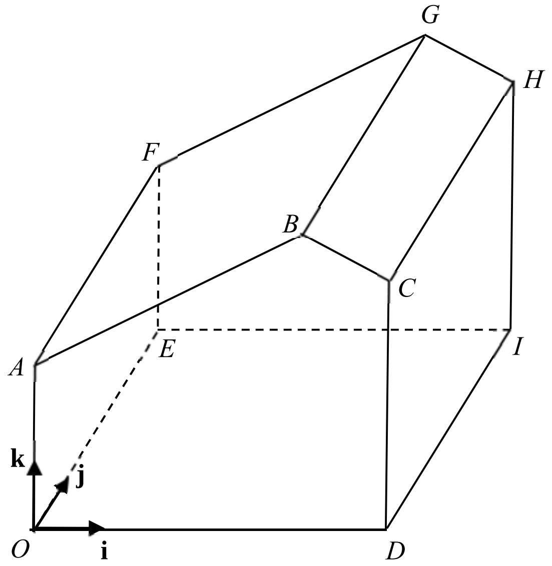
2. **(i)** Express $\dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}$ as partial fractions. [2]

(ii) Hence find an expression for $\displaystyle\sum_{r=1}^{n}\frac{2r+3}{(r+1)(r+2)(r+3)}$ in terms of n. [4]

(iii) Using the result in part **(ii)**, show that the following series is convergent and find its sum to infinity.

$$\frac{21}{3\times4\times5}+\frac{27}{4\times5\times6}+\frac{33}{5\times6\times7}+\dots\dots\dots$$ [4]

3. The diagram below shows the structure of an indoor farm to be built. Edges *OA*, *DC*, *IH* and *EF* are vertical, and edges *OA*, *AB*, *BC* and *CD* are equal to *EF*, *FG*, *GH* and *HI* respectively. Diagram is not drawn to scale.



Taking *O* as the origin and the vectors **i**, **j** and **k** as unit vectors along the *OD*, *OE* and *OA* respectively, the coordinates of *A* and *C* are (0, 0, 20) and (50, 0, 25) respectively. All lengths are measured in metres.

(i) Given that $-\mathbf{i} + 4\mathbf{k}$ is a vector perpendicular to the flat roof *ABGF*, and $\mathbf{i} + 2\mathbf{k}$ is perpendicular to the flat roof *BCHG*, find the obtuse angle between the two roofs. [2]

(ii) A humidity sensor is to be installed along the line *BG*, 50 metres away from *B*. Find the coordinates of the location of the sensor. [4]

(iii) A metal cable is anchored on the ground outside the farm in front of *OABCD*, at a point 25 metres away from *O* along *OD* and 10 metres in front of *OD*. The cable is attached to a point on *AB* to secure the roof *ABGF*. Find the length of the shortest cable needed. [3]

(iv) State an assumption for the above calculations to be valid. [1]

4. The function f is defined by $\mathrm{f} : x \mapsto \dfrac{1}{(x-2)^2}$ for $x < m$ where $m < 2$.

(i) Find the range of values of *m* such that both f^{-1} and f^2 exist. [4]

(ii) Suppose the range of values of *m* found in part **(i)** holds, find the functions f^{-1} and f^2, and state their domains and range. [6]

Section B: Statistics [60 marks]

5. Two fair six-sided dice are thrown. The random variable X is the smaller of the two scores if they are different, and their common value if they are the same.

(i) Show that $\mathrm{P}(X=2)=\dfrac{1}{4}$ and find the probability distribution of X. [2]

(ii) Hence find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

A game is played with Ivan and Jon taking turns to throw the two dice each. Ivan throws the dice first and the player who first obtain the value of X equals to 2 wins the game. If Ivan wins the game, Jon pays him \$7. Otherwise, Ivan pays Jon \$10.

(iii) Find the player who has the higher expected gain. Justify your answer. [3]

6. **(a)** During a class reunion, 4 men and 6 women decide to stand in a row to take a class photograph. Find the number of ways that they can do so if

(i) there is no restriction, [1]

(ii) at least 2 men stand together, [2]

(iii) exactly 5 women stand together. [3]

(b) The 10 persons then sit at a round table for dinner. Find the probability that 2 particular women sit opposite each other. [2]

7. At a supermart, the option of using cashless payment for purchases is available to customers. On average, p% of the customers use cashless payment.

The manager of the supermart selects a sample of 30 customers for a survey.

(i) The probability of not more than 1 of customers surveyed use cashless payment for their purchases is 0.245. Write down an equation in terms of p. Find p, giving your answer correct to 3 decimal places. [3]

For the remaining parts of this question, use $p=30$.

(ii) Find the probability that the last customer of the 30 selected for survey is the 10^{th} customer who uses a cashless payment for his purchase. [2]

Another manager of the supermart also selects a sample of 30 customers.

(iii) Given that there are a total of less than 16 customers using cashless payment for their purchases from both samples, find the probability that one of the sample has more than 12 customers using cashless payment for their purchases. [3]

(iv) State one assumption you have made in the calculation in part **(iii)**. [1]

8. Verde wants to investigate the time taken for different volumes of water to cool to room temperature. He prepared a few samples of different volume and heated the samples to their boiling point, and then recorded the time taken for the water to cool to room temperature. The results are given in the table.

Volume (x/cm^3)	100	200	300	400	500	600	700
Time (t/min)	14	23	47	83	a	172	293

(i) It is known that the regression line of t on x is $t = -65.1429 + 0.4318x$. Show that a = 121. [2]

(ii) Draw a scatter diagram for the data and find the product moment correlation coefficient between x and t. [2]

(iii) Comment whether the regression line is appropriate based on

(a) the scatter diagram; [1]

(b) the context. [1]

Verde considers using one of the following two models:

$$\text{A: } t = a + bx^2\text{, B: } t = ae^{bx}\text{,}$$

where $a, b \in \mathbb{R}$, for the relationship between x and t.

(iv) Explain which is the better model and find the equation of a suitable regression line for that model. [2]

(v) Estimate the time taken for 450 cm^3 of water to cool from boiling point to room temperature. Comment on the reliability of the estimate. [3]

9. A newspaper report claims that the mean starting monthly salary of a fresh university graduate is \$3500. However a human resource manager believes that this claim is incorrect. A random sample of 60 fresh university graduates is surveyed and their starting monthly salaries, \$$x$, are summarized by

$$\sum x = 207000, \quad \sum (x-3400)^2 = 5450000.$$

(i) Calculate the unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 5% significance level, whether the human resource manager's belief is correct. [4]

(iii) State, in the context of the question, the meaning of the p-value found in part **(ii)**. [1]

A second sample of 50 fresh university graduates is surveyed and the sample mean and standard deviation of their starting salaries are found to be \$$\overline{y}$ and \$342 respectively.

(iv) Find the range of values of $\overline{y}$ such that this second sample would result in accepting the manager's belief at the 5% significance level. [3]

(v) Suppose the claim by the newspaper is correct and the standard deviation of the starting salaries of fresh university graduates is \$543, find the probability that a random sample of 50 fresh university graduates have its mean salary below \$3350. [2]

10. Gary likes to take part in duathlons where there is a total running distance of 15 km and a total cycling distance of 36 km. His timings, in minutes, for running and cycling are modelled as having independent normal distributions with means and standard deviations as shown in the table.

	Mean	Standard Deviation
Timing for running (min)	100	15
Timing for cycling (min)	μ	σ

(i) The probability that he completes his cycling in less than an hour is equal to the probability that he completes his cycling after 2 hours. Write down the value of μ. [1]

(ii) The probability that he completes his cycling under 80 minutes is 0.158655. Show that $\sigma = 10$. [2]

(iii) Find the probability that thrice the time needed for him to complete cycling is more than an hour from twice the time needed for him to complete the running. [2]

(iv) Find the probability that his timing for cycling is less than 80 minutes and his timing for running is less than 90 minutes. [2]

(v) Show that the probability of Gary finishing the duathlon under 170 minutes is 0.134. Give a reason why this probability is greater than the probability calculated in part **(iv)**. [2]

Gary's target timing for completing a duathlon is under 2 hours 50 minutes. Over the past two years, he has already completed 9 duathlons.

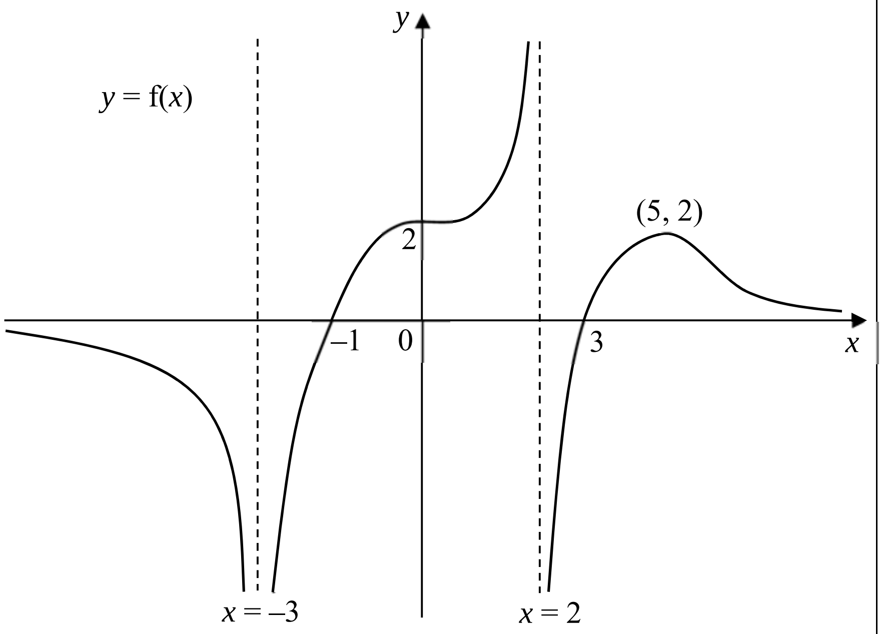
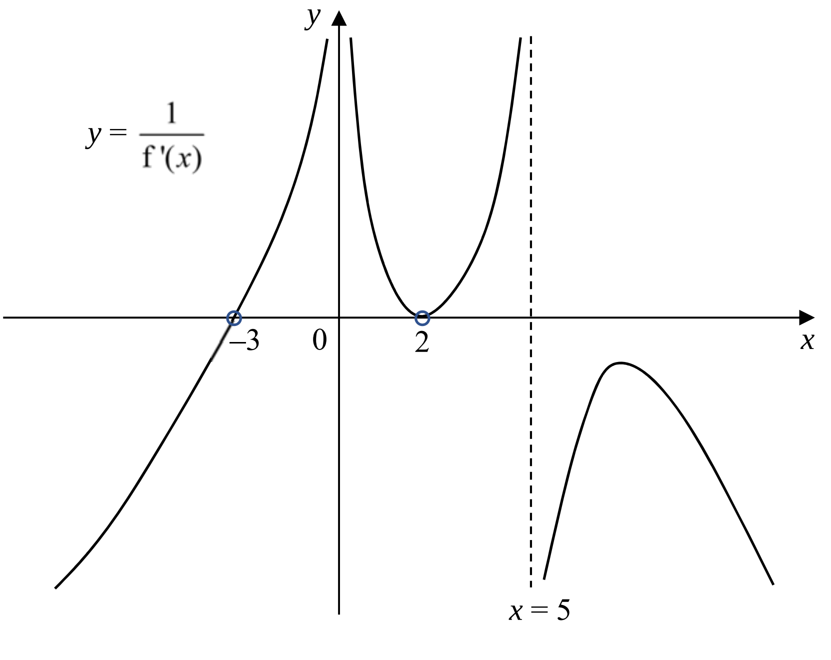
(vi) Find the probability that he achieves his target timing in at least 3 but fewer than 6 of the duathlons over the last two years. State an assumption needed for your calculation to be valid. [4]

END OF PAPER

BLANK PAGE

Solutions to 2018 Y6 H2 Maths Prelim Exam P1

1.	Solution [6 marks]	
	Given that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular, $\begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0$ $\Rightarrow p + 2q + r = 0$ ---- (1) Let $\mathbf{n} = \overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -r \\ r \\ p-q \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ ---- (2) From (2): $\begin{cases} -r = -k \\ r = k \\ p - q = 2k \end{cases} \Rightarrow p - q - 2r = 0$ ---- (3) Let l be the line passing through the point D parallel to the vector $\overrightarrow{OB}$. $l : \mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ where $\beta \in \Re$ Since A lies on l, $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for some $\beta \in \Re$ ---- (4) From (4): $\begin{cases} p = \beta \\ q = -3 + 2\beta \Rightarrow 2p - q = 3 \text{ or } 2r - q = 3 \text{ ---- (5)} \\ r = \beta \end{cases}$ Solving (1), (3) and (5), $p = 1$, $q = -1$, $r = 1$.	

2.	Solution [6 marks]	
	(i)  	
	(ii)  	

3.	Solution [7 marks]	
	(i) $E=\dfrac{0.9qQ}{r}$ diff wrt t, $\dfrac{\mathrm{d}E}{\mathrm{d}t}=\dfrac{-0.9qQ}{r^2}\dfrac{\mathrm{d}r}{\mathrm{d}t}$	
	(ii) $0.2\le r\le 1.4$ (since $-1\le\cos\left(\dfrac{1}{2}t\right)\le 1$)	
	(iii) $r=0.8+0.6\cos\left(\dfrac{1}{2}t\right)$ Given: Q = 4.0 , q = 1.5, r = 1.1. Subst r = 1.1 into eqn: $1.1=0.8+0.6\cos\left(\dfrac{1}{2}t\right)$ $\cos\left(\dfrac{1}{2}t\right)=\dfrac{1}{2}$ $\dfrac{1}{2}t=\dfrac{\pi}{3}+2k\pi$ or $\dfrac{1}{2}t=\dfrac{5\pi}{3}+2k\pi$, for $k=0,1,2,3...$ $\therefore t=\dfrac{2\pi}{3}+4k\pi$ or $\dfrac{10\pi}{3}+4k\pi$ $\dfrac{\mathrm{d}r}{\mathrm{d}t}=-0.3\sin\left(\dfrac{1}{2}t\right)$ At the instant where the particle is 1.1 units from the conductor, $\sin\left(\dfrac{1}{2}t\right)=\dfrac{\sqrt{3}}{2}$ or $-\dfrac{\sqrt{3}}{2}$ $\therefore\dfrac{\mathrm{d}E}{\mathrm{d}t}=\dfrac{-0.9qQ}{r^2}\dfrac{\mathrm{d}r}{\mathrm{d}t}$ $=\left(\dfrac{-0.9(1.5)(4.0)}{1.1^2}\right)\left(-0.3\dfrac{\sqrt{3}}{2}\right)$ or $\left(\dfrac{-0.9(1.5)(4.0)}{1.1^2}\right)\left(0.3\dfrac{\sqrt{3}}{2}\right)$ $=\pm 1.15947$ $=\pm 1.16$ (to 3sf)	

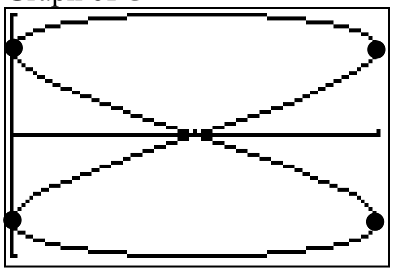
4.	Solution [8 marks]	
	(i)(a) Note that origin is on plane Π. When O, A, B and C are coplanar, the line l lies on the plane Π, therefore the line l and plane Π intersect along the line l. (Points of intersection are points on l.) (i)(b) When O, A, B and C are not coplanar the line l and plane Π intersect at the point A.	
	(ii) $\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ gives the area of a triangle formed with two sides parallel and equal in magnitude to $\mathbf{a}$ and $\mathbf{b}$.	
	(iii) $\lvert\mathbf{c}\cdot\mathbf{p}\rvert$ gives the length of projection of $\mathbf{c}$ on $\mathbf{p}$ (<u>or</u> on $\mathbf{a}\times\mathbf{b}$, <u>or</u> on normal of Π).	
	(iv) Area of triangle OAB $=\frac{1}{2}\left\lvert\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert$ $=\frac{\sqrt{3}}{2}$ units^2 Using (iii) result, height of pyramid = height of point C relative to plane OAB $=\lvert\mathbf{c}\cdot\mathbf{p}\rvert$ where $\mathbf{p}$ is a unit vector in the direction of $\mathbf{a}\times\mathbf{b}$. $=\left\lvert\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rvert=\frac{2}{\sqrt{3}}$ Volume of the pyramid $OABC$ $=\frac{1}{3}\times\frac{\sqrt{3}}{2}\times\frac{2}{\sqrt{3}}=\frac{1}{3}$ unit^3	

5.	Solution [8 marks]	
	(i) Note that $x = 2\sin^2\theta$ $\Rightarrow \dfrac{dx}{d\theta} = 2(2\sin\theta)\cos\theta = 4\sin\theta\cos\theta$. Also, when $x = 0$, $2\sin^2\theta = 0 \Rightarrow \theta = 0$; when $x = 1$, $2\sin^2\theta = 1 \Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}$ $\therefore \int_0^1 \sqrt{2x - x^2}\ \mathrm{d}x$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta - (2\sin^2\theta)^2} \times 4\sin\theta\cos\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta - 4\sin^4\theta} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta(1 - \sin^2\theta)} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta\cos^2\theta} \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 2\sin\theta\cos\theta \times 2\sin 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 2\sin^2 2\theta\ \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{4}} 1 - \cos 4\theta\ \mathrm{d}\theta$ $= \left[\theta - \dfrac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{4}}$ $= \dfrac{\pi}{4} - 0 = \dfrac{\pi}{4}$	
	(ii) $\int_0^1 \dfrac{x(1-x)}{\sqrt{2x - x^2}}\ \mathrm{d}x = \dfrac{1}{2}\int_0^1 \dfrac{x(2-2x)}{\sqrt{2x - x^2}}\ \mathrm{d}x$ $= \dfrac{1}{2}\int_0^1 (x) \times \dfrac{(2-2x)}{\sqrt{2x - x^2}}\ \mathrm{d}x$ $= \dfrac{1}{2}\left[\left[x \ \mathrm{g} 2\sqrt{2x - x^2}\right]_0^1 - \int_0^1 2\sqrt{2x - x^2}\ \mathrm{g}\,(1)\ \mathrm{d}x\right]$ $= \dfrac{1}{2}\left[2 - 0 - 2\int_0^1 \sqrt{2x - x^2}\ \mathrm{d}x\right]$ $= \dfrac{1}{2}\left[2 - 2 \times \dfrac{\pi}{4}\right] = 1 - \dfrac{\pi}{4}$	

	Alternatively, $\int_0^1 \frac{x(1-x)}{\sqrt{2x-x^2}}\,dx = \int_0^1 \sqrt{2x-x^2} - \frac{x}{\sqrt{2x-x^2}}\,dx$ $= \int_0^1 \sqrt{2x-x^2}\,dx + \frac{1}{2}\int_0^1 \frac{-2x+2}{\sqrt{2x-x^2}} - \frac{2}{\sqrt{2x-x^2}}\,dx$ $= \frac{\pi}{4} + \frac{1}{2}\int_0^1 \frac{-2x+2}{\sqrt{2x-x^2}}\,dx - \int_0^1 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$ $= \frac{\pi}{4} + \frac{1}{2}\left[2\sqrt{2x-x^2}\right]_0^1 - \left[\sin^{-1}(x-1)\right]_0^1$ $= \frac{\pi}{4} + \frac{1}{2}(2) - \left(\sin^{-1}0 - \sin^{-1}(-1)\right)$ $= \frac{\pi}{4} + 1 - \left(0 - \left(-\frac{\pi}{2}\right)\right)$ $= 1 - \frac{\pi}{4}$	

6.	Solution [9 marks]	
	(i) $x = 1 - \sin 2t$ and $y = \cos t$ $\frac{dx}{dt} = -2\cos 2t$ and $\frac{dy}{dt} = -\sin t$ $\frac{dy}{dx} = \frac{\sin t}{2\cos 2t}$	
	(ii) For stationary points, let $\frac{dy}{dx}=0$ sin t =0 $t = 0, \pi$ *correspondingly* $x = 1,\ 1$ $y = 1,\ -1$ Thus the two stationary points are (1,1) and (1, –1).	

(iii)
Graph of C



Normal at point P is parallel to x-axis implies tangent at point P is vertical (grad is undefined).

Set denominator of $\frac{\mathrm{d}y}{\mathrm{d}x}$ to be 0

$\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $\frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

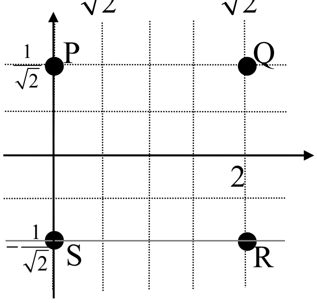
respectively,

$x = 0$, 2 , 0 or 2

$y = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$

Thus the normal occur at four points

$P(0, \frac{1}{\sqrt{2}})$, $Q(2, \frac{1}{\sqrt{2}})$, $R(2, -\frac{1}{\sqrt{2}})$ or $S(0, -\frac{1}{\sqrt{2}})$



The points form a rectangle with length 2 and breadth $\frac{2}{\sqrt{2}}$.

7.	Solution [9 marks]	
	(i) $\sqrt{1-x^2}\dfrac{dy}{dx}=\dfrac{y}{2}------(1)$ Diff Implicitly wrt x: $\sqrt{1-x^2}\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)\left(\dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}\right)(-2x)=\dfrac{1}{2}\dfrac{dy}{dx}$ $\sqrt{1-x^2}\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)\left(\dfrac{-x}{\sqrt{1-x^2}}\right)=\dfrac{1}{2}\dfrac{dy}{dx}------(2)$ $\times\sqrt{1-x^2}$ throughout, $(1-x^2)\dfrac{d^2y}{dx^2}-x\left(\dfrac{dy}{dx}\right)=\dfrac{1}{2}\dfrac{dy}{dx}\sqrt{1-x^2}$ $(1-x^2)\dfrac{d^2y}{dx^2}-x\left(\dfrac{dy}{dx}\right)=\dfrac{y}{4}$ (from (1)) Diff Implicitly wrt x: $(1-x^2)\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}(-2x)-\left[x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}\right]=\dfrac{1}{4}\dfrac{dy}{dx}$ $(1-x^2)\dfrac{d^3y}{dx^3}-3x\dfrac{d^2y}{dx^2}=\dfrac{5}{4}\dfrac{dy}{dx}$ ---------(3) Given that $x=0$, $y=1$, From (1), (2) and (3) respectively $\dfrac{dy}{dx}=\dfrac{1}{2}$, $\dfrac{d^2y}{dx^2}=\dfrac{1}{4}$, $\dfrac{d^3y}{dx^3}=\dfrac{5}{8}$ $\therefore\ y=1+x(\dfrac{1}{2})+\dfrac{x^2}{2!}(\dfrac{1}{4})+\dfrac{x^3}{3!}(\dfrac{5}{8})+...$ $=1+\dfrac{x}{2}+\dfrac{x^2}{8}+\dfrac{5x^3}{48}+...$(up to the x^3 term) <u>Alternatively</u>: $\left(1-x^2\right)\left(\dfrac{dy}{dx}\right)^2=\dfrac{y^2}{4}$ $\left(1-x^2\right)2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)-2x\left(\dfrac{dy}{dx}\right)^2=\dfrac{y}{2}\dfrac{dy}{dx}$ $4\left(1-x^2\right)\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}=y$ $4\left(1-x^2\right)\dfrac{d^3y}{dx^3}-8x\dfrac{d^2y}{dx^2}-4x\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}=\dfrac{dy}{dx}$ $4\left(1-x^2\right)\dfrac{d^3y}{dx^3}-12x\dfrac{d^2y}{dx^2}-5\dfrac{dy}{dx}=0$...... (the rest is the same as abv)	

	(ii) From (i)'s answer, diff term by term $y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} + \frac{x}{4} + \frac{5x^2}{16} + \ldots$(up to the x^2 term) Using the standard Maclaurin's formula for e^x, $\mathrm{e}^{y'} \approx \mathrm{e}^{\frac{1}{2}+\frac{x}{4}+\frac{5x^2}{16}}$ $= \mathrm{e}^{\frac{1}{2}} e^{\frac{x}{4}+\frac{5x^2}{16}}$ $\approx \mathrm{e}^{\frac{1}{2}}\left(1+\frac{x}{4}\right)$ (up to the x term) Hence $p = \sqrt{\mathrm{e}}$ and $q = \frac{\sqrt{\mathrm{e}}}{4}$	

8.	Solution [11 marks]	
	(a)(i) $\lvert z^* \rvert = 4$ since $\lvert z \rvert = 4$ $\left\lvert \frac{(1+ai)^3}{(1-i)^2} \right\rvert = 4$ $\frac{\lvert(1+ai)\rvert^3}{\lvert(1-i)\rvert^2} = 4$ $\frac{\left(\sqrt{1+a^2}\right)^3}{\left(\sqrt{2}\right)^2} = 4$ $\left(\sqrt{1+a^2}\right)^3 = 8$ $\sqrt{1+a^2} = 2$ $1+a^2 = 4$ $a = \pm\sqrt{3}$ $a = \sqrt{3}$ (reject -ve as a is given to be positive)	

	(a)(ii) $\arg z^* = \arg \dfrac{(1+ai)^3}{(1-i)^2}$ $= 3\arg(1+\sqrt{3}i) - 2\arg(1-i)$ $= 3(\frac{\pi}{3}) - 2(-\frac{\pi}{4})$ $= \dfrac{3\pi}{2}$ $= -\dfrac{\pi}{2}$ (principal value) $\arg z^* = -\arg z$ $\therefore \arg z = \dfrac{\pi}{2}$ (principal value) hence $z = 4e^{\frac{i\pi}{2}}$	
	(b) (i) since $i\pi$ is a root, $(i\pi)^4 + 2(i\pi)^3 + (\pi^2+e^2)(i\pi)^2 + p(i\pi) + q = 0$ $\pi^4 + 2\pi^3(-i) + (\pi^2+e^2)(-\pi^2) + i(p\pi) + q = 0$ $(q - e^2\pi^2) + i(p\pi - 2\pi^3) = 0$ Equating real and imaginary parts, $q = e^2\pi^2$ and $p = 2\pi^2$	

	(b)(ii) Now, $\mathrm{f}(z) = z^4 + 2z^3 + (\pi^2 + e^2)z^2 + 2\pi^2 z + e^2\pi^2$ As the polynomial f(z) has real coefficients, $-i\pi$ is also a root. Thus $(z - i\pi)$ and $(z + i\pi)$ are factors of f(z) i.e., $z^2 + \pi^2$ is a factor $\mathrm{f}(z) = \left(z^2 + \pi^2\right)\left(z^2 + mz + e^2\right)$ where m is to be determined Equating coeff of z^3 or z in f(z), $m = 2$ $\mathrm{f}(z) = \left(z^2 + \pi^2\right)\left(z^2 + 2z + e^2\right)$ for $\mathrm{f}(z) = 0$ $\left(z^2 + \pi^2\right)\left(z^2 + 2z + e^2\right) = 0$ $z = \pm i\pi$ or $z = \dfrac{-2 \pm \sqrt{4 - 4e^2}}{2}$ $= -1 \pm \sqrt{1 - e^2}$ $= -1 \pm i\sqrt{e^2 - 1}$ ($NOTE: 1 - e^2$ is negative) Hence the remaining roots are $-i\pi$, $-1 + i\sqrt{e^2 - 1}$ or $-1 + i\sqrt{e^2 - 1}$	

9.	Solution [11 marks]	
	(i) Let y_3, y_4 and y_6 be the 3rd, 4th and 6th terms of the arithmetic series. $y_3 = a$, $y_4 = ar$, $y_6 = ar^2$ $y_6 - y_4 = 2d = 2(y_4 - y_3)$ $\therefore ar^2 - ar = 2(ar - a)$ $r^2 - 3r + 2 = 0$ (shown) $r = 2$ or $r = 1$ Reject since $r = 1$ implies $d = 0$.	

(ii)

$\frac{y_4}{y_3} = r = 2$

$\Rightarrow y_4 = 2y_3$

$\Rightarrow a + 3d = 2(a + 2d)$

$\Rightarrow a = -d$ ---- (1)

<u>OR</u> $\frac{y_6}{y_4} = r = 2$

$\Rightarrow y_6 = 2y_4$

$\Rightarrow a + 5d = 2(a + 3d)$

$\Rightarrow a = -d$ ---- (1)

$S_4 = -6$

$\frac{4}{2}\left[2a + (4-1)d\right] = -6$ ---- (2)

Sub $a = -d$ into (2):

$2d = -6$

$d = -3$ and $a = 3$

<u>Alternatively</u>

$\frac{y_4}{y_3} = \frac{y_6}{y_4} \Rightarrow \frac{a+3d}{a+2d} = \frac{a+5d}{a+3d}$

$\Rightarrow ......\ d(a+d) = 0$

i.e. $d = 0$ (n.a.) or $d = -a$ ---- (1)

Solving (1) and (2),

$d = -3$ and $a = 3$

(iii)

Let g_m and a_m denote the m^{th} terms of G and A respectively.

$|g_m - a_m| > 10000$ ---- (3)

Since $A: 3, 0, -3, -6, \ldots$

while $G: -3, -6, -12, -24, \ldots$, $a_m > g_m$ for all $m \in \mathbb{Z}^+$.

$|g_m - a_m| > 10000$

$a_m - g_m > 10000$

$\left[3 + (m-1)(-3)\right] - (-3)(2)^{m-1} > 10000$

Using GC,

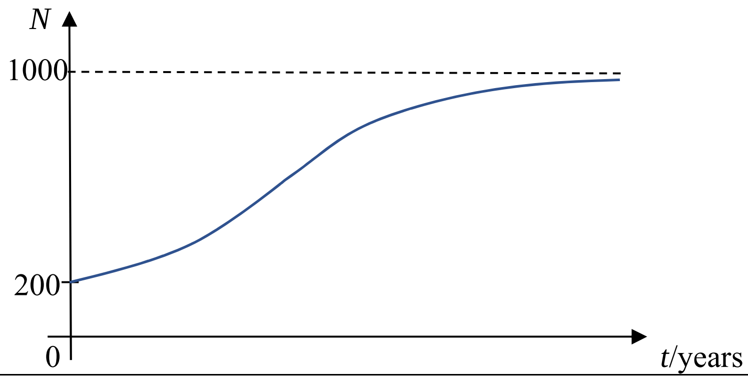
when $m = 12$, $a_m - g_m = 6114 < 10000$

when $m = 13$, $a_m - g_m = 12255 > 10000$

Hence, the least value of $m = 13$.

	(iv) $a_n = 3+(n-1)(-3)$ $x_n = (2k)^{a_n}$ $x_n = (2k)^{3+(n-1)(-3)}$ $x_n = (2k)^3\left[(2k)^{-3}\right]^{(n-1)}$ is a GP with first term $(2k)^3$ and common ratio $(2k)^{-3}$. The series $\sum_{n=1}^{\infty} x_n$ is the sum to infinity of the GP and it converges when $\left\|(2k)^{-3}\right\| < 1$ $\left\|\frac{1}{(2k)^3}\right\| < 1$ $(2k)^3 > 1$ or $(2k)^3 < -1$ $2k > 1$ or $2k < -1$ $k > \frac{1}{2}$ or $k < -\frac{1}{2}$	

10.	Solution [12 marks]	
	(i) Based on the proposed growth model by the scientists, $\frac{dN}{dt} = kN(1000-N)$, $k>0$. Given that $\frac{dN}{dt} = 125$ when $N = 500$, $125 = k(500)(1000-500)$ $\Rightarrow k = \frac{125}{500\times 500} = \frac{1}{2000}$ Thus, the required differential equation is $\frac{dN}{dt} = \frac{1}{2000}N(1000-N)$	

	(ii) Solving the D.E. $\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{1}{2000}N(1000-N)$, we have $\int\frac{1}{N(1000-N)}\,\mathrm{d}N = \int\frac{1}{2000}\,\mathrm{d}t$ $\frac{1}{1000}\int\frac{1}{N}+\frac{1}{1000-N}\,\mathrm{d}N = \int\frac{1}{2000}\,\mathrm{d}t$ $\int\frac{1}{N}+\frac{1}{1000-N}\,\mathrm{d}N = \int\frac{1000}{2000}\,\mathrm{d}t= \int\frac{1}{2}\,\mathrm{d}t$ Then, $\ln\lvert N\rvert-\ln\lvert 1000-N\rvert=\frac{1}{2}t+c$ Simplifying, $\ln\left\lvert\frac{N}{1000-N}\right\rvert=\frac{1}{2}t+c$ $\Rightarrow\left\lvert\frac{N}{1000-N}\right\rvert=e^{\frac{1}{2}t+c}\Rightarrow\frac{N}{1000-N}=\pm\,e^{\frac{1}{2}t+c}$ $\Rightarrow\frac{N}{1000-N}=Ae^{\frac{1}{2}t}$ where $A=\pm\mathrm{e}^{c}$ Given that when t = 0, N = 200, we have $A=\frac{200}{1000-200}=\frac{1}{4}$ Then, $\frac{N}{1000-N}=\frac{1}{4}e^{\frac{1}{2}t}$ Making N the subject: $4N=(1000-N)e^{\frac{1}{2}t}$ $\Rightarrow(4+e^{\frac{1}{2}t})N=1000$ $\Rightarrow N=\frac{1000e^{\frac{1}{2}t}}{4+e^{\frac{1}{2}t}}=\frac{1000}{4e^{-\frac{1}{2}t}+1}$	
		

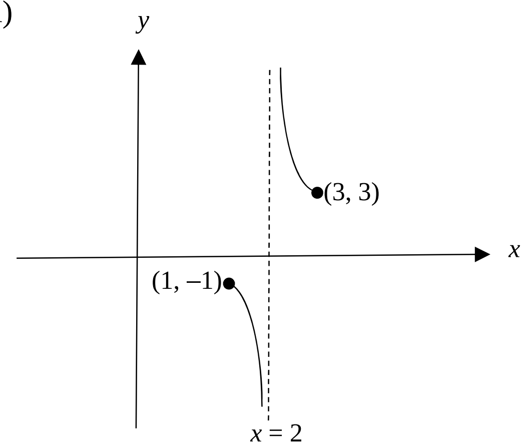
	(iii) Based on the graph, we can deduce that the value of N tends towards 1000 as t tends to infinity. Thus, it is not possible for the scientist to achieve their target of exceeding 1200 plants in the area after 5 years.	
	(iv) (1) With severe climate changes in the wildlife area, there may be harsh conditions to deter growth of the plant and thus causing decline in their numbers at times; or (2) With the news of their cure for terminal illnesses, it may attract further unwelcomed visitors to the area for harvesting of the plant for private profits, thus again causing significant decline in their numbers.	

11.	Solution [13 marks]	
	(i) For the x-coordinates of intersection points P and Q, we let $\dfrac{1}{1+4x^2}=\dfrac{1}{\sqrt{5-4x^2}}$ Using GC, we have $x=\dfrac{1}{2}$ for point P and $x=-\dfrac{1}{2}$ for point Q The area of region R $=\displaystyle\int_{-1/2}^{1/2} \frac{1}{1+4x^2}-\frac{1}{\sqrt{5-4x^2}}\ \mathrm{d}x$ $=\dfrac{1}{2}\displaystyle\int_{-1/2}^{1/2} \frac{2}{1+(2x)^2}\ \mathrm{d}x-\frac{1}{2}\int_{-1/2}^{1/2} \frac{2}{\sqrt{5-(2x)^2}}\ \mathrm{d}x$ $=\dfrac{1}{2}\Big[\tan^{-1}(2x)\Big]_{-\frac{1}{2}}^{\frac{1}{2}}-\dfrac{1}{2}\left[\sin^{-1}\left(\dfrac{2x}{\sqrt{5}}\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $=\dfrac{1}{2}\left[\dfrac{\pi}{4}-\left(-\dfrac{\pi}{4}\right)\right]-\dfrac{1}{2}\left[\sin^{-1}\dfrac{1}{\sqrt{5}}-\sin^{-1}\left(-\dfrac{1}{\sqrt{5}}\right)\right]$ $=\dfrac{\pi}{4}-\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right).$	

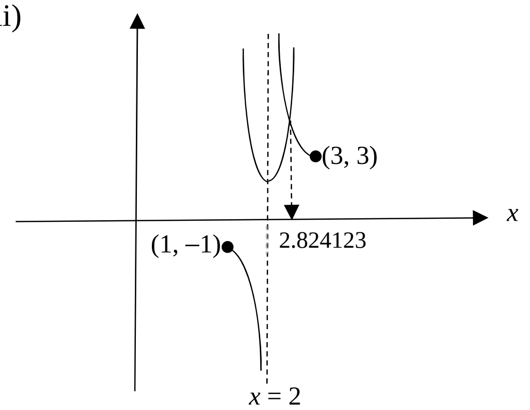
	(ii)(a) $y=\dfrac{1}{4x^2-8x+5}=\dfrac{1}{4(x-1)^2+1}$ and $y=\dfrac{1}{\sqrt{1+8x-4x^2}}=\dfrac{1}{\sqrt{5-4(x-1)^2}}$ Thus, the original curves are translated 1 unit in the positive x-direction. Thus, the exact area is still $\dfrac{\pi}{4}-\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$ unit^2. (ii)(b) $y=\dfrac{2}{1+4x^2}=2\left(\dfrac{1}{1+4x^2}\right)$, and $y=\dfrac{1}{\sqrt{\frac{5}{4}-x^2}}=\dfrac{1}{\sqrt{\frac{5-4x^2}{4}}}=\dfrac{2}{\sqrt{5-4x^2}}=2\left(\dfrac{1}{\sqrt{5-4x^2}}\right)$ Thus, the original curves undergoes scaling by scale factor 2 parallel to the y-axis. The area of bounded region $=2\times\displaystyle\int_{-1/2}^{1/2}\dfrac{1}{1+4x^2}-\dfrac{1}{\sqrt{5-4x^2}}\,\mathrm{d}x$ $=\dfrac{\pi}{2}-2\sin^{-1}\left(\dfrac{1}{\sqrt{5}}\right)$ unit^2.	
	(iii) For $y=\dfrac{1}{\sqrt{5-4x^2}}$: when $x=0$, $y=\dfrac{1}{\sqrt{5}}$, and $x^2=\dfrac{5}{4}-\dfrac{1}{4y^2}$ For $y=\dfrac{1}{1+4x^2}$: when $x=0$, $y=1$, and $x^2=\dfrac{1}{4y}-\dfrac{1}{4}$ Also, $Q=\left(\dfrac{1}{2},\dfrac{1}{2}\right)$	

	Then the required volume $= \pi \int_{\frac{1}{\sqrt{5}}}^{\frac{1}{2}} \frac{5}{4} - \frac{1}{4} y^{-2} \, \mathrm{d}y \; + \; \pi \int_{\frac{1}{2}}^{1} \frac{1}{4y} - \frac{1}{4} \, \mathrm{d}y$ $= 0.174$ unit^3	

Solutions to 2018 Y6 H2 Maths Prelim Exam P2

1.	Solution [10 marks]	
	(i) $y=\dfrac{x^2-3x+3}{x-2}=x-1+\dfrac{1}{x-2}$ $\dfrac{dy}{dx}=1-\dfrac{1}{(x-2)^2}$ For $x=1$ or 3, $\dfrac{dy}{dx}=0$ $\Rightarrow C$ has stationary points at $x=1$ and $x=3$. For $1<x<2$ or $2<x<3$, $\dfrac{dy}{dx}<0$ $\Rightarrow C$ is decreasing on $1<x<2$ and $2<x<3$. Alternatively, Since $1\le x\le 3$, $0\le (x-2)^2\le 1$ $\therefore \dfrac{1}{(x-2)^2}\ge 1$ $\therefore \dfrac{dy}{dx}\le 0$ i.e. the gradient is always non-positive.	
	(ii)  	

(iii)



$$\frac{x^2-4x+5}{x-2} \geq 3\left(x-2\right)^2$$

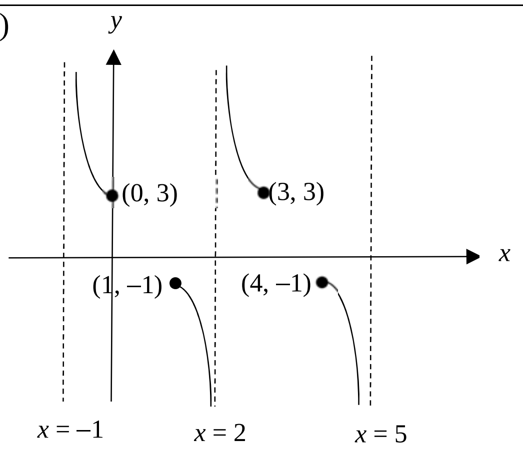
$$\frac{x^2-3x+3-x+2}{x-2} \geq 3\left(x-2\right)^2$$

$$\frac{x^2-3x+3}{x-2} \geq 3\left(x-2\right)^2+1$$

Sketch the addition graph $y=3\left(x-2\right)^2+1$.

Thus, $2 < x \leq 2.82$.

(iv)



2.	Solution [10 marks]	
	(i) Let $\dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}+\dfrac{C}{r+3}$. By cover up rule, $A=\dfrac{2(1)}{(1)(2)}=1;\ B=\dfrac{2(-1)}{(-1)(1)}=2;\ C=\dfrac{2(-3)}{(-2)(-1)}=-3;$ $\therefore \dfrac{2(2r+3)}{(r+1)(r+2)(r+3)}=\dfrac{1}{r+1}+\dfrac{2}{r+2}-\dfrac{3}{r+3}$	
	(ii) $\displaystyle\sum_{r=1}^{n}\frac{2r+3}{(r+1)(r+2)(r+3)}=\frac{1}{2}\sum_{r=1}^{n}\frac{2(2r+3)}{(r+1)(r+2)(r+3)}$ $\displaystyle=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}\right)$ $=\dfrac{1}{2}\left[\begin{array}{ccccc}\frac{1}{2}&+&\frac{2}{3}&-&\frac{3}{4}\\ \frac{1}{3}&+&\frac{2}{4}&-&\frac{3}{5}\\ \frac{1}{4}&+&\frac{2}{5}&-&\frac{3}{6}\\ \cdots&&\cdots&&\cdots\\ \frac{1}{n-1}&+&\frac{2}{n}&-&\frac{3}{n+1}\\ \frac{1}{n}&+&\frac{2}{n+1}&-&\frac{3}{n+2}\\ \frac{1}{n+1}&+&\frac{2}{n+2}&-&\frac{3}{n+3}\end{array}\right]$ $=\dfrac{1}{2}\left[\dfrac{3}{2}-\dfrac{1}{n+2}-\dfrac{3}{n+3}\right]$ $=\dfrac{3}{4}-\dfrac{1}{2(n+2)}-\dfrac{3}{2(n+3)}$	

	(iii) $\frac{21}{3\times4\times5}+\frac{27}{4\times5\times6}+\frac{33}{5\times6\times7}+......$ $=3\left(\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\frac{11}{5\times6\times7}+....\right)$ $=3\sum_{r=2}^{\infty}\frac{2r+3}{(r+1)(r+2)(r+3)}$ $=3\left[\sum_{r=1}^{\infty}\frac{2r+3}{(r+1)(r+2)(r+3)}-\frac{2(1)+3}{(1+1)(1+2)(1+3)}\right]$ $=3\lim_{n\to\infty}\left[\frac{3}{4}-\frac{1}{2(n+2)}-\frac{3}{2(n+3)}\right]-\frac{5}{8}$ As $n\to\infty$, $\frac{3}{2(n+2)}\to0$ and $\frac{9}{2(n+3)}\to0$. Hence the series is convergent, and the sum = $3\left[\frac{3}{4}-0-0\right]-\frac{5}{8}=\frac{13}{8}$	

3.	Solution [10 marks]	
	(i) $\cos^{-1}\left(\frac{\begin{pmatrix}-1\\0\\4\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}{\sqrt{17}\sqrt{5}}\right)=\cos^{-1}\frac{7}{\sqrt{17}\sqrt{5}}=40.6013°$ Angle between roofs $=180°-40.6013°=139.4°$	

(ii) Plane *ABGF*: $\mathbf{r}\cdot\begin{pmatrix}-1\\0\\4\end{pmatrix}=\begin{pmatrix}0\\0\\20\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\4\end{pmatrix}=80$

Plane *BCHG*: $\mathbf{r}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}50\\0\\25\end{pmatrix}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}=100$

Taking intersection and using GC,

$l_{BG}:\mathbf{r}=\begin{pmatrix}40\\0\\30\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$

When $y = 50$, $\lambda = 50$.
Thus coordinates of sensor is (40, 50, 30).

(iii) Coordinates of *B* is (40, 0, 30).

Cable is anchored to the ground at the point
P(25, −10, 0).

Required length = perpendicular distance from *P* to line *AB*.

$\overrightarrow{PA}=\begin{pmatrix}0\\0\\20\end{pmatrix}-\begin{pmatrix}25\\-10\\0\end{pmatrix}=\begin{pmatrix}-25\\10\\20\end{pmatrix}$

$\overrightarrow{AB}=\begin{pmatrix}40\\0\\30\end{pmatrix}-\begin{pmatrix}0\\0\\20\end{pmatrix}=\begin{pmatrix}40\\0\\10\end{pmatrix}$

Length of shortest cable

$=\left|\overrightarrow{PA}\times\begin{pmatrix}4\\0\\1\end{pmatrix}\right|=\left|\begin{pmatrix}-25\\10\\20\end{pmatrix}\times\begin{pmatrix}4\\0\\1\end{pmatrix}\right|$

$=\left|\frac{\begin{pmatrix}10\\105\\-40\end{pmatrix}}{\sqrt{17}}\right|=27.4$ metres

	(iv) The thickness of materials used for building is negligible.	

4.	Solution [10 marks]	
	(i) f^{-1} exists because f is one-one on $x < m < 2$. $D_f = (-\infty, m)$, $R_f = \left(0, \frac{1}{(m-2)^2}\right)$ For f^2 to exist, $R_f \subseteq D_f$. $0 < \frac{1}{(m-2)^2} \le m$ $m(m-2)^2 \ge 1$ $m^3 - 4m^2 + 4m - 1 \ge 0$ Solving with GC, $0.382 \le m \le 1$ or $m \ge 2.84$ (reject since $m < 2$)	
	(ii) $y = \frac{1}{(x-2)^2} \Rightarrow x = 2 \pm \frac{1}{\sqrt{y}}$ Since $x < m < 2$, $x = 2 - \frac{1}{\sqrt{y}}$. $f^{-1}(x) = 2 - \frac{1}{\sqrt{x}}$ $D_{f^{-1}} = \left(0, \frac{1}{(m-2)^2}\right)$ $R_{f^{-1}} = (-\infty, m)$ $f^2(x)$ $= f\left(\frac{1}{(x-2)^2}\right)$ $= \frac{1}{\left(\frac{1}{(x-2)^2} - 2\right)^2}$ $= \frac{(x-2)^4}{\left(1 - 2(x-2)^2\right)^2}$	

$D_{f^2} = (-\infty, m)$

$$(-\infty, m) \xrightarrow{f} \left(0, \frac{1}{(m-2)^2}\right) \xrightarrow{f} \left(\frac{1}{4}, \frac{(m-2)^4}{\left(1-2(m-2)^2\right)^2}\right) = R_{f^2}$$

5. Solution [7 marks]

(i) Using the following possibility table, we have:

2nd die (columns), **1st die** (rows)

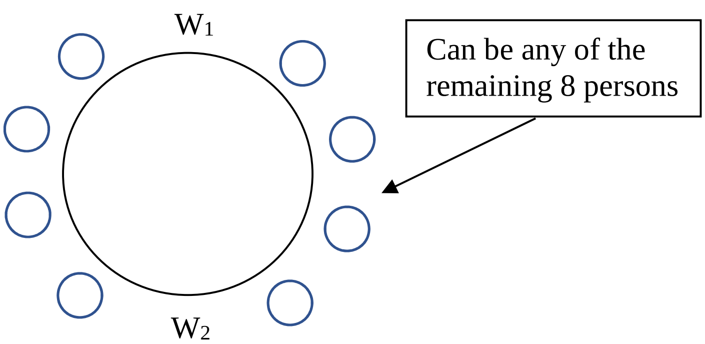
	1	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

From the table, $\mathrm{P}(X = 2) = \frac{9}{36} = \frac{1}{4}$

Probability distribution of *X*:

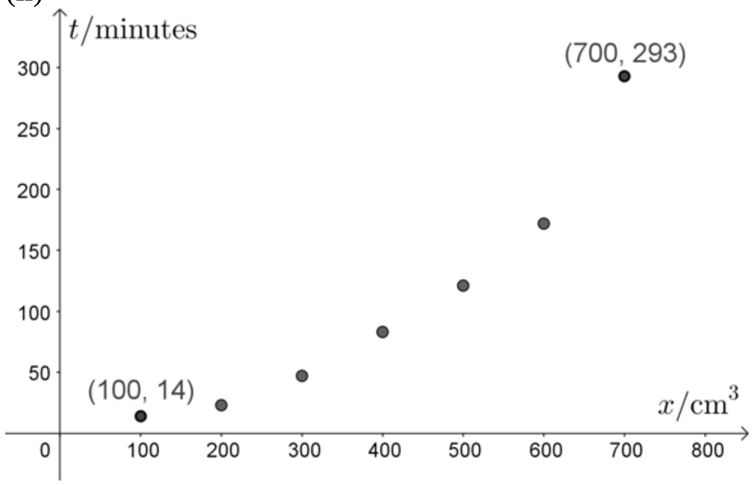
x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

	(ii) $\mathrm{E}(X)=\sum_{x=1}^{6} x\mathrm{P}\left(X=x\right)=\frac{91}{36}$ (or 2.53) $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ $=\left(\sum_{x=1}^{6} x^2\mathrm{P}\left(X=x\right)\right)-\left(\sum_{x=1}^{6} x\mathrm{P}\left(X=x\right)\right)^2$ $=\frac{301}{36}-\left(\frac{91}{36}\right)^2$ $=\frac{2555}{1296}$ (or 1.97)	
	(iii) P(Ivan wins) $=\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)+\ldots$ $=\frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^2}=\frac{4}{7}$ E(gain) for Ivan $=\frac{4}{7}\times 7+\frac{3}{7}\times(-10)=-\$0.29$ Since Ivan is expecting to lose 29 cents, Jon has a higher expected gain. <u>Alternatively</u>, P(Jon wins) $=\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)+\ldots$ $=\frac{\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}{1-\left(\frac{3}{4}\right)^2}=\frac{3}{7}$ E(gain) for Jon $=\frac{3}{7}\times 10+\frac{4}{7}\times(-7)=\$0.29$ Since Jon is expecting to win 29 cents (i.e. Ivan is expected to lose 29 cents), he has a higher expected gain.	

6.	Solution [8 marks]	
	(a) (i) Number of ways = 10! = 3 628 800 ways (ii) $\uparrow W_1 \uparrow W_2 \uparrow W_3 \uparrow W_4 \uparrow W_5 \uparrow W_6 \uparrow$ $\uparrow$ Position for each of the men Number of ways = 10! – No. of ways all men are separated $=10! - 6! \times \binom{7}{4} 4!$ = 3 628 800 – 604 800 = 3 024 000 ways (iii) $\uparrow M_1 \uparrow M_2 \uparrow M_3 \uparrow M_4 \uparrow$ $\uparrow$ Position for the following "items" $\boxed{W_1W_2W_3W_4W_5}$, W_6 Number of ways = $4! \times \binom{5}{2} 2! \times \overbrace{\binom{6}{5}}^{\text{the w group}} 5!$ = 345 600 ways	
	(b) Let particular women be W_1, W_2  Fix W_1 and W_2, then arrange the remaining 8 persons. $\Rightarrow$ No. of ways = 8! $\therefore$ Required probability $= \frac{8!}{(10-1)!} = \frac{1}{9}$	

7.	Solution [10 marks]	
	(i) Let X be the r.v. "no of customers who uses cashless payment" out of 30. $X \sim \mathrm{B}\left(30, \frac{p}{100}\right)$ $\mathrm{P}(X \le 1) = 0.245$ $\mathrm{P}(X = 0) + \mathrm{P}(X = 1) = 0.245$ $\binom{30}{0}\left(1-\frac{p}{100}\right)^{30} + \binom{30}{1}\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^{29} = 0.245$ $\left(1-\frac{p}{100}\right)^{30} + 0.3p\left(1-\frac{p}{100}\right)^{29} - 0.245 = 0$ Using Solver in GC, $p = 8.811$	
	(ii) Let Y be the r.v. "no of customers who use cashless payment" out of 29. $Y \sim \mathrm{B}(29, 0.3)$ Required Probability $= \mathrm{P}(Y = 9) \times 0.3$ $= 0.0471872337$ ≈ 0.0472	
	(iii) $X \sim \mathrm{B}(30, 0.3)$ Let W be the r.v. "no of customers who use cashless payment" out of 60. $W \sim \mathrm{B}(60, 0.3)$ Requires probability $= \frac{\mathrm{P}(X_1 = 13, X_2 \le 2) + \mathrm{P}(X_1 = 14, X_2 \le 1) + \mathrm{P}(X_1 = 15, X_2 = 0)}{\mathrm{P}(W < 16)}$ $= \frac{2 \times \left(\mathrm{P}(X = 13)\mathrm{P}(X \le 2) + \mathrm{P}(X = 14)\mathrm{P}(X \le 1) + \mathrm{P}(X = 15)\mathrm{P}(X = 0)\right)}{\mathrm{P}(W \le 15)}$ = 0.0008312264247 ≈ 0.000831	

	(iv) The assumption is that the people being surveyed are all different in both samples. (<u>OR</u> Samples are independently and identically distributed.)	

8.	Solution [10 marks]	
	(i) $\overline{x}=400$ $\overline{t}=-65.1429+0.4318(400)$ $\overline{t}=107.5771$ $14+23+47+83+a+172+293=7\times\overline{t}$ $632+a=753.0397$ $a\approx 121$ (shown)	
	(ii)  $r=0.9414677299\approx 0.941$ (3 sig fig)	
	(iii) (a) Although the r value is 0.941 which is close to 1 suggesting that there is a strong linear correlation, the scatter diagram shows that the relationship is more of a curvilinear one, with t increasing increasingly as x increases. Thus, the regression line is not appropriate. (b) The regression line predicts that for volume less than a certain amount, it takes negative time for the water to cool to room temperature which is not possible in the context. Thus, the regression line is not appropriate.	

	(iv) Calculating the r- values for models A and B, Model A: r-value = 0.987 while Model B: $t = ae^{bx} \Rightarrow \ln t = \ln a + bx$, r-value = 0.994, Since the $\|r\|$-value for model B is closer to 1 compared to model A, model B is the better model. Regression line of ln t on x is $\ln t = 2.224858298 + 0.0050332105x$ $\ln t = 2.22 + 0.00503x$ (3 sig fig)	
	(v) When x = 450, $\ln t = 2.224858298 + 0.0050332105(450)$ $t = 89.10389273$ ≈ 89.1 minutes (3 sig fig) The estimate is reliable as the appropriate regression line is used (t being the dependent variable); r = 0.994 which is close to 1, suggesting a strong linear correlation between ln t and x; and the estimated point where x = 450 is within the data range of 100 to 700.	

9.	Solution [12 marks]	
	(i) $\bar{x} = \dfrac{207000}{60} = 3450$ $\sum(x-3400) = 3000$ $s^2 = \dfrac{1}{59}\left[5450000 - \dfrac{3000^2}{60}\right] = 89830.50847 \approx 89800$	

	(ii) Let μ be the mean starting salaries of fresh graduates. Test $H_0:\ \mu = 3500$ (i.e. manager's belief is incorrect) Against $H_1:\ \mu \neq 3500$ (i.e. manager's belief is correct) Perform a 2-tailed test at 5% level of significance. Test Statistics: Under H_0, $\overline{X} \sim \mathrm{N}\left(3500, \frac{s^2}{60}\right)$ approximately $Z = \frac{\overline{X} - 3500}{s/\sqrt{60}} \sim \mathrm{N}(0,1)$ approximately Using GC, *p*-value = 0.196284 (>0.05) Do not reject H_0. There is insufficient evidence at 5% level of significance that the human resource manager's belief is correct.	
	(iii) *p*-value of 0.196284 is the smallest value of significance level for which the claim that the mean starting monthly salary of a fresh university graduate is \$3500 would be rejected. Or *p*-value of 0.196284 is the probability of obtaining a sample mean as extreme as the one obtained, assuming the claim that the mean starting monthly salary of a fresh university graduate is \$3500 is true.	

	(iv) Test $H_0:\ \mu = 3500$ Against $H_1:\ \mu \neq 3500$ Perform a 2-tailed test at 5% level of significance. $s^2 = \frac{50}{49} \times 342^2 = 119351.0204$ Test Statistics: Under H_0, $\overline{Y} \sim \mathrm{N}\left(3500, \frac{119351.0204}{50}\right)$ approximately $Z = \frac{\overline{Y} - 3500}{\sqrt{119024.4898/50}} \sim \mathrm{N}(0,1)$ approximately Since H_0 is rejected, $\frac{\overline{y} - 3500}{\sqrt{119351.0204/50}} < -1.95996$ $\overline{y} < 3404.241954$ <u>Or</u> $\frac{\overline{y} - 3500}{\sqrt{119351.0204/50}} > 1.95996$ $\overline{y} > 3595.758046$ Answer: $\overline{y} < 3400$ or $\overline{y} > 3600$ (to 3sf)	
	(v) $\overline{X} \sim \mathrm{N}\left(3500, \frac{543^2}{50}\right)$ approximately by CLT since sample size (50) is large. $\mathrm{P}(\overline{X} < 3350) = 0.0254$	

10.	Solution [13 marks]	
	(i) $\mu = \frac{60+120}{2} = 90$ minutes	

	(ii) Let C be Gary's timing for cycling. $C \sim N\left(90, \sigma^2\right)$ $Z = \dfrac{C-90}{\sigma} \sim N(0,1)$ $\mathrm{P}(C<80) = 0.158655$ $\mathrm{P}\left(Z < \dfrac{80-90}{\sigma}\right) = 0.158655$ $\dfrac{80-90}{\sigma} = -1.000001057$ $\sigma = \dfrac{-10}{-1.000001057} \approx 10$	
	(iii) Let R be Gary's timing for running. $R \sim N\left(100, 15^2\right)$ $3C - 2R \sim N(70, 1800)$ $\mathrm{P}(3C-2R>60) = 0.5931680976 \approx 0.593$ (3 sig fig)	
	(iv) Required probability $= \mathrm{P}(C<80) \times \mathrm{P}(R<90)$ $= 0.0400592579$ ≈ 0.0401 (3 sig fig)	
	(v) $C + R \sim N(190, 325)$ $\mathrm{P}(C+R<170) = 0.1336287896 \approx 0.134$ (3 sig fig)(shown) The event in part (iv) is but a proper subset of that in this part, hence the probability is here is greater than that in (iv).	

	(vi) Let X denote the number of duathlons which Gary manages to achieve his target timing, out of 9 duathlons. $X \sim B(9, 0.134)$ $\mathrm{P}(3 \leq X < 6) = \mathrm{P}(X \leq 5) - \mathrm{P}(X \leq 2)$ $= 0.1080999944$ ≈ 0.108 (3 sig fig) The timings for each of the duathlons are independent.	